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Dirichlet Problem for Elliptic Equations in Weighted Sobolev Spaces on Unbounded Domains

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Abstract. In this review article we give an overview on some known results recently obtained within the study of the Dirichlet problem for a class of second-order linear elliptic equations in weighted Sobolev spaces on unbounded domains.

Key Words and Phrases: Elliptic equations, Discontinuous coefficients, A priori bounds, Weighted Sobolev spaces

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1. Introduction

In this review article we are interested in the study, in an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$, of the Dirichlet problem

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)\,, \\ \\ Lu = f\,, \quad f \in L_s^p(\Omega)\,, \end{cases} \tag{1.1}$$

where L is the uniformly elliptic second order linear differential operator with discontinuous coefficients

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a,$$

$p > 1$, $s \in \mathbb{R}$, and $W_s^{2,p}(\Omega)$, $\overset{\circ}{W}{}_s^{1,p}(\Omega)$ and $L_s^p(\Omega)$ are certain classes of weighted Sobolev and Lebesgue spaces, recently introduced in [5].

In particular, we consider a weight ρ^s that is a power of a function ρ of class $C^2(\bar{\Omega})$ such that $\rho : \bar{\Omega} \to \mathbb{R}_+$ and

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall\, |\alpha| \leq 2,$$

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$$\lim_{|x|\to+\infty}\left(\rho(x)+\frac{1}{\rho(x)}\right)=+\infty \quad \text{and} \quad \lim_{|x|\to+\infty}\frac{\rho_x(x)+\rho_{xx}(x)}{\rho(x)}=0.$$

To fix the ideas one can think of the function

$$\rho(x)=(1+|x|^2)^t, \quad t\in\mathbb{R}\backslash\{0\}.$$

We assume that the leading coefficients satisfy hypotheses of Miranda's type, having in mind the classical paper [4] where the a_{ij} have derivatives in $L^n(\Omega)$. To be more precise, we suppose that the $(a_{ij})_{x_h}$ belong to suitable Morrey type spaces that are a generalization to unbounded domains of the classical Morrey spaces. This hypothesis is of crucial relevance in our analysis since it allows to rewrite the operator in variational form in order to exploit some non-weighted a priori estimates, proved for divergence form problems in [6, 7, 8]. The main result of this work, proved in [10], is the following weighted $W^{2,p}$-bound, $p>1$,

$$\|u\|_{W^{2,p}_s(\Omega)}\le c\|Lu\|_{L^p_s(\Omega)}, \quad \forall\, u\in W^{2,p}_s(\Omega)\cap \overset{\circ}{W}{}^{1,p}_s(\Omega), \tag{1.2}$$

that allows to deduce, via the method of continuity along a parameter, the solvability of the Dirichlet problem (1.1). We refer the reader also to [5] where, for $p=2$, the corresponding non-weighted and weighted cases have been studied.

2. A Class of Weighted Sobolev Spaces

Let us introduce the class of weighted spaces we are interested in. They have been recently defined in [5], where detailed descriptions of all the properties below are provided.

To this aim, given an open subset Ω of $\mathbb{R}^n$, not necessarily bounded, $n\ge 2$, and $k\in\mathbb{N}_0$, we consider a weight function $\rho:\bar\Omega\to\mathbb{R}_+$ such that $\rho\in C^k(\bar\Omega)$ and

$$\sup_{x\in\Omega}\frac{|\partial^\alpha\rho(x)|}{\rho(x)}<+\infty, \quad \forall\,|\alpha|\le k. \tag{2.1}$$

An example is given by

$$\rho(x)=(1+|x|^2)^t, \quad t\in\mathbb{R}.$$

For $k\in\mathbb{N}_0$, $p\in[1,+\infty[$ and $s\in\mathbb{R}$, and given ρ satisfying (2.1), we define the weighted Sobolev space $W^{k,p}_s(\Omega)$ as the space of distributions u on Ω such that

$$\|u\|_{W^{k,p}_s(\Omega)}=\sum_{|\alpha|\le k}\|\rho^s\partial^\alpha u\|_{L^p(\Omega)}<+\infty. \tag{2.2}$$

Furthermore, $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ denotes the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ and $L^p_s(\Omega)=W^{0,p}_s(\Omega)$.

Let us state now a useful lemma obtained from the more general result of [5]. We refer the reader to [1] for the definition of sets having the segment property.

Lemma 2.1. *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$. If Ω has the segment property and assumption* (2.1) *is satisfied, then*

$$C_o^k(\Omega) \subset \overset{\circ}{W}{}_s^{k,p}(\Omega).$$

We now prove the following fundamental result (see also Lemmas 2.1, 2.2 and 2.5 of [5]):

Lemma 2.2. *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$. If Ω has the segment property and assumption* (2.1) *is satisfied, then the map*

$$u \longrightarrow \rho^s u$$

defines a topological isomorphism from $W_s^{k,p}(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$.

Proof. Let us start by observing that, due to assumption (2.1), one has

$$\sup_{x\in\Omega} \frac{|\partial^\alpha \rho^s(x)|}{\rho^s(x)} < +\infty \quad \forall s \in \mathbb{R}, \quad \forall\, |\alpha| \le k. \tag{2.3}$$

This can be proved by induction. Indeed, (2.1) implies

$$|(\rho^s)_{x_i}| = |s\rho^{s-1}\rho_{x_i}| \le c_1 \rho\rho^{s-1} = c_1\rho^s, \quad i = 1, \dots, n,$$

with $c_1 = c_1(s)$ positive constant. Therefore (2.3) is true for $|\,\alpha| = 1$.◀

Assume now that (2.3) holds for any β such that $|\beta| < |\alpha|$ and any $s \in \mathbb{R}$, and fix a β such that $|\beta| = |\alpha| - 1$. Then, using (2.1) and by the induction hypothesis written for $s - 1$, we have

$$|\partial^\alpha \rho^s| = |\partial^\beta (\rho^s)_{x_i}| = |\partial^\beta (s\rho^{s-1}\rho_{x_i})| \le$$
$$c_2 \sum_{\gamma\le\beta} |\partial^{\beta-\gamma}\rho_{x_i}\ \partial^\gamma \rho^{s-1}| \le c_3\rho\rho^{s-1} = c_3\rho^s, \text{ for } i = 1, \dots, n,$$

with $c_2 = c_2(s)$ and $c_3 = c_3(s)$ positive constants. That is, (2.3) is true also for α.

In view of (2.3) we have that

$$|\partial^\alpha(\rho^s u)| \le c_4 \sum_{\gamma\le\alpha} |\partial^{\alpha-\gamma}\rho^s\ \partial^\gamma u| \le c_5 \sum_{\gamma\le\alpha} |\rho^s \partial^\gamma u|, \quad \forall\, |\alpha| \le k,$$

with $c_4 = c_4(s)$ and $c_5 = c_5(s)$ positive constants. Therefore, there exists a positive constant $c_6 = c_6(s)$ such that

$$||\rho^s u||_{W^{k,p}(\Omega)} \le c_6 ||u||_{W_s^{k,p}(\Omega)}. \tag{2.4}$$

Moreover, there exists also a positive constant $c_7 = c_7(s)$ such that

$$||u||_{W_s^{k,p}(\Omega)} \le c_7 ||\rho^s u||_{W^{k,p}(\Omega)}. \tag{2.5}$$

In order to prove (2.5), let us show that

$$|\rho^s \partial^\alpha u| \leq c_8 \sum_{\delta \leq \alpha} |\partial^\delta (\rho^s u)|, \quad \forall\, |\alpha| \leq k, \tag{2.6}$$

with $c_8 = c_8(s)$ positive constant. Again, this will be done by induction. From (2.3) one has

$$|\rho^s u_{x_i}| = |(\rho^s u)_{x_i} - (\rho^s)_{x_i} u| \leq c_9\big((\rho^s u)_x + \rho^s |u|\big),$$

for $i = 1, \ldots, n$, with $c_9 = c_9(s)$. Hence, (2.6) holds for $|\alpha| = 1$.

If (2.6) holds for any δ such that $|\delta| < |\alpha|$, then, using again (2.3) and by the induction hypothesis, we have

$$|\rho^s \partial^\alpha u| \leq |\partial^\alpha (\rho^s u)| + c_{10} \sum_{\delta < \alpha} |\partial^{\alpha-\delta} \rho^s| |\partial^\delta u| \leq$$
$$|\partial^\alpha(\rho^s u)| + c_{11} \sum_{\delta < \alpha} |\rho^s \partial^\delta u| \leq c_{12} \sum_{\delta \leq \alpha} |\partial^\delta (\rho^s u)|,$$

with $c_{10} = c_{10}(s)$, $c_{11} = c_{11}(s)$ and $c_{12} = c_{12}(s)$ positive constants. Combining (2.4) and (2.5) we obtain the first part of the claim.

To finish our proof we have to show that $u \in \overset{\circ}{W}{}^{k,p}_s(\Omega)$ if and only if $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$.

If $u \in \overset{\circ}{W}{}^{k,p}_s(\Omega)$, there exists a sequence $(\phi_h)_{h\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to u in $W^{k,p}_s(\Omega)$. Therefore, fixed $\varepsilon \in \mathbb{R}_+$, by (2.4) there exists $h_0 \in \mathbb{N}$ such that

$$\|\rho^s(\phi_h - u)\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall h > h_0. \tag{2.7}$$

Fix $h_1 > h_0$. Clearly $\rho^s \phi_{h_1} \in \overset{\circ}{W}{}^{k,p}(\Omega)$, because of its compact support. Therefore, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to $\rho^s\phi_{h_1}$ in $W^{k,p}(\Omega)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\|\psi_n - \rho^s \phi_{h_1}\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{2.8}$$

Putting together (2.7) and (2.8) we get

$$\|\psi_n - \rho^s u\|_{W^{k,p}(\Omega)} \leq \|\psi_n - \rho^s\phi_{h_1}\|_{W^{k,p}(\Omega)} + \|\rho^s\phi_{h_1} - \rho^s u\|_{W^{k,p}(\Omega)} < \varepsilon,$$

$\forall\, n > n_0$. Thus $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$.

Vice-versa, if we assume that $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$, we find a sequence $(\phi_h)_{h\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to $\rho^s u$ in $W^{k,p}(\Omega)$. Hence, by (2.5) there exists $h_0 \in \mathbb{N}$ such that

$$\|\rho^{-s}\phi_h - u\|_{W^{k,p}_s(\Omega)} < \frac{\varepsilon}{2}, \quad \forall h > h_0. \tag{2.9}$$

Fix $h_1 > h_0$. Since $\rho^{-s}\phi_{h_1} \in C^k_o(\Omega)$, which is contained in $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ by Lemma 2.1, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to $\rho^{-s}\phi_{h_1}$ in $W^{k,p}_s(\Omega)$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\|\psi_n - \rho^{-s}\phi_{h_1}\|_{W^{k,p}_s(\Omega)} < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{2.10}$$

From (2.9) and (2.10) we get

$$\|\psi_n - u\|_{W^{k,p}_s(\Omega)} \leq \|\psi_n - \rho^{-s}\phi_{h_1}\|_{W^{k,p}_s(\Omega)} + \|\rho^{-s}\phi_{h_1} - u\|_{W^{k,p}_s(\Omega)} < \varepsilon,$$

$\forall\, n > n_0$. So that $u \in \overset{\circ}{W}{}^{k,p}_s(\Omega)$. □

From now on, let us suppose that the weight ρ satisfies (2.1) with $k = 2$ and the additional assumptions holds:

$$\lim_{|x|\to+\infty} \left(\rho(x) + \frac{1}{\rho(x)}\right) = +\infty \quad \text{and} \quad \lim_{|x|\to+\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0. \tag{2.11}$$

As an example, we can then consider

$$\rho(x) = (1 + |x|^2)^t, \quad t \in \mathbb{R}\backslash\{0\}.$$

We associate to ρ the function σ defined by

$$\begin{cases} \sigma = \rho & \text{if } \rho \to +\infty \quad \text{for } |x| \to +\infty, \\ \sigma = \dfrac{1}{\rho} & \text{if } \rho \to 0 \quad\quad \text{for } |x| \to +\infty. \end{cases} \tag{2.12}$$

It is easily seen that σ verifies (2.1) too, and moreover

$$\lim_{|x|\to+\infty} \sigma(x) = +\infty, \qquad \lim_{|x|\to+\infty} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \tag{2.13}$$

Now, fix a cutoff function $f \in C^\infty_\circ(\bar{\mathbb{R}}_+)$ such that

$$0 \leq f \leq 1, \quad f(t) = 1 \text{ if } t \in [0,1], \quad f(t) = 0 \text{ if } t \in [2, +\infty[$$

and set

$$\zeta_k : x \in \bar{\Omega} \longrightarrow f\left(\frac{\sigma(x)}{k}\right), \quad k \in \mathbb{N}$$

and

$$\Omega_k = \{x \in \Omega \ : \ \sigma(x) < k\ \}, \quad k \in \mathbb{N}. \tag{2.14}$$

If we define the sequence

$$\eta_k : x \in \bar{\Omega} \longrightarrow 2k\,\zeta_k(x) + (1 - \zeta_k(x))\sigma(x), \quad k \in \mathbb{N},$$

one has that, for any $k \in \mathbb{N}$,

$$\sigma \sim \eta_k. \tag{2.15}$$

Furthermore, concerning the derivatives, one can show that, for any $k \in \mathbb{N}$,

$$\frac{(\eta_k)_x}{\eta_k} \leq c'_1 \sup_{\Omega\backslash\Omega_k} \frac{\sigma_x}{\sigma} \qquad \text{in } \overline{\Omega}, \tag{2.16}$$

$$\frac{(\eta_k)_{xx}}{\eta_k} \quad \leq \quad c_2' \sup_{\Omega\setminus\Omega_k} \frac{\sigma_x^2 + \sigma\sigma_{xx}}{\sigma^2} \qquad \text{in } \overline{\Omega}. \tag{2.17}$$

Moreover,

$$\lim_{k\to+\infty} \sup_{\Omega\setminus\Omega_k} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \tag{2.18}$$

3. The Spaces of the Coefficients

Here we recall the definitions and the main properties of Morrey type spaces. These spaces were first introduced in [11] (see also [2] for further investigations) to extend the classical notion of Morrey spaces to the case of unbounded domains. Let us consider, then, an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$. By $\Sigma(\Omega)$ we denote the σ-algebra of all Lebesgue measurable subsets of Ω. For $E \in \Sigma(\Omega)$, χ_E is its characteristic function, $|E|$ its Lebesgue measure and $E(x,r) = E \cap B(x,r)$ ($x \in \mathbb{R}^n$, $r \in \mathbb{R}_+$), where $B(x,r)$ is the open ball centered at x with radius r. The class of restrictions to $\bar{\Omega}$ of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ is $\mathfrak{D}(\bar{\Omega})$. For $q \in [1,+\infty[$, $L^q_{\text{loc}}(\bar{\Omega})$ is the class of all functions $g : \Omega \to \mathbb{R}$ such that $\zeta\, g \in L^q(\Omega)$ for any $\zeta \in \mathfrak{D}(\bar{\Omega})$.

For $q \in [1,+\infty[$ and $\lambda \in [0,n[$, the Morrey type space $M^{q,\lambda}(\Omega)$ is made up of all the functions g in $L^q_{loc}(\bar{\Omega})$ such that

$$||g||_{M^{q,\lambda}(\Omega)} = \sup_{\substack{\tau\in]0,1] \\ x\in\Omega}} \tau^{-\lambda/q} ||g||_{L^q(\Omega(x,\tau))} < +\infty. \tag{3.1}$$

The closures of $C_\circ^\infty(\Omega)$ and $L^\infty(\Omega)$ in $M^{q,\lambda}(\Omega)$ are denoted by $M_\circ^{q,\lambda}(\Omega)$ and $\tilde{M}^{q,\lambda}(\Omega)$, respectively.

One has the inclusion

$$M_\circ^{q,\lambda}(\Omega) \subset \tilde{M}^{q,\lambda}(\Omega).$$

Moreover,

$$M^{q,\lambda}(\Omega) \subseteq M^{q_0,\lambda_0}(\Omega) \quad \text{ if } q_0 \leq q \text{ and } \frac{\lambda_0 - n}{q_0} \leq \frac{\lambda - n}{q}.$$

We put $M^q(\Omega) = M^{q,0}(\Omega)$, $\tilde{M}^q(\Omega) = \tilde{M}^{q,0}(\Omega)$ and $M_\circ^q(\Omega) = M_\circ^{q,0}(\Omega)$.

Now, let us define the moduli of continuity of functions belonging to $\tilde{M}^{q,\lambda}(\Omega)$ or $M_\circ^{q,\lambda}(\Omega)$. For $h \in \mathbb{R}_+$ and $g \in M^{q,\lambda}(\Omega)$, we set

$$F[g](h) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_{x\in\Omega} |E(x,1)| \leq \frac{1}{h}}} \|g\, \chi_E\|_{M^{q,\lambda}(\Omega)}.$$

Given a function $g \in M^{q,\lambda}(\Omega)$, the following characterizations hold:

$$g \in \tilde{M}^{q,\lambda}(\Omega) \iff \lim_{h\to+\infty} F[g](h) = 0,$$

and

$$g \in M_{\circ}^{q,\lambda}(\Omega) \iff \lim_{h \to +\infty} \big(F[g](h) + ||(1-\zeta_h) g||_{M^{q,\lambda}(\Omega)} \big) = 0,$$

where ζ_h denotes a function of class $C_o^\infty(\mathbb{R}^n)$ such that

$$0 \le \zeta_h \le 1\,, \qquad \zeta_h|_{\overline{B(0,h)}} = 1\,, \qquad \operatorname{supp} \zeta_h \subset B(0,2h).$$

Thus, if g is a function in $\tilde{M}^{q,\lambda}(\Omega)$, a *modulus of continuity* of g in $\tilde{M}^{q,\lambda}(\Omega)$ is a map $\widetilde{\sigma}^{q,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) \le \widetilde{\sigma}^{q,\lambda}[g](h), \qquad \lim_{h \to +\infty} \widetilde{\sigma}^{q,\lambda}[g](h) = 0.$$

While, if g belongs to $M_o^{q,\lambda}(\Omega)$, a *modulus of continuity* of g in $M_o^{q,\lambda}(\Omega)$ is an application $\sigma_o^{q,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) + \|(1-\zeta_h)\, g\|_{M^{q,\lambda}(\Omega)} \le \sigma_o^{q,\lambda}[g](h),$$

$$\lim_{h \to +\infty} \sigma_o^{q,\lambda}[g](h) = 0.$$

We end this section with some boundedness results for the multiplication operator

$$u \longrightarrow g\, u, \tag{3.2}$$

where the function g belongs to suitable Morrey type spaces (see also [3] for more details).

Theorem 3.1. *If $g \in M^{q,\lambda}(\Omega)$, with $q > 2$ and $\lambda = 0$ for $n = 2$, and $q \in]2, n]$ and $\lambda = n - q$ for $n > 2$, then the operator in (3.2) is bounded from $\overset{\circ}{W}{}^{1,p}(\Omega)$ to $L^2(\Omega)$. Moreover, there exists a constant $C \in \mathbb{R}_+$ such that*

$$\|g\, u\|_{L^2(\Omega)} \le C\, \|g\|_{M^{q,\lambda}(\Omega)}\, \|u\|_{W^{1,2}(\Omega)} \quad \forall\, u \in \overset{\circ}{W}{}^{1,p}(\Omega),$$

with $C = C\,(n, q)$.

Let $p > 1$ and $r,\, t \in [p, +\infty[$. If Ω is an open subset of $\mathbb{R}^n$ having the cone property and $g \in M^r(\Omega)$, with $r > p$ for $p = n$, then the operator in (3.2) is bounded from $W^{1,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$ such that

$$\|g\, u\|_{L^p(\Omega)} \le c\, \|g\|_{M^r(\Omega)}\, \|u\|_{W^{1,p}(\Omega)} \quad \forall\, u \in W^{1,p}(\Omega),$$

with $c = c\,(\Omega, n, p, r)$.

If $g \in M^t(\Omega)$, with $t > p$ for $p = n/2$, then the operator in (3.2) is bounded from $W^{2,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c' \in \mathbb{R}_+$ such that

$$\|g\, u\|_{L^p(\Omega)} \le c'\, \|g\|_{M^t(\Omega)}\, \|u\|_{W^{2,p}(\Omega)} \quad \forall\, u \in W^{2,p}(\Omega)$$

with $c' = c'(\Omega, n, p, t)$.

4. Preliminary Results

From now on Ω will be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$, satisfying the uniform $C^{1,1}$-regularity property (see [1] for details).

Let us start by recalling some known results, recently proved in the works [6, 7, 8, 9], that play an essential role in the study of our weighted problem.

The first one is an L^p-bound, $p > 1$, obtained within the study of the Dirichlet problem

$$\begin{cases} u \in \overset{\circ}{W}{}^{1,p}(\Omega)\,, \\ \bar{L}u = f,\ f \in W^{-1,2}(\Omega)\,, \end{cases} \tag{4.1}$$

where $\bar{L}$ is a second order linear differential operator in divergence form

$$\bar{L} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(\bar{a}_{ij}\frac{\partial}{\partial x_i} + d_j\Big) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{4.2}$$

whose coefficients are such that

$$\begin{cases} \bar{a}_{ij} \in L^\infty(\Omega), \quad i,j = 1,\dots,n\,, \\ \\ \bar{a}_{ij} = \bar{a}_{ji}, \quad i,j = 1,\dots,n, \\ \\ \exists\, \nu > 0 \quad : \quad \displaystyle\sum_{i,j=1}^{n} \bar{a}_{ij}\,\xi_i\,\xi_j \;\geq\; \nu|\xi|^2 \quad \text{a.e. in } \Omega\,,\ \forall \xi \in \mathbb{R}^n\,, \end{cases} \tag{h_1}$$

$$\begin{cases} b_i, d_i \in M_o^{2t,\lambda}(\Omega)\,, \quad i = 1,\dots,n\,, \\ c \in M^{t,\lambda}(\Omega)\,, \\ \ \text{with } t > 1 \text{ and } \lambda = 0 \text{ if } n = 2\,, \\ \ \text{with } t \in]1, n/2] \text{ and } \lambda = n - 2t \text{ if } n > 2\,, \end{cases} \tag{h_2}$$

$$c - \sum_{i=1}^{n} (d_i)_{x_i} \geq \mu, \tag{h_3}$$

in the distributional sense on Ω, with μ positive constant.

Let us associate to $\bar{L}$ the bilinear form

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{n} (\bar{a}_{ij}u_{x_i} + d_j\ u)\ v_{x_j}\ + (\sum_{i=1}^{n} b_i u_{x_i} + cu)v \Big)\, dx, \tag{4.3}$$

$u, v \in \overset{\circ}{W}{}^{1,p}(\Omega)$, and note that, as a consequence of Theorem 3.1, the form a is continuous on $\overset{\circ}{W}{}^{1,p}(\Omega)\times \overset{\circ}{W}{}^{1,p}(\Omega)$ and then the operator $\bar{L} : \overset{\circ}{W}{}^{1,p}(\Omega) \to W^{-1,2}(\Omega)$ is continuous as well. The above mentioned bound is the assertion of the following theorem:

Theorem 4.1. *Assume that hypotheses $(h_1) - (h_3)$ are satisfied. If $f \in L^2(\Omega) \cap L^p(\Omega)$, for some $p \in]1, +\infty[$, then the solution u of problem (4.1) is in $L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \tag{4.4}$$

where C is a constant depending on $n, t, p, \nu, \mu, ||b_i - d_i||_{M^{2t,\lambda}(\Omega)}$, $i = 1, \dots, n$.

This last bound plays a crucial role in the proof of a non-weighted estimate for the non variational problem below.

Let

$$L = - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \,, \tag{4.5}$$

with the following conditions on the coefficients:

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega), \ \ i, j = 1, \dots, n \,, \\ \\ \exists\, \nu > 0 \ \ : \ \ \displaystyle\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \ \geq \ \nu |\xi|^2 \ \ \text{a.e. in } \Omega \,, \ \forall \xi \in \mathbb{R}^n, \\ \\ (a_{ij})_{x_h} \in M_o^{q,\lambda}(\Omega) \,, \ \ i, j, h = 1, \dots, n, \ \text{with} \\ q > 2 \text{ and } \lambda = 0 \quad \text{for } n = 2 \,, \\ q \in]2, n] \text{ and } \lambda = n - q \quad \text{for } n > 2 \,. \end{cases} \tag{h_1'}$$

$$\begin{cases} a_i \in M_o^r(\Omega) \,, \ \ i = 1, \dots, n \,, \ \text{with} \\ r > 2 \text{ if } p \leq 2 \text{ and } r = p \text{ if } p > 2 \quad \text{for } n = 2, \\ r \geq p \text{ and } r \geq n, \ \text{with } r > p \text{ if } p = n \quad \text{for } n > 2, \end{cases} \tag{h_2'}$$

$$\begin{cases} a \in \tilde{M}^{\,t}(\Omega) \,, \ \text{with} \\ t = p \quad \text{for } n = 2 \,, \\ t \geq p \text{ and } t \geq \dfrac{n}{2}, \ \text{with } t > p \text{ if } p = \dfrac{n}{2} \quad \text{for } n > 2, \\ \operatorname*{ess\,inf}_{\Omega} a = a_0 > 0. \end{cases} \tag{h_3'}$$

In view of Theorem 3.1, under the assumptions (h_1') - (h_3'), the operator $L : W^{2,p}(\Omega) \to L^p(\Omega)$ is also bounded.

The existence of the derivatives of the a_{ij} is a crucial hypothesis that allows to rewrite the operator L in divergence form and exploit (4.4) to obtain the following theorem, whose proof can be found in Theorem 3.2 of [9].

Theorem 4.2. *Let L be defined in (4.5). If hypotheses (h_1')-(h_3') are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c \|Lu\|_{L^p(\Omega)} \quad \forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega) \,, \tag{4.6}$$

with $c = c(\Omega, n, \nu, p, r, t, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.

As we will see in the last section, this last preliminary estimate together with the topological isomorphism in Lemma 2.2 will allow us to show the analogous weighted estimate.

5. Main Theorem

Here we present the main result of this work, proved in Theorems 4.2 and 5.2 of [10]. The proof of this theorem will be performed in two steps: in the first one we show a weighted a priori bound and in the second we derive the solvability of a weighted Dirichlet problem associated to the operator L.

Theorem 5.1. *Let L be defined in (4.5). If hypotheses (h_1')-(h_3') are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \le c\|Lu\|_{L_s^p(\Omega)} \quad \forall\, u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)\,, \tag{5.1}$$

with $c = c(\Omega, n, s, \nu, p, r, t, ||a_{ij}||_{L^\infty(\Omega)}, ||a_i||_{M^r(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.
Moreover, the problem

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)\,, \\ \\ Lu = f\,, \quad f \in L_s^p(\Omega)\,, \end{cases} \tag{5.2}$$

is uniquely solvable.

Proof. Step 1. Fix $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)$. If $\rho \to +\infty$ for $|x| \to +\infty$, then $\sigma = \rho$. Thus, in view of the isomorphism of Lemma 2.2, one has that $\sigma^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$. Now if we write $\eta_k = \eta$ for a fixed $k \in \mathbb{N}$, since η and σ are equivalent, one also has that $\eta^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$. Hence, the estimate in Theorem 4.2 implies that there exists $c_0 \in \mathbb{R}_+$, such that

$$||\eta^s\, u||_{W^{2,p}(\Omega)} \le c_0 ||L(\eta^s u)||_{L^p(\Omega)}, \tag{5.3}$$

with $c_0 = c_0(\Omega, n, \nu, p, r, t, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.
Easy calculations provide that

$$\begin{aligned} L(\eta^s u) &= \eta^s Lu - s \sum_{i,j=1}^n a_{ij}\Big((s-1)\eta^{s-2}\eta_{x_i}\eta_{x_j}u + \eta^{s-1}\eta_{x_i x_j}u + \\ &+ 2\eta^{s-1}\eta_{x_i}u_{x_j}\Big) + s\sum_{i=1}^n a_i \eta^{s-1}\eta_{x_i} u. \end{aligned} \tag{5.4}$$

By (5.3) and (5.4) we get

$$||\eta^s\, u||_{W^{2,p}(\Omega)} \le c_1\Big(||\eta^s\, Lu||_{L^p(\Omega)} + \sum_{i,j=1}^n \big(||\eta^{s-2}\eta_{x_i}\eta_{x_j} u||_{L^p(\Omega)} +$$

$$+ \quad ||\eta^{s-1}\eta_{x_i x_j} u||_{L^p(\Omega)} + ||\eta^{s-1}\eta_{x_i} u_{x_j}||_{L^p(\Omega)}\big) +$$
$$+ \quad \sum_{i=1}^{n} ||a_i \eta^{s-1}\eta_{x_i} u||_{L^p(\Omega)}\Big), \tag{5.5}$$

where $c_1 \in \mathbb{R}_+$ depends on the same parameters as c_0 and on s.
Moreover, from Lemma 3.1 and (2.16) we obtain

$$||a_i \eta^{s-1}\eta_{x_i} u||_{L^p(\Omega)} \le c_2 \sup_{\Omega\setminus\Omega_k} \frac{\sigma_x}{\sigma} \; ||a_i||_{M^r(\Omega)} ||\eta^s u||_{W^{1,p}(\Omega)}, \tag{5.6}$$

with $c_2 = c_2(\Omega, n, p, r)$.
Combining (2.16), (2.17), (5.5) and (5.6), we have

$$\begin{aligned} ||\eta^s\, u||_{W^{2,p}(\Omega)} \quad &\le \quad c_3\Bigg[||\eta^s\, Lu||_{L^p(\Omega)} + \Bigg(\sup_{\Omega\setminus\Omega_k} \frac{\sigma_x^2 + \sigma\,\sigma_{xx}}{\sigma^2} + \\ &+ \quad \sup_{\Omega\setminus\Omega_k} \frac{\sigma_x}{\sigma}\Bigg) \; ||\eta^s\, u||_{W^{2,p}(\Omega)}\Bigg], \end{aligned} \tag{5.7}$$

where c_3 depends on the same parameters as c_1 and on $\|a_i\|_{M^r(\Omega)}$.
Furthermore, by (2.18) it follows that there exists $k_o \in \mathbb{N}$ such that

$$\Bigg(\sup_{\Omega\setminus\Omega_{k_o}} \frac{\sigma_x^2 + \sigma\,\sigma_{xx}}{\sigma^2} + \sup_{\Omega\setminus\Omega_{k_o}} \frac{\sigma_x}{\sigma}\Bigg) \le \frac{1}{2\,c_3}. \tag{5.8}$$

Therefore, if we denote by η the function η_{k_o}, putting together (5.7) and (5.8) we obtain

$$||\eta^s\, u||_{W^{2,p}(\Omega)} \le 2c_3 ||\eta^s\, Lu||_{L^p(\Omega)}. \tag{5.9}$$

This last estimate, (2.15) with for $k = k_o$ and the topological isomorphism in Lemma 2.2 give

$$\sum_{|\alpha|\le 2} ||\sigma^s \partial^\alpha u||_{L^p(\Omega)} \le c_4 ||\sigma^s Lu||_{L^p(\Omega)}, \tag{5.10}$$

with c_4 depending on the same parameters as c_3 and on k_o.

If $\rho \to 0$ for $|x| \to +\infty$, then $\sigma = \rho^{-1}$. Thus, again by Lemma 2.2, one has that $\sigma^{-s} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$. Therefore, arguing as to get (5.10), one obtains

$$\sum_{|\alpha|\le 2} ||\sigma^{-s} \partial^\alpha u||_{L^p(\Omega)} \le c_5 ||\sigma^{-s} Lu||_{L^p(\Omega)}. \tag{5.11}$$

This concludes the proof of (5.1).

Step 2. For each $\tau \in [0,1]$ we put

$$L_\tau = \tau(L) + (1-\tau)(-\Delta + b)\,,$$

with b given by

$$b = 1 + \Big| - s(s+1) \sum_{i=1}^{n} \frac{\sigma_{x_i}^2}{\sigma^2} + s \sum_{i=1}^{n} \frac{\sigma_{x_i x_i}}{\sigma} \Big| \tag{5.12}$$

if $\rho \to +\infty$ for $|x| \to +\infty$, or

$$b = 1 + \Big| - s(s-1) \sum_{i=1}^{n} \frac{\sigma_{x_i}^2}{\sigma^2} - s \sum_{i=1}^{n} \frac{\sigma_{x_i x_i}}{\sigma} \Big| \tag{5.13}$$

if $\rho \to 0$ for $|x| \to +\infty$.

By (5.1) one gets

$$||u||_{W_s^{2,p}(\Omega)} \leq c||L_\tau u||_{L_s^p(\Omega)},$$

$$\forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)\,,\ \forall \tau \in\ [0,1]\,.$$

Therefore, using the result of Lemma 5.1 of [10] and the method of continuity along a parameter we obtain the solvability of problem (5.2). ◀

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