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On the 2-Generator p-Groups with Non-cyclic Commutator Subgroup

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Abstract. A complete classification of p-groups of every nilpotency class is given by R.J. Miech in 1975 where the commutator subgroup is cyclic. M.R. Bacon in 1993 and L.-C. Kappe in 1999 studied and classified 2-generated p-groups based on the nilpotency 2 groups which have the cyclic commutator subgroups. In this paper, we attempt to study the finite 2-generator p-groups of nilpotency class 3, where the commutator subgroup is non-cyclic, and identify the structure of one class of such p-groups for every prime $p \neq 2, 3$.

Key Words and Phrases: 2-Generator p-groups, Nilpotency Class 3
2000 Mathematics Subject Classifications: 20D15, 20E99

1. Introduction

Most of the effort of finite group theory has been directed at the classification of p-groups and there are still many outstanding problems in this area. In the present paper, we want to concentrate on 2-generator p-groups of nilpotency class 3, where the commutator subgroup is non-cyclic. First of all we give a short history of the finite 2-generator p-groups of class 2. In 1975, Miech classified all finite 2-generator p-groups with cyclic commutator subgroup for odd p [6]. In 1989, Trebenko attempted to classify all 2-generator groups of nilpotency class two, along very different lines from those of Miech [8]. This attempt was flawed. Later, Bacon and Kappe tried to correct Trebenko's paper to produce a classification of finite 2-generator p-groups of class two, where p is an odd prime, but their classification was still incomplete [2]. In 1999, Kappe, Visscher, and Sarmin extended the classification to the case of 2-groups, but again with incomplete descriptions [4]. Finally the classification of infinite 2-generator groups of class two has been done by Sarmin, also along the same lines [7]. All of these classifications were still incomplete until recently in [1], a new classification for the 2-generator p-groups of nilpotency class two is given that corrects and simplifies previous classifications for these groups. These classifications have been used to compute the nonabelian tensor squares of these groups (see [2, 4, 7]) and determine those that are capable (see [3, 5]). All of these attempts consider cyclic commutator subgroups.

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The goal of this paper is to study the finite 2-generator p-groups of nilpotency class 3 with the non-cyclic commtator subgroups and identify the structure of one class where $p \neq 2, 3$. Our notation are fairly simple and standard. Let G be a group. The *commutator* $[x, y]$ is defined by $x^{-1}y^{-1}xy$, for the elements x and y of G. The group generated by all commutators $[x, y]$, where $x, y \in G$, is denoted by $[G, G]$ or G'. We define the *lower central series*

$$\gamma_1(G) \geq \gamma_2(G) \geq ... \geq \gamma_i(G) \geq ...$$

of G inductively as follows:

$$\gamma_1(G) = G,\ \gamma_2(G) = [G, G],\ \gamma_i(G) = [\gamma_{i-1}(G), G]\ (i = 3, 4, ..).$$

If there exists an integer c such that $\gamma_{c+1}(G) = 1$, then G is said to be *nilpotent*, and if c is the smallest such integer, c is called the *nilpotency class* of G. Also, if $x_1, x_2, ..., x_n$ are elements of G we define

$$[x_1, x_2, ..., x_n] = [[x_1, x_2, ..., x_{n-1}], x_n]\ (n = 3, 4, ...).$$

2. Preliminary Results

In this section, we prove some results concerning the nilpotent groups of class 3. The first lemma is well-known.

Lemma 1. *Let G be a group and $x, y, z \in G$. Then*
(i) $[xy, z] = [x, z][x, z, y][y, z]$.
(ii) $[x, yz] = [x, z][x, y][x, y, z]$.
(iii) $[x^{-1}, y] = ([x, y]^{-1})^{x^{-1}} = [x, y]^{-1}[[x, y]^{-1}, x^{-1}]$.
(iv) $[x, y^{-1}] = ([x, y]^{-1})^{y^{-1}} = [x, y]^{-1}[[x, y]^{-1}, y^{-1}]$.

Lemma 2. *Let G be a nilpotent group of class 3 and $x, y, z \in G$. Then*
(i) $[G', G] \leq Z(G)$.
(ii) $[xy, z] = [x, z][y, z][x, z, y]$ *and* $[x, yz] = [x, z][x, y][x, y, z]$.
(iii) $[x, y, z]^{-1} = [y, x, z]$.

Proof. It may be easily proved by using the previous lemma and the nilpotency class of G. ◄

Lemma 3. *Let G be a nilpotent group of class 3 and $x, y, z \in G$. Then for every positive integers m, n and k,*
(i) $[x, y, z]^n = [[x, y]^n, z]$.
(ii) $[x, y, z]^{mnk} = [x^m, y^n, z^k]$.
(iii) $[x^m, y^n] = [x, y]^{mn}[x, y, x]^{n\binom{m}{2}}[x, y, y]^{m\binom{n}{2}}$.
(iv) $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}[y, x, x]^{\frac{n(n-1)(n-2)}{6}}[y, x, y]^{\frac{n(n-1)(2n-1)}{6}}$.

Proof. We can use the inductive method to prove the lemma. For example, by assuming (i), (ii) and (iii), we give the complete proof of (iv). Obviously, the result is true for $n = 1$. Assume that it is true for $n \geq 1$. Then

$$\begin{aligned}
(xy)^{n+1} &= (xy)^n(xy) \\
&= x^n y^n [y,x]^{\binom{n}{2}} [y,x,x]^{\frac{n(n-1)(n-2)}{6}} [y,x,y]^{\frac{n(n-1)(2n-1)}{6}} (xy) \\
&= x^n y^n ([y,x]^{\binom{n}{2}} x) y [y,x,x]^{\frac{n(n-1)(n-2)}{6}} [y,x,y]^{\frac{n(n-1)(2n-1)}{6}} \\
&= x^n (y^n x) [y,x]^{\binom{n}{2}} y [y,x,x]^{\frac{n(n-1)(n-2)}{6}+\binom{n}{2}} [y,x,y]^{\frac{n(n-1)(2n-1)}{6}} \\
&= x^{n+1} y^n [y,x]^{\binom{n}{2}+n} y [y,x,x]^{\frac{(n+1)n(n-1)}{6}} [y,x,y]^{\frac{n(n-1)(2n-1)}{6}+\binom{n}{2}} \\
&= x^{n+1} y^n ([y,x]^{\binom{n+1}{2}} y) [y,x,x]^{\frac{(n+1)n(n-1)}{6}} [y,x,y]^{\frac{(n+1)n(n-1)}{3}} \\
&= x^{n+1} y^{n+1} [y,x]^{\binom{n+1}{2}} [y,x,x]^{\frac{(n+1)n(n-1)}{6}} [y,x,y]^{\frac{(n+1)n(n-1)}{3}+\binom{n+1}{2}} \\
&= x^{n+1} y^{n+1} [y,x]^{\binom{n+1}{2}} [y,x,x]^{\frac{(n+1)n(n-1)}{6}} [y,x,y]^{\frac{(n+1)n(2n+1)}{6}}.
\end{aligned}$$

◀

The following two lemmas are directly for the groups of nilpotency class 3 and some parts of the assertions are almost similar to those of used for the groups of nilpotency class 2 in [4].

Lemma 4. *Let $G = \langle a, b\rangle$ be a nilpotent group of nilpotency class 3. Then*

(i) *$G' = \langle [a, b], [a, b, a], [a, b, b]\rangle$.*

(ii) *if $n = l.c.m(|[a, b, a]|, |[a, b, b]|)$, then $\langle [a, b]\rangle \cap Z(G) = \langle [a, b]^n\rangle$.*

(iii) *if $n = l.c.m(|[a, b]|, |[a, b, a]|, |[a, b, b]|)$ and n is odd number, then $\langle a\rangle \cap Z(G) = \langle a^n\rangle$ and $\langle b\rangle \cap Z(G) = \langle b^n\rangle$.*

Proof. (i) Using previous lemma and the relations $[b, a, a] = [a, b, a]^{-1}$ and $[b, a, b] = [a, b, b]^{-1}$, we get the validity of this assertion.

(ii) Let $\langle [a, b]\rangle \cap Z(G) = \langle [a, b]^k\rangle$, where $k \in \mathbb{N}$. Since $[a, b]^k \in Z(G)$, we have $[a, b, a]^k = [a, b, b]^k = 1$. Thus $n \leq k$. On the other hand, $[a, b, a]^n = [a, b, b]^n = 1$. This implies $[a, b]^n \in Z(G)$. Consequently, $k \leq n$.

(iii) This is proved similar to (ii). ◀

Lemma 5. *Let $G = \langle a, b\rangle$ be a p-group of nilpotency class 3.*

(i) *If $l.c.m(|[a, b, a]|, |[a, b, b]|) = p^\sigma$ and $\langle a\rangle \cap G' = \langle a^{p^k}\rangle$, then $k \geq \sigma$.*

(ii) *If $l.c.m(|[a, b, a]|, |[a, b, b]|) = p^\sigma$, $|[a, b]| = p^\gamma$ and $\langle [a, b, a], [a, b, b]\rangle \leq \langle [a, b]\rangle$, then $\gamma \geq 2\sigma$.*

Proof. (i) As $a^{p^k} \in G'$, we have $[a^{p^k}, b] \in Z(G)$. So $[a, b, a]^{p^k} = [a, b, b]^{p^k} = 1$. Thus $k \geq \sigma$.

(ii) We know that $[a, b, a]$ and $[a, b, b]$ are the central elements. Now by using the hypothesis, we have $\langle [a, b, a], [a, b, b]\rangle \leq Z(G) \cap \langle [a, b]\rangle = \langle [a, b]^{p^\sigma}\rangle$. This implies $\gamma \geq 2\sigma$. ◀

Due to the following lemma, the cases $p \geq 5$, $p = 2$ and $p = 3$ have to be handled differently.

Lemma 6. *Let G be a finite 2-generator p-group of nilpotency class 3. Further, let $b \in G$ be an element of minimal order not in $\Phi(G)$, the Frattini subgroup of G, and a be an element of minimal order such that $\langle a, b\rangle = G$. If $\langle a\rangle \cap \langle b\rangle \neq 1$, then $p = 2$ or $p = 3$.*

Proof. Choose a and b as in the hypothesis. Let $p \neq 2, 3$ and $\langle d\rangle = \langle a\rangle \cap \langle b\rangle$. Then there exist integers u and v such that $d = a^u = b^v$. Since G is not cyclic, we have $u = sp^i, v = tp^j$, with $i, j \in \mathbb{N}$, and $(s, p) = (t, p) = 1$. Now $|b| \leq |a|$ implies $j \leq i$. Set $b_1 = a^{-sp^{i-j}}b^t$. Since $(t, p) = 1$, it follows that $\langle b_1, a\rangle = G$. Now by Lemma 3 we have

$$
\begin{aligned}
b_1^{p^j} &= (a^{-sp^{i-j}}b^t)^{p^j} \\
&= a^{-sp^i}b^{tp^j}[b^t, a^{-sp^{i-j}}]^{\binom{p^j}{2}}[b^t, a^{-sp^{i-j}}, a^{-sp^{i-j}}]^{\frac{p^j(p^j-1)(p^j-2)}{6}} \\
&\quad [b^t, a^{-sp^{i-j}}, b^t]^{\frac{p^j(p^j-1)(2p^j-1)}{6}} \\
&= [b^t, a^{-sp^{i-j}}]^{\binom{p^j}{2}}[b^{tp^j}, a^{-sp^{i-j}}, a^{-sp^{i-j}}]^{\frac{(p^j-1)(p^j-2)}{6}} \\
&\quad [b^t, a^{-sp^i}, b^t]^{\frac{(p^j-1)(2p^j-1)}{6}} \\
&= [b^t, a^{-sp^{i-j}}]^{p^j(\frac{p^j-1}{2})} \\
&= ([b^t, a^{-sp^i}][b^t, a^{-sp^{i-j}}, a^{-sp^{i-j}}]^{-\binom{p^j}{2}})^{\frac{p^j-1}{2}} \\
&= [b^t, a^{-sp^{i-j}}, a^{-sp^{i-j}}]^{-p^j(\frac{p^j-1}{2})^2} \\
&= [b^{tp^j}, a^{-sp^{i-j}}, a^{-sp^{i-j}}]^{-(\frac{p^j-1}{2})^2} \\
&= 1.
\end{aligned}
$$

By choice of b, $|b| \leq |b_1|$, and then $b^{p^j} = 1$. Thus $d = 1$. ◄

3. Result

In this section, let G be a finite 2-generator p-group of nilpotency class 3. Choose b an element of minimal order not in $\Phi(G)$ and a an element of minimal order such that $\langle a, b\rangle = G$. Also, let $p \neq 2, 3$. By Lemma 6, $\langle a\rangle \cap \langle b\rangle = 1$. So the following cases occur:

$$\langle b\rangle \cap G' = 1, \langle a\rangle \cap G' = 1, \tag{1}$$
$$\langle b\rangle \cap G' = 1, \langle a\rangle \cap G' = G', \tag{2}$$
$$\langle b\rangle \cap G' = 1, 1 \neq \langle a\rangle \cap G' \subset G', \tag{3}$$
$$\langle a\rangle \cap G' = 1, \langle b\rangle \cap G' = G', \tag{4}$$
$$\langle a\rangle \cap G' = 1, 1 \neq \langle b\rangle \cap G' \subset G', \tag{5}$$
$$1 \neq \langle a\rangle \cap G' \subset G', 1 \neq \langle b\rangle \cap G' \subset G'. \tag{6}$$

In the cases (2) and (4), the commutator subgroup is cyclic and they have been surveyed by R.J. Miech in [6]. Interchanging a and b in (5) gives the case (3). So, three cases-(1), (3) and (6) remain. We here investigate the case (1) and identify the structure of group G.

Proposition 1. *By the above hypothesis, if $\langle a\rangle \cap G' = \langle b\rangle \cap G' = 1$, then*

$$G \cong (G' \rtimes \langle a\rangle) \rtimes \langle b\rangle,$$

where, $|a| = p^{\alpha}, |b| = p^{\beta}, |[a,b]| = p^{\gamma}, |[a,b,b]| = p^{\delta_1}, |[a,b,a]| = p^{\delta_2}$, and the integers $\alpha, \beta, \gamma, \delta_1$ and δ_2 satisfy the conditions $\alpha \geq \beta \geq \gamma \geq 1, \gamma \geq \delta_1, \gamma \geq \delta_2$ and $\delta_1 + \delta_2 \geq 1$.

Proof. Let $|a| = p^{\alpha}, |b| = p^{\beta}, |[a,b]| = p^{\gamma}, |[a,b,b]| = p^{\delta_1}$ and $|[a,b,a]| = p^{\delta_2}$. Then by choosing b (b is of minimal order), we get $\alpha \geq \beta$. Using Lemma 3(iii), we deduce that $[a,b]^{p^{\beta}} = [a,b^{p^{\beta}}][a,b,b]^{-\binom{p^{\beta}}{2}} = [a,b,b^{p^{\beta}}]^{-(\frac{p^{\beta}-1}{2})} = 1$. So, $\beta \geq \gamma$. Since $G' \neq 1$ we have $\gamma \geq 1$ and therefore, $\alpha \geq \beta \geq \gamma \geq 1$. Again using Lemma 3(i), we have $[a,b,b]^{p^{\gamma}} = [[a,b]^{p^{\gamma}}, b] = 1$ and $[a,b,a]^{p^{\gamma}} = [[a,b]^{p^{\gamma}}, a] = 1$ and so, $\gamma \geq \delta_1$ and $\gamma \geq \delta_2$. The assertion $\delta_1 + \delta_2 \geq 1$ comes from the nilpotency class of G. The decomposition of G in the form given is now quite easy by considering the properties of a and b. ◄

We give two examples of p-groups whose the commutator subgroup is non-cyclic and which satisfy the above proposition.

Example 1. *Suppose $p \neq 2, 3$ and*

$$G = \langle a, b, c, d | a^{p^2} = b^p = c^p = d^p = 1, [b,a] = c, [c,a] = d,$$

$$[a,d] = [b,c] = [b,d] = [c,d] = 1\rangle.$$

The group G is of order p^5 and nilpotency class 3. By the relations of the group, we can see that $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$. So, G' is non-cyclic. Also, $\langle a\rangle \cap G' = \langle b\rangle \cap G' = 1$. Therefore, $G \cong (G' \rtimes \langle a\rangle) \rtimes \langle b\rangle$.

Example 2. *Suppose $p \neq 2$ and*

$$G = \langle a, b, c, d, e | a^p = b^p = c^p = d^p = e^p = 1, [a,b] = c, [c,a] = d, [c,b] = e,$$

$$[a,d] = [a,e] = [b,d] = [b,e] = [c,d] = [c,e] = [d,e] = 1\rangle.$$

This group is of order p^5 and nilpotency class 3. It is easily seen that $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ (G' is non-cyclic) and $\langle a\rangle \cap G' = \langle b\rangle \cap G' = 1$. Hence, $G \cong (G' \rtimes \langle a\rangle) \rtimes \langle b\rangle$.

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