Azerbaijan Journal of Mathematics
V. 4, No 1, 2014, January
ISSN 2218-6816

On Direct And Inverse Theorems Of Approximation Theory In Variable Lebesgue And Sobolev Spaces

Idris I. Sharapudinov

Abstract. We consider the Lebesgue space $L^{p(x)}_{2\pi}$ with variable exponent $p(x)$. It consists of measurable functions $f(x)$ for which the integral $\int_0^{2\pi} |f(x)|^{p(x)}dx$ exists. We establish an analogue of Jackson's second theorem in the case when the 2π-periodic variable exponent $p(x) \geq 1$ satisfies the condition

$$|p(x) - p(y)| \cdot \ln \frac{2\pi}{|x-y|} \leq d, \quad x, y \in [0, 2\pi].$$

Results obtained in present paper radically differ from other authors' results on this subject because we don't require from variable exponent $p(x)$ the fulfillment of additional condition $p(x) \geq \underline{p} > 1$, which is closely related with boundedness of Hardy - Littlewood maximal function $M(f)$ in $L^{p(x)}_{2\pi}$. In the definition of the modulus of continuity of a function $f(x) \in L^{p(x)}_{2\pi}$, we replace the ordinary shift $f^h(x) = f(x+h)$ by an averaged shift determined by Steklov's function $s_h(f)(x) = \frac{1}{h}\int_0^h f(x+t)dt$.

Key Words and Phrases: Variable exponent Lebesgue spaces, approximation theory, direct and inverse theorems

2000 Mathematics Subject Classifications: 42A10, 42B25, 46E30

1. Introduction

In 1976, the year when we began studying the topology of space $L^{p(x)}(E)$, there was no theory of variable exponent Lebesgue spaces. There was only example of measurable functions set noted by Orlicz in [1]. Common modular spaces theory was being developed by the Japanese mathematicians (H. Nakano [2],[3]), and functional modular spaces theory - by the Polish mathematicians (J. Musielak and W. Orlicz [4], [5]). Also note the work of Russian mathematician I. V. Tsenov [6].

But in these theories there was no consideration of a special theory of $L^{p(x)}(E)$ spaces. Such spaces were noted only as exotic examples of modular spaces. Spaces of functions integrable with an exponent ceased to play the role of exotic examples of modular spaces and set off on their path of development once the topology of these spaces was shown to be normable, with one of the equivalent norms given by Kolmogorov's well-known theorem on the normability of linear topological spaces having a bounded balanced convex neighbourhood of zero [7]. A.N.Kolmogorov [7] introduced a norm on such spaces by means of the Minkowski functional. In the same direction, the author showed in 1976 (but published [8] only in 1979) that the Lebesgue space $L_{\mu}^{p(x)}(E)$ with variable exponent $p(x) \geq 1$ (this space consists of measurable functions $f(x)$ on E such that $|f(x)|^{p(x)}$ is integrable on E) is a normed space with the norm of $f \in L_{\mu}^{p(x)}(E)$ given by

$$\|f\|_{p(\cdot)}(E) = \inf\{\alpha > 0 \,|\, \int_E \left|\frac{f(x)}{\alpha}\right|^{p(x)} \mu(dx) \leq 1\}. \tag{1.1}$$

For unknown reasons, many authors call such norms Luxemburg norms instead of Kolmogorov norms.

In [8], conditions on variable exponent $p(x)$ for the $L^{p(x)}(E)$ space to be a linear topological space, were found. It was shown that $L^{p(x)}(E)$ will be a linear topological space if and only if $p(x)$ is essentially bounded function, i.e. $0 < p(x) \leq \overline{p}$ for almost every $x \in E$.

The case when $p(x)$ is not essentially bounded was considered in [8]. Such case is arising in the problem of finding conjugate space $[L^{p(x)}(E)]^*$ (space of continuous linear functionals) when $\operatorname{ess\,inf} p(x) = 1$. Moreover, there can be cases when $p(x) = \infty$ on set with nonzero measure. In all such cases, the corresponding spaces $[L^{p(x)}(E)]^*$ were found in [8].

Results and methods developed in [8] have been used in the sequel by many authors (quoting or not quoting the paper [8]) and they represent now a kind of folklore in the theory of spaces $L^{p(x)}(E)$.

The next stage in the development of the theory of the spaces $L_{\mu}^{p(x)}(E)$ was the imposition of stronger conditions on the variable exponent $p(x)$ and obtaining $L_{\mu}^{p(x)}(E)$ analogues of classical results that were well known in the case of constant $p(x)$. The first step in this direction was made by the author [9] who showed that if μ is the ordinary Lebesgue measure on the line, then Haars system forms a basis for $L^{p(x)}([0,1])$ if and only if the variable exponent $p(x) \geq 1$ satisfies the Dini-Lipschitz condition on $[0,1]$:

$$|p(x) - p(y)| \log \frac{1}{|x-y|} \leq C \quad (|x-y| \leq \frac{1}{2}).$$

Under the same hypotheses, the author [10] proved that some families of convolution operators are uniformly bounded in $L_{\mu}^{p(x)}([0,2\pi])$. This covers in particular a large class of classical operators, including the operators of Fejr, de la Valle-Poussin, Abel, Steklov and many others.

Substantial contributions to the theory of the spaces $L_{\mu}^{p(x)}(E)$ were made by V. V. Zhikov [11]–[13] and L. Diening in [14]-[17]. The best result obtained in [14]-[17] is as follows. Suppose that Ω is a bounded domain in $\mathbb{R}^n$, μ is the ordinary Lebesgue measure on $\mathbb{R}^n$, and $p(x)$ is defined on Ω and satisfies the conditions $1 < p_-(\Omega) \leq p(x) \leq p^-(\Omega) < \infty$, $|p(x) - p(y)| \log \frac{1}{|x-y|} \leq C \quad (|x-y| \leq \frac{1}{2} \quad x, y \in \Omega)$. Then the operator $M(f)$ of the

Hardy-Littlewood maximal function acts boundedly on $L_\mu^{p(x)}(\Omega)$. As a corollary, it was shown in [15] that under the same restrictions on p(x) and some additional condition on p(x) outside some ball, the well-known Calderon-Zygmund operators act boundedly in $L_\mu^{p(x)}(\mathbb{R}^n)$. In particular, for $n = 1$ it follows that the Hilbert transform is bounded in $L_\mu^{p(x)}(\mathbb{R})$ provided that $1 < p_1 \le p(x) \le p_2 < \infty$, $|p(x) - p(y)| \log \frac{1}{|x-y|} \le C \quad (|x - y| \le \frac{1}{2} \quad x, y \in \mathbb{R})$ and $p(x)$ coincides with a constant outside some interval. Thus, the connection between the Dini-Lipschitz condition for the variable exponent $p(x)$ and the uniform boundedness in $L_\mu^{p(x)}(\mathbb{E})$ of families of classical operators, described by the author in [8], [9], turned out to be characteristic in the construction of a deep theory of integral operators in the spaces $L_\mu^{p(x)}(\mathbb{R})$. Numerous recent results obtained by specialists in the theory of differential equations show that a similar situation arises when constructing a deep theory of differential equations in Sobolev spaces with variable exponent. Many references can be found in the recent monograph [17]. Among them, a special place belongs to the papers [11][13], where the spaces $L_\mu^{p(x)}(\mathbb{E})$ were used for the first time to study problems arising in the multidimensional calculus of variations. The properties of singular integrals in the spaces $L_\mu^{p(x)}(\mathbb{E})$) were studied in [18][25] under the same logarithmic Dini-Lipschitz condition on the variable exponent $p(x)$.

Here we consider the problem of the approximation of functions by trigonometric polynomials in the metric of $L_\mu^{p(x)}([0, 2\pi])$. Suppose that $p = p(x)$ is a measurable 2π-periodic function, $p_- = \inf\{p(x) : x \in \mathbb{R}\}$, $p^- = \sup\{p(x) : x \in \mathbb{R}\}$, $1 \le p_- \le p^- < \infty$, $L_{2\pi}^{p(x)}$ is the space of measurable 2π-periodic functions $f(x)$ with $\int_0^{2\pi} |f(x)|^{p(x)} dx < \infty$. Putting

$$\|f\|_{p(\cdot)} = \inf \left\{ \alpha > 0 : \int_0^{2\pi} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \le 1 \right\}, \tag{1.1}$$

we turn $L_{2\pi}^{p(x)}$ into a Banach space. We write $\mathcal{P}_{2\pi}$ for the set of all 2π-periodic variable exponents $p = p(x) \ge 1$ satisfying the condition

$$|p(x) - p(y)| \ln \frac{2\pi}{|x-y|} \le d \quad (x, y \in [0, 2\pi]). \tag{1.2}$$

The subclass of all $p = p(x) \in \mathcal{P}_{2\pi}$ satisfying the additional condition $p_- > 1$, is denoted by $\hat{\mathcal{P}}_{2\pi}$. The author proved [26] that if $p(x) \in \hat{\mathcal{P}}_{2\pi}$, then the trigonometric system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ forms a basis for the space $L_{2\pi}^{p(x)}$. In other words, putting

$$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt, \quad k \in \mathbb{Z},$$

$$S_n(f) = S_n(f, x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}, \tag{1.3}$$

we have the estimate

$$\|S_n(f)\|_{p(\cdot)} \le c(p)\|f\|_{p(\cdot)} \quad (n = 0, 1, \ldots). \tag{1.4}$$

It follows that the Fourier series of a function $f \in L_{2\pi}^{p(x)}$ converges to it in the norm (1.1), that is,

$$\|f - S_n(f)\|_{p(\cdot)} \to 0 \quad (n \to \infty).$$

Moreover, if $p(x) \in \hat{\mathcal{P}}_{2\pi}$, then the order of approximation of $f \in L_{2\pi}^{p(x)}$ by the partial sums (1.3) in the norm (1.1) as $n \to \infty$ coincides with the order of best approximation

$$E_n(f)_{p(\cdot)} = \inf_{T_n} \|f - T_n\|_{p(\cdot)}, \tag{1.5}$$

where the infimum is taken over all trigonometric polynomials

$$T_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}. \tag{1.6}$$

We may now ask how the rate of decay of $E_n(f)_{p(\cdot)}$ as $n \to \infty$ depends on the properties of $f \in L_{2\pi}^{p(x)}$. In other words, we want to define the modulus of continuity of a function $f \in L_{2\pi}^{p(x)}$ and estimate $E_n(f)_{p(\cdot)}$ in terms of it. As mentioned in [27], the quantity

$$\omega(f, \delta)_{p(\cdot)} = \sup_{0<h\le\delta} \|f - f(* + h)\|_{p(\cdot)}$$

cannot play the role of the modulus of continuity of $f \in L_{2\pi}^{p(x)}$ in the case of a variable exponent $p = p(x)$ because, generally speaking, the equation $\lim_{\delta\to 0} \omega(f, \delta)_{p(\cdot)} = 0$ does not hold for all such f. If $p(x)$ is not equal to a constant almost everywhere on $[0, 2\pi]$, then the shift $f_h(x) = f(x + h)$ of a function $f(x)$ in $L_{2\pi}^{p(x)}$ need not belong to $L_{2\pi}^{p(x)}$. Quite the contrary, the integral $\int_0^{2\pi} |f(x + h)|^{p(x)} dx$ usually diverges for $h \ne 0$. This was the main obstacle in the way of transferring the main theorems of the theory of approximation by trigonometric polynomials to the case of spaces $L_{2\pi}^{p(x)}$. We give one of the possible ways to overcome this obstacle by using certain types of Steklov functions. We put

$$f_h(x) = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(x + t)dt, \quad s_h(f)(x) = f_h\left(x + \frac{h}{2}\right) = \frac{1}{h} \int_0^h f(x + t)dt \tag{1.7}$$

and consider the quantity

$$\Omega(f, \delta)_{p(\cdot)} = \sup_{0<h\le\delta} \|f - f_h(* + \frac{h}{2})\|_{p(\cdot)} = \sup_{0<h\le\delta} \|f - s_h(f)\|_{p(\cdot)}. \tag{1.8}$$

It follows from the author's results in [26] that if $p(x) \in \mathcal{P}_{2\pi}$, then the function $\Omega(f,\delta)_{p(\cdot)}$ is continuous on $[0,\infty)$ and $\lim\limits_{\delta\to 0}\Omega(f,\delta)_{p(\cdot)} = 0$. It also follows from the definition (1.8) that $\Omega(f,\delta)_{p(\cdot)}$)is a non-decreasing function of δ. We call $\Omega(f,\delta)_{p(\cdot)}$ the modulus of continuity of a function $f \in L_{2\pi}^{p(x)}$.

It was proved in author's works [28] – [30] that if the variable exponent $p(x) \in \mathcal{P}_{2\pi}$ and $f \in L_{2\pi}^{p(x)}$, then the following Jackson-type inequality holds:

$$E_n(f)_{p(\cdot)} \le c(p)\Omega(f,\frac{1}{n})_{p(\cdot)}. \tag{1.9}$$

Moreover, if $\Omega(f,\delta)_{p(\cdot)} \le c\delta^\alpha$ $(0 < \alpha < 1)$, then the converse assertion holds. Namely, if $E_n(f)_{p(\cdot)} \le c/n^\alpha$ $(n = 1,2,\ldots)$, then $\Omega(f,\delta)_{p(\cdot)} = O(\delta^\alpha)$. We note that in [26] we considered the quantity

$$\Omega^\gamma(f,0)_{p(\cdot)} = 0, \quad \Omega^\gamma(f,\delta)_{p(\cdot)} = \sup_{\substack{h,\tau \\ |\tau|^{1/\gamma}\le h\le\delta}} \|f - f_h(*+\tau)\|_{p(\cdot)}, \tag{1.10}$$

where $\gamma > 0$. We call it the γ-*modulus of continuity* of a function $f(x) \in L_{2\pi}^{p(x)}$. It follows from (1.8) and (1.10) that

$$\Omega(f,\delta)_{p(\cdot)} = \sup_{0<h\le\delta} \|f - f_h(*+\frac{h}{2})\|_{p(\cdot)} \le \Omega^1(f,\delta)_{p(\cdot)}. \tag{1.11}$$

On the other hand, the following result was proved in [26]:

Theorem A. *If $p(x) \in \mathcal{P}_{2\pi}$, $f(x) \in L_{2\pi}^{p(x)}$, then the function $g(\delta) = \Omega^\gamma(f,\delta)_{p(\cdot)}$ is non-decreasing on $[0,\infty]$ and continuous at the point $\delta = 0$.*

In particular, Theorem A and the estimate (1.11) yield the equation

$$\lim_{\delta\to 0}\Omega(f,\delta)_{p(\cdot)} = 0, \tag{1.12}$$

mentioned above.

The proof of Theorem A is based on the uniform boundedness in $L_{2\pi}^{p(x)}$ $0 < h \le 1$, $|\tau| \le \pi h^\gamma$ of the family of shifts of the Steklov functions

$$\mathcal{S}_{h,\tau}(f) = \mathcal{S}_{h,\tau}(f)(x) = f_{h,\tau}(x) = f_h(x+\tau) = \frac{1}{h}\int_{x+\tau-\frac{h}{2}}^{x+\tau+\frac{h}{2}} f(t)dt.$$

Namely, it was proved in [26] that if $p(x) \in \mathcal{P}_{2\pi}$, then

$$\|\mathcal{S}_{h,\tau}(f)\|_{p(\cdot)} \le c(d)(2\pi+1)^{p^-}\|f\|_{p(\cdot)} \quad 0 < h \le 1, |\tau| \le \pi h^\gamma, \tag{1.13}$$

where d is the constant in the inequality (1.2).

We mention that the direct and inverse theorems of approximation theory in the spaces $L^{p(x)}_{2\pi}$ were obtained in [31][34] under the assumption that $p(x) \in \hat{\mathcal{P}}_{2\pi}$. The principal difference between our results and those in [31][34] is that we are able to get rid of the restriction $p_- > 1$ and prove the direct and inverse theorems of approximation theory in $L^{p(x)}_{2\pi}$ under the natural assumption $p_- \geq 1$, where $p_- = \inf\{p(x) : x \in \mathbb{R}\}$ (by the definition above). The results in [31][34] were obtained for $p_- > 1$, and we stress that this is not accidental. The methods used in those papers to study the direct and inverse problems of approximation theory $L^{p(x)}_{2\pi}$ (and even in the more general weighted spaces $L^{p(x)}_{2\pi,\rho}$ with variable exponent)are based, either directly or indirectly, on the boundedness in $L^{p(x)}_{2\pi}$ of the operator $M(f)$ given by the Hardy-Littlewood maximal function (or of its analogues and generalizations in $L^{p(x)}_{2\pi,\rho}$), and it is well known that this holds only for $p_- > 1$. For example, in [31] the proof of a direct Jackson-type theorem for $L^{p(x)}_{2\pi}$ under the assumption that $p(x) \in \hat{\mathcal{P}}_{2\pi}$ is based on the facts that the operator of conjugation (of functions) is bounded in $L^{p(x)}_{2\pi}$ and the trigonometric system forms a basis there. These facts were established by the author [26] using the boundedness in $L^{p(x)}_{2\pi}$ of the Hilbert transform under the assumption that $p(x) \in \hat{\mathcal{P}}_{2\pi}$, and this boundedness was deduced in [15] from that of the maximal function, which was proved in [14]. To obtain direct and inverse theorems of approximation theory in $L^{p(x)}_{2\pi}$, where the variable exponent $p(x) \in \mathcal{P}_{2\pi}$ satisfies the Dini-Lipschitz condition (1.2) and may be equal to 1 at some points (that is, $p^- = 1$), it is required to develop essentially new approaches which do not use the properties of the maximal function $M(f)$. In author's works [28] – [30] we make an attempt to solve the part of this problem that concerns Jackson's first theorem. One of the instruments in the proof of Jackson's first theorem in $L^{p(x)}_{2\pi}$ is Jackson's well-known operator (trigonometric polynomial of degree $2n-2$)

$$\mathcal{D}_n(f) = \mathcal{D}_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)J_n(t)dt \quad (n = 1, 2, \ldots),$$

where

$$J_n(x) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right)^4.$$

We proved in [29],[30] that $\mathcal{D}_n(f)(x)$ approximates every $f(x) \in L^{p(x)}_{2\pi}$ with accuracy $O(\Omega(f, \frac{1}{n})_{p(\cdot)})$. In other words, if $f(x) \in L^{p(x)}_{2\pi}$ with $p(x) \in \mathcal{P}_{2\pi}$, then

$$\|f - \mathcal{D}_n(f)\|_{p(\cdot)} \leq c(p)\Omega(f, \frac{1}{n})_{p(\cdot)},$$

which again gives the inequality (1.9).

The proof of the inequality (analogue of Jackson's second theorem)

$$E_n(f)_{p(\cdot)} \leq c(p)\frac{1}{n^r}\Omega(f^{(r)}, \frac{1}{n})_{p(\cdot)} \tag{1.14}$$

encounters additional difficulties, and in [28]–[30] we have not been able to overcome them in the general case when $p(x) \in \mathcal{P}_{2\pi}$. Therefore in [28]-[30] we only give it for $p(x) \in \hat{\mathcal{P}}_{2\pi}$.

But in present work we consider the general case when $p(x) \in \mathcal{P}_{2\pi}$. We succeeded in proving that the inequality (1.14) holds for every function $f(x) \in W^r_{p(\cdot)}$, where $p(x) \in \mathcal{P}_{2\pi}$, $W^r_{p(\cdot)}$ is the Sobolev space of 2π-periodical functions $f(x)$ such that $f^{(r-1)}(x)$ is absolutely continuous in $[0, 2\pi]$ and $f^{(r)}(x) \in L^{p(x)}_{2\pi}$. In the author's works [35] – [36] it is shown that one of instruments in the proof of inequality (1.14) is the Valle - Poussin's well-known means

$$V^n_m(f) = V^n_m(f, x) = \frac{1}{m+1} \sum_{l=0}^{m} S_{n+l}(f, x),$$

where Fourier sums $S_k(f, x)$ are defined in (1.3). Namely, in [35]–[36] the following inequality

$$\|f - V^n_m(f)\|_{p(\cdot)} \le \frac{c(p)}{n^r} E_n(f^{(r)})_{p(\cdot)} \tag{1.15}$$

is proved, where $p = p(x) \in \mathcal{P}_{2\pi}$, $r \ge 0$, $f(x) \in W^r_{p(\cdot)}$, $m \in \{n-1, n\}$. The estimate (1.14) (analogue of Jackson's second theorem) follows from (1.9) and (1.15) as a corollary.

The complete proof of inequality (1.15) is given in a next section.

2. The approximation of functions in $L^{p(x)}_{2\pi}$ by Vallee-Poussin means

We will consider in $L^{p(x)}$ Sobolev type classes $W^r_{p(\cdot)}(M)$, which consist of 2π-periodical $r-1$ times continuously differentiable functions $f(x)$, whose derivative $f^{(r-1)}(x)$ is absolutely continuous in $[0, 2\pi]$ and $f^{(r)}(x) \in L^{p(x)}_{2\pi}$, $\|f^{(r)}\|_{p(\cdot)} \le M$. Let us assume

$$W^r_{p(\cdot)} = \bigcup_{M>0} W^r_{p(\cdot)}(M), \quad W^0_{p(\cdot)} = L^{p(x)}_{2\pi}.$$

We can consider the Fourier series for $f \in L^{p(x)}_{2\pi}$:

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{2.1}$$

and partial sum of Fourier series

$$S_n(f) = S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx, \tag{2.2}$$

where

$$a_k = a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos ktdt, \quad b_k = b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin ktdt.$$

If $r \geq 1$, $p(x) \geq 1$ and $f \in W^r_{p(\cdot)}$, then [37, p. 75]

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t)B_r(t-x)dt, \tag{2.3}$$

where

$$B_r(u) = \sum_{k=1}^{\infty} \frac{\cos(ku + \frac{\pi r}{2})}{k^r} \tag{2.4}$$

is the Bernoulli function. Since $S_n^{(r)}(f,x) = S_n(f^{(r)},x)$, then we conclude from (2.3) and (2.4) an equality for $f \in W^r_{p(\cdot)}$

$$f(x) - S_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t)R_{r,n}(t-x)dt, \tag{2.5}$$

$$R_{r,n}(u) = \sum_{k=n+1}^{\infty} \frac{\cos(ku + \frac{\pi r}{2})}{k^r}. \tag{2.6}$$

We will define Vallee-Poussin means $V_m^n(f) = V_m^n(f,x)$ by equality

$$V_m^n(f) = V_m^n(f,x) = \frac{1}{m+1}\sum_{l=0}^{m} S_{n+l}(f,x). \tag{2.7}$$

Matching equalities (2.5) and (2.6) with (2.7) we notice

$$f(x) - V_m^n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t)\frac{1}{m+1}\sum_{l=0}^{m} R_{r,n+l}(t-x)dt. \tag{2.8}$$

We will assume

$$\mathfrak{K}_{r,m+1}^n(u) = (m+1)^{r-1}\sum_{l=0}^{m} R_{r,n+l}(n) \tag{2.9}$$

and transcribe (2.8)

$$f(x) - V_m^n(f,x) = \frac{1}{\pi(m+1)^r}\int_{-\pi}^{\pi} f^{(r)}(t)\mathfrak{K}_{r,m+1}^n(t-x)dt. \tag{2.10}$$

Since, by (2.9), $\mathfrak{K}_{r,m+1}^r(x)$ is orthogonal to all trigonometric polynomials of degree not greater than n, then we obtain from (2.10)

$$f(x) - V_m^n(f,x) = \frac{1}{\pi(m+1)^r}\int_{-\pi}^{\pi} (f^{(r)}(t) - T_n(t))\mathfrak{K}_{r,m+1}^n(t-x)dt, \tag{2.11}$$

where $T_n(x)$ is an arbitrary trigonometric polynomial of degree n. Now we can state the next result.

Theorem 2.1. *Let* $p = p(x) \in \mathcal{P}_{2\pi}$, $r \geq 0$, $f(x) \in W^r_{p(\cdot)}$. *Then the following estimates hold:*

$$\|f - V^n_{n-1}(f)\|_{p(\cdot)} \leq \frac{c(p)}{n^r} E_n(f^{(r)})_{p(\cdot)}, \tag{2.12}$$

$$\|f - V^n_n(f)\|_{p(\cdot)} \leq \frac{c(p)}{n^r} E_n(f^{(r)})_{p(\cdot)}. \tag{2.13}$$

Proof of the theorem 2.1. is based on a number of auxiliary assertions concerning functions $\mathfrak{K}^n_{r,m+1}(u)$.

Lemma 2.1. *We have the following equalities*

$$\kappa^n_{2s,n}(u) =$$

$$(-1)^s n^{2s-1} \sum_{l=0}^{n-2} \sum_{k=0}^{\infty} \frac{\sin \frac{l+1}{2} u \sin \frac{k+1}{2} u \cos(2n+k+l+2)\frac{u}{2}}{2\sin^2 \frac{u}{2}} \Delta^2 g_s(n+1+k+l)$$

$$+(-1)^{s-1} n^{2s-1} \sum_{k=0}^{\infty} \frac{\sin \frac{n}{2} u \sin \frac{k+1}{2} u \cos(3n+k+1)}{2\sin^2 \frac{u}{2}} \Delta q_s(2n+k), \tag{2.14}$$

$$\kappa^n_{2s-1,n}(u) =$$

$$(-1)^s n^{2s-2} \sum_{l=0}^{n-2} \sum_{k=0}^{\infty} \frac{\sin \frac{l+1}{2} u \sin \frac{k+1}{2} u \cos(2n+k+l+2)\frac{u}{2}}{2\sin^2 \frac{u}{2}} \Delta^2 g_s(n+1+k+l)$$

$$+(-1)^{s-1} n^{2s-2} \sum_{k=0}^{\infty} \frac{\sin \frac{n}{2} u \sin \frac{k+1}{2} u \cos(3n+k+1)}{2\sin^2 \frac{u}{2}} \Delta q_s(2n+k), \tag{2.15}$$

where $g_s(t) = t^{-2s}$, $q_s(t) = t^{-2s+1}$, $\Delta\varphi(t) = \varphi(t+1) - \varphi(t)$, $\Delta^2\varphi(t) = \varphi(t+2) - 2\varphi(t+1) + \varphi(t)$.

Proof. From (2.6) and (2.9) we have

$$\kappa^n_{r,m+1}(u) = (m+1)^{r-1} \sum_{l=0}^{m} \sum_{k=0}^{\infty} \frac{\cos\left[(n+k+l+1)u + \frac{\pi r}{2}\right]}{(n+k+l+1)^r},$$

So, with the help of Abel transform, we can write

$$\kappa^n_{r,m+1}(u) = (m+1)^{r-1} \sum_{l=0}^{m} \sum_{k=0}^{\infty} \left[\frac{1}{(n+1+k+l)^r} - \frac{1}{(n+2+k+l)^r}\right] v^n_{k,l}(u), \tag{2.16}$$

where

$$v^n_{k,l}(u) = \sum_{j=0}^{k} \cos[(n+1+l+j)u + \frac{\pi r}{2}]. \tag{2.17}$$

We will consider the case when $m = n-1$ and the two cases of r, even and odd. If $r = 2s$, then $\cos(\mu u + \frac{\pi r}{2}) = (-1)^s \cos \mu u$. Therefore, (2.17) takes the form

$$v_{k,l}^n(u) = (-1)^s \sum_{j=0}^{k} \cos(n+1+l+j)u =$$

$$= (-1)^s \cdot \frac{\sin(2(n+1+l+k)+1)\frac{u}{2} - \sin(2(n+l)+1)\frac{u}{2}}{2\sin\frac{u}{2}} \tag{2.18}$$

From (2.16) and (2.18) we have

$$\kappa_{2s,n}^n(u) = (-1)^{s-1} n^{2s-1} \times$$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{n-1} \Delta g_s(n+1+k+l) \frac{\sin(2(n+1+l+k)+1)\frac{u}{2} - \sin(2(n+l)+1)\frac{u}{2}}{2\sin\frac{u}{2}}. \tag{2.19}$$

We apply Abel transform to the inner sum again. From (2.19) we get

$$\kappa_{2s,n}^n(u) = (-1)^{s-1} n^{2s-1} \sum_{k=0}^{\infty}\sum_{l=0}^{n-2} \Delta g_s(n+1+k+l) \frac{W_{k,l}^n(u) - W_{0,l}^n(u)}{2\sin\frac{u}{2}} +$$

$$+(-1)^{s-1} n^{2s-1} \sum_{k=0}^{\infty} \Delta g_s(2n+k) \frac{W_{k,n-1}^n(u) - W_{0,n-1}^n(u)}{2\sin\frac{u}{2}}, \tag{2.20}$$

where

$$W_{k,l}^n(u) = \sum_{\mu=0}^{l} \sin(2(n+1+\mu+k)+1)\frac{u}{2} =$$

$$\frac{\sin^2(n+l+k+2)\frac{u}{2} - \sin^2(n+k+1)\frac{u}{2}}{\sin\frac{u}{2}} =$$

$$= \frac{1}{\sin\frac{u}{2}}(\sin(n+l+k+2) - \sin(n+k+1)\frac{u}{2})(\sin(n+l+k+2) + \sin(n+k+1)\frac{u}{2}) =$$

$$\frac{4}{\sin\frac{u}{2}} \sin\frac{l+1}{4}u \cdot \cos(n+k+1+\frac{l+1}{2})\frac{u}{2} \cdot \sin(n+k+1+\frac{l+1}{2})\frac{u}{2} \cos\frac{l+1}{4}. \tag{2.21}$$

From (2.21) we get

$$W_{k,l}^n(u) - W_{0,l}^n(u) =$$

$$= \frac{4}{\sin\frac{u}{2}} \sin\frac{l+1}{2} \cos\frac{l+1}{2} \left(\sin(n+k+1+\frac{l+1}{2})\frac{u}{2} \cos(n+k+1+\frac{l+1}{2})\frac{u}{2} - \right.$$

$$\left. - \sin(n+\frac{l+1}{2})\frac{u}{2} \cos(n+\frac{l+1}{2})\frac{u}{2}\right) =$$

$$\frac{\sin\frac{l+1}{2}}{\sin\frac{u}{2}}\Big(\sin(2(n+k+1)+l+1)\frac{u}{2}-\sin(2n+l+1)\frac{u}{2}\Big)=$$

$$\frac{1}{\sin\frac{u}{2}}\sin\frac{l+1}{2}u\sin\frac{k+1}{2}u\cos(2n+k+l+2)\frac{u}{2}. \tag{2.22}$$

So, the equality (2.14) follows from (2.20) and (2.22). Equality (2.15) is proved similarly. Lemma 2.1 is proved.

Lemma 2.2. *Suppose* $0\le k\le l$. *Then*

$$A_{k,l}=\int\limits_0^{\pi}\frac{|\sin\frac{k+1}{2}u\sin\frac{l+1}{2}u|}{\sin\frac{u}{2}}du\le 2(k+1)(2+\ln\frac{l+1}{k+1})+\frac{\pi}{3-\frac{\pi^2}{8}}.$$

Proof. We have

$$A_{k,l}=2\int\limits_0^{\frac{\pi}{2}}\frac{|\sin(k+1)u\sin(l+1)|}{\sin^2 u}du=2\int\limits_0^{\frac{\pi}{2}}\frac{|\sin(k+1)u\sin(l+1)|}{u^2}du+$$

$$2\int\limits_0^{\frac{\pi}{2}}|\sin(k+1)u\sin(l+1)u|\varphi(u)du, \tag{2.23}$$

where

$$\varphi(u)=\frac{1}{\sin^2 u}-\frac{1}{u^2}=\frac{u^2-\sin^2 u}{u^2\sin^2 u}.$$

Suppose $0<u<\frac{\pi}{2}$. Then

$$\varphi(u)=\frac{(u+\sin u)(u-\sin u)}{u^2\sin^2 u}=\frac{(2u-\frac{u^3}{3!}+\frac{u^5}{5!}-\ldots)(\frac{u^3}{3!}-\frac{u^5}{5!}+\ldots)}{u^2\sin^2 u}=$$

$$\frac{(2-\frac{u^2}{3!}+\frac{u^4}{5!}-\ldots)(\frac{1}{3!}-\frac{u^2}{5!}+\ldots)}{\left(\frac{\sin u}{u}\right)^2}=\frac{(2-\frac{u^2}{3!}+\frac{u^4}{5!}-\ldots)(\frac{1}{3!}-\frac{u^2}{5!}+\ldots)}{(1-\frac{u^2}{3!}+\frac{u^4}{5!}-\ldots)^2}<$$

$$<\frac{\frac{2}{3!}}{\left(1-\frac{u^2}{3!}\right)^2}=\frac{1}{3(1-\frac{\pi^2}{24})}=\frac{1}{3-\frac{\pi^2}{8}}$$

and, therefore,

$$2\int\limits_0^{\frac{\pi}{2}}|\sin(k+1)u\sin(l+1)u|\varphi(u)du\le\frac{1}{3-\frac{\pi^2}{8}}. \tag{2.24}$$

On the other hand,

$$\int\limits_0^{\frac{\pi}{2}}\frac{|\sin(k+1)u\sin(l+1)u|}{u^2}du=(k+1)\int\limits_0^{\frac{\pi}{2}(k+1)}\frac{|\sin u\sin\frac{l+1}{k+1}u|}{u^2}du\le$$

$$(k+1)\int\limits_0^1 \frac{|\sin\frac{l+1}{k+1}u|}{u}du+(k+1)\int\limits_1^{\frac{\pi}{2}(k+1)}\frac{du}{u^2}=$$

$$(k+1)\int\limits_0^{\frac{l+1}{k+1}}\frac{|\sin u|}{u}du+(k+1)=(k+1)\left[\int\limits_0^1\frac{\sin u}{u}du+\int\limits_0^{\frac{l+1}{k+1}}\frac{du}{u}+1\right]<$$

$$(k+1)\left(2+\ln\frac{l+1}{k+1}\right). \tag{2.25}$$

The statement of the lemma follows from equality (2.23) and inequalities (2.24) and (2.25).◄

Lemma 2.3. *If $r\ge 1$, then*

$$\int\limits_{-\pi}^{\pi}|\kappa_{r,n}^n|du\le c(r).$$

Proof. Consider the case of even $r=2s$. Then, from Lemma 2.1 we have

$$\int\limits_{-\pi}^{\pi}|\kappa_{2s,n}^n|du\le n^{2s-1}\sum_{l=0}^{n-2}\sum_{k=0}^{\infty}\Delta^2 g_s(n+k+l)\frac{1}{2}\int\limits_{-\pi}^{\pi}\frac{|\sin\frac{k+1}{2}u\sin\frac{l+1}{2}u|}{\sin^2\frac{u}{2}}du+$$

$$n^{2s-1}\sum_{k=0}^{\infty}\Delta g_s(2n+k)\frac{1}{2}\int\limits_{-\pi}^{\pi}\frac{|\sin\frac{n}{2}u\sin\frac{k+1}{2}u|}{\sin^2\frac{u}{2}}du. \tag{2.26}$$

Because of the Lemma 2.2,

$$\frac{1}{2}\int\limits_{-\pi}^{\pi}\frac{|\sin\frac{k+1}{2}u\sin\frac{l+1}{2}u|}{\sin^2\frac{u}{2}}du=\begin{cases}2(k+1)(2+\ln\frac{l+1}{k+1})+\frac{\pi}{3-\frac{\pi^2}{8}}, k\le l,\\ 2(k+1)(2+\ln\frac{k+1}{l+1})+\frac{\pi}{3-\frac{\pi^2}{8}}, l<k.\end{cases} \qquad (2.27))$$

Next, since the $\Delta^2 g_s(t)=g_s''(\bar t)$ $(t\le\bar t\le t+2)$, then

$$\Delta^2 g_s(n+1+k+l)=g_s''(\bar t)=\frac{2s(2s+1)}{\bar t^{2s+2}}\le\frac{2s(2s+1)}{(n+1+k+l)^{2s+2}}, \tag{2.28}$$

where $n+1+k+l<\bar t<n+1+k+l+2$ and, similarly,

$$\Delta g_s(2n+k)\le\frac{2s}{(2n+k)^{2s+1}}. \tag{2.29}$$

From (2.27) and (2.28) we have $(l\le n-2)$

$$\sum_{k=l}^{\infty}\Delta^2 g_s(n+1+k+l)\frac{1}{2}\int\limits_{-\pi}^{\pi}\frac{|\sin\frac{k+1}{2}u\sin\frac{l+1}{2}u|}{\sin^2\frac{u}{2}}du\le$$

$$\leq \sum_{k=l}^{\infty} \frac{4s(2s+1)[2+\ln\frac{k+1}{l+1}] + \frac{\pi}{3-\frac{\pi^2}{8}}}{(n+1+k+l)^{2s+2}} \leq$$

$$\leq \sum_{k=l}^{\infty} \frac{4s(2s+1)[2(l+1)+k-l+\frac{\pi}{3-\frac{\pi^2}{8}}]}{(n+1+k+l)^{2s+2}} \leq c(s)n^{-2s}, \tag{2.30}$$

$$\sum_{k=0}^{l} \frac{2s(2s+1)[(k+1)(2+\ln\frac{l+1}{k+1}+\frac{\pi}{3-\frac{\pi^2}{8}})]}{(n+1+k+l)^{2s+2}} \leq c(s)n^{-2s}, \tag{2.31}$$

therefore

$$n^{2s-1}\sum_{l=0}^{n-2}\sum_{k=0}^{\infty}\Delta^2 g_s(n+1+k+l)\frac{1}{2}\int_{-\pi}^{\pi}\frac{|\sin\frac{k+1}{2}u\sin\frac{l+1}{2}u|}{\sin^2\frac{u}{2}}du \leq$$

$$c(s)n^{2s-1}\sum_{l=0}^{n-2} n^{-2s} \leq c(s). \tag{2.32}$$

Next, from (2.27) and (2.29) we have

$$n^{2s-1}\sum_{k=0}^{\infty}\Delta g_s(2n+k)\frac{1}{2}\int_{-\pi}^{\pi}\frac{|\sin\frac{n}{2}u\sin\frac{k+1}{2}u|}{\sin^2\frac{u}{2}}du \leq$$

$$n^{2s-1}\sum_{k=0}^{n}\frac{4s\left[(k+1)(2+\ln\frac{n}{k+1})+\frac{\pi}{3+\frac{\pi^2}{8}}\right]}{(2n+k)^2s+1}+$$

$$n^{2s-1}\sum_{k=n+1}^{\infty}\frac{4s\left[n(\ln\frac{n}{k+1}+2)+\frac{\pi}{3+\frac{\pi^2}{8}}\right]}{(2n+k)^2s+1}=$$

$$n^{2s-1}\sum_{k=n+1}^{\infty}\frac{4s(2n+\frac{\pi}{3+\frac{\pi^2}{8}})}{2n+k}^{2s+1}+n^{2s-1}\sum_{k=n+1}^{\infty}\frac{4sn\ln\frac{k+1}{n}}{(2n+k)^{2s+1}} \leq$$

$$c_1(s)+c_2(s)n^{2s}\sum_{k=n+1}^{\infty}\frac{\ln(1+\frac{k-n+1}{n})}{(2n+k)^{2s+1}}=$$

$$c_1(s)+c_2(s)n^{2s}\sum_{j=1}^{\infty}\frac{\ln(1+\frac{j}{n})}{(3n-1+j)^{2s+1}}=c(s)+\frac{c(s)}{n}\sum_{j=1}^{\infty}\frac{\ln(1+\frac{j}{n})}{(3-\frac{1}{n}+\frac{j}{n})^{2s+1}} \leq$$

$$c(s)(1+\int_0^{\infty}\frac{\ln(1+x)dx}{(2+x)^{2s+1}}) \leq c(s). \tag{2.33}$$

Comparing (2.32) and (2.33) with (2.26), we complete the proof of Lemma (2.3) for even $r \geq 1$. Lemma (2.3) is proved similarly in case of odd $r = 2s-1$.◄

Lemma 2.4. *For $n^{-\frac{1}{2}} \leq u \leq 2\pi - n^{-\frac{1}{2}}$ we have inequality*

$$|\kappa^n_{r,n}(u)| \leq c(r).$$

Proof. If $n^{-\frac{1}{2}} \leq u \leq 2\pi - n^{-\frac{1}{2}}$, then $\frac{1}{\sin^2 \frac{u}{2}} \leq \frac{\pi^2}{4} n$ and, therefore, from Lemma 2.1 and inequalities (2.28) and (2.29) we have

$$|\kappa^n_{2s,n}(u)| \leq c(s) n^{2s} \left(\sum_{l=0}^{n-2} \sum_{k=0}^{\infty} \frac{1}{(n+1+k+l)^{2s+2}} + \sum_{k=0}^{\infty} \frac{1}{(2n+k)^{2s+1}} \right) \leq c(s)$$

and, similarly, $|\kappa^n_{2s-1}(u)| \leq c(s)$. Lemma 2.4 is proved.◄

Lemma 2.5. *We have the estimate*

$$\max_u |\kappa^n_{r,n}(u)| \leq c(r) n \quad (n = 1, 2, \ldots).$$

Proof. Consider the case $r = 1$. So, from (2.6) we have $(0 \leq u \leq 2\pi)$

$$R_{1,n}(u) = \sum_{k=n+1}^{\infty} \frac{\sin ku}{k} = \frac{\pi - u}{2} - \sum_{k=1}^{n} \frac{\sin ku}{k}. \tag{2.34}$$

It is well known [37, p.105], that

$$\left| \sum_{k=1}^{n} \frac{\sin ku}{k} \right| \leq c \quad (n = 1, 2, \ldots),$$

Assertion of Lemma 2.5 follows from (2.9), (2.35) and (2.36).◄

Now we need one result established in author's paper [10]. Define for every $\lambda \geq 1$ a measurable 2π-periodical essentionally confined function (kernel) $\kappa_\lambda = \kappa_\lambda(x)$. Then we can define linear operator

$$\mathcal{K}_\lambda(f) = \mathcal{K}_\lambda(f)(x) = \int_{-\pi}^{\pi} f(t) \kappa_\lambda(t-x) dt, \tag{2.37}$$

functional in space $L^{p(x)}_{2\pi}$. We will say that the kernel family $\{\kappa_\lambda(x)\}_{1 \leq \lambda < \infty}$ satisfies the conditions $A)$, $B)$ $C)$, respectively, if the following estimates hold:

$$A) \quad \int_{-\pi}^{\pi} |\kappa_\lambda(t)| dt \leq c_1,$$

$$B) \quad \sup_x |\kappa_\lambda(x)| \leq c_2 \lambda^v,$$

$$C) \quad |\kappa_\lambda(x)| \le c_3 \quad \lambda^{-\gamma} \le |x| \le \pi,$$

where $v, \gamma, c_j > 0$ are independent of λ. The theorem below was proved in [10].

Theorem I. *Let $\kappa_\lambda = \kappa_\lambda(x)$ $(1 \le \lambda < \infty)$ satisfy the conditions $A)$—$C)$. If $p(x) \in \mathcal{P}_{2\pi}$, then the operator family convolution $\{\mathcal{K}_\lambda(f)\}_{\lambda \ge 1}$, defined by the equality (2.37), is uniformly bounded in $L_{2\pi}^{p(x)}$.*

Now we can formulate the following auxiliary assertion:

Lemma 2.6. *Let $p(x) \in \mathcal{P}_{2\pi}$, $f \in L_{2\pi}^{p(x)}$,*

$$\mathcal{K}_n(f) = \mathcal{K}_n(f)(x) = \int_{-\pi}^{\pi} f(t)\kappa_{r,n}^n(t-x)dt, \quad (n = 1, 2, \ldots). \tag{2.38}$$

Then we have the estimate

$$\|\mathcal{K}_n(f)\|_{p(\cdot)} \le c_r(p)\|f\|_{p(\cdot)}.$$

The assertion of this Lemma follows directly from Theorem I, because in view of Lemmas 2.3—2.5 the kernel family $\kappa_{r,n}^n(x)$ $(n = 1, 2, \ldots)$ satisfies the conditions $A)$—$C)$.

Let's return to the Proof of Theorem 2.1. From the equality (2.11) and Lemma 2.6 we have

$$\|f - V_{n-1}^n(f)\|_{p(\cdot)} \le \frac{c_r(p)}{n^r}\|f^{(r)} - T_n\|_{p(\cdot)}, \tag{2.39}$$

where $T_n = T_n(x)$ is an arbitrary trigonometric polynomial of degree n. The estimate (2.12) follows from (2.39). As for estimate (2.13), its proof is quite similar. The Theorem 2.1 is proved.

Now let's mention the theorem proved in [29]:

Theorem J. *Let $p = p(x) \in \mathcal{P}_{2\pi}$, $f(x) \in L_{2\pi}^{p(x)}$. Then the following estimate holds:*

$$E_n(f)_{p(\cdot)} \le c(p)\Omega(f, \frac{1}{n})_{p(\cdot)}.$$

Combined, Theorem J and Theorem 2.1 make it possible to formulate

Consequence 2.1. *Let $p = p(x) \in \mathcal{P}_{2\pi}$, $r \ge 0$, $f(x) \in W_{p(\cdot)}^r$. Then the following estimates hold:*

$$\|f - V_{n-1}^n(f)\|_{p(\cdot)} \le \frac{c(p)}{n^r}\Omega(f^{(r)}, \frac{1}{n})_{p(\cdot)}, \tag{2.40}$$

$$\|f - V_n^n(f)\|_{p(\cdot)} \le \frac{c(p)}{n^r}\Omega(f^{(r)}, \frac{1}{n})_{p(\cdot)}. \tag{2.41}$$

Consequence 2.2. *Let $p = p(x) \in \mathcal{P}_{2\pi}$, $r \ge 0$, $f(x) \in W_{p(\cdot)}^r$. Then the following estimate holds $(m = 1, 2, \ldots)$:*

$$E_m(f)_{p(\cdot)} \le \frac{c(p)}{m^r}\Omega(f^{(r)}, \frac{2}{m})_{p(\cdot)}. \tag{2.42}$$

Proof. If $m = 2n$, then estimate (2.42) follows from (2.40). If $m = 2n - 1$, then (2.42) follows from (2.41). Consequence 2.2 is proved.◄

References

[1] W. Orlicz.Uber konjugierte Exponentenfolgen// Studia Math. 3 (1931), 200212.

[2] H. Nakano. Modulared Semi-ordered Linear Spaces. – Tokyo.: Maruzen Co., Ltd., 1950,

[3] H. Nakano. Topology and Topological Linear Spaces. – Tokyo.: Maruzen Co., Ltd., 1951.

[4] J. Musielak. Orlicz Spaces and Modular Spaces. – Berlin.: Springer-Verlag, 1983,

[5] J. Musielak and W. Orlicz. On modular spaces// Studia Math. 18 (1959), 4965.

[6] I. V. Tsenov. Generalization of the problem of best approximation of a function in the space L^s// (Russian) Uch. Zap. Dagestan Gos. Univ. 7 (1961), 2537.

[7] A. N. Kolmogorov, Zur Normierbarkeit eines allgemeinen topologischen linearen Raumes, Studia Math. 5 (1934), 2933.

[8] Sharapudinov I.I. On the topologi of the space $L^{p(x)}([0,1])$// Math. Notes. - 1979. - V. 26. 4. P. 796-806.

[9] Sharapudinov I.I. The basis property of the Haar sistem in the space $L^{p(x)}([0,1])$ and the principle of localization in the mean // Math. Sb. - 1986. -Vol. 130(172), No 2(6). P. 275–283,

[10] Sharapudinov I.I. Uniform boundedness in L^p $(p = p(x))$ of some families of convolution operators// Mathematical Notes. February 1996, Volume 59, Issue 2, pp 205-212

[11] V. V. Zhikov. Averaging of functionals of the calculus of variations and elasticity theory// Math. USSR Izv. 29 (1987), no. 1, 3366. [Translation of Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), no. 4, 675710, 877.]

[12] V. V. Zhikov. Meyer-type estimates for solving the nonlinear Stokes system// Differ. Equ. 33 (1997), no. 1, 108115. [Translation of Differ. Uravn. 33 (1997), no. 1, 107114, 143.]

[13] V. V. Zhikov. On some variational problems //Russian J. Math. Phys. 5 (1997), no. 1, 105116 (1998).

[14] L. Diening. Maximal function on generalized Lebesgue spaces Lp(). Math. Inequal. Appl., 7:245253, 2004.

[15] L. Diening and M. Ružička. Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics// J. Reine Angew. Math. 563 (2003), 197220.

[16] L. Diening, P. Hästö and A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces. //*Function Spaces, Differential Operators and Nonlinear Analysis*, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004. Math. Inst. Acad. Sci. Czech Republick, Praha.

[17] Lars Diening, Petteri Harjulehto, Peter Hasto, Michael Ružička. Lebesgue and Sobolev spaces with variable exponent. – Lecture Notes in Mathematics 2017. Springer-Verlag Berlin and Heidelberg. 2011.

[18] V. Kokilashvili, N. Samko, and S. Samko. Singular operators in variable spaces $L^{p(\cdot)}(\Omega, \rho)$ with oscillating weights. Math. Nachr., 280:11451156, 2007.

[19] V. Kokilashvili and S. Samko. Singular integral equations in the Lebesgue spaces with variable exponent. Proc. A. Razmadze Math. Inst., 131:6178, 2003.

[20] V. Kokilashvili and S. Samko. Singular integrals in weighted Lebesgue spaces with variable exponent. Georgian Math. J., 10:145156, 2003.

[21] V. Kokilashvili and S. Samko. Weighted boundedness in Lebesgue spaces with variable exponents of classical operators on Carleson curves. Proc. A. Razmadze Math. Inst., 138:106110, 2005.

[22] V. Kokilashvili. and Samko S. Singular Integrals in Weigted Lebesgue Spaces with Variable Exponent// Georgian Math.J. 2003. Vol. 10. No 1. 145–156.

[23] S. Samko. Convolution type operators in Lp(x). Integr. Transform. Spec. Funct., 7:no. 1.2, 123.144, 1998.

[24] S. Samko. Hardy inequality in the generalized Lebesgue spaces. Fract. Calc. Appl. Anal., 6:355.362, 2003.

[25] S. Samko. On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. Integral Transforms Spec. Funct., 16:461.482, 2005.

[26] Sharapudinov I.I. Several questions of approximation theory in spaces $L^{p(x)}$ // Analysis Mathematica. 2007, . 33, 2, . 135-153

[27] Sharapudinov I.I. Approximation of function in metric space $L^{p(x)}([a, b])$ and quadrature formulas //Constructive function theory'81. Procedings of the International Conference on Constructive Function Theory. Varna, June 1-5, 1981. . 189–193,

[28] Sharapudinov I.I. Several questions of approximation theory by trigonometric polynomials in $L_{2\pi}^{p(x)}$// Itogi nauki. South of Russia. Mathematical forum. Studies on mathematical analysis and differetial equation – Vladikavkaz. 2011. V.5. pp. 108 – 118.

[29] Sharapudinov I.I. Approximation of functions in $f \in L_{2\pi}^{p(x)}$ by trigonometric polynomials//Izvestiya: Mathematics 77:2 407434.2013 (translated from Izvestiya RAN: Ser. Mat. 77:2 197224. 2013)

[30] Sharapudinov I.I. Several questions of approximation theory in variable exponent Lebesgue and Sobolev spaces. – Itogi nauki. South of Russia. Mathematical monograph. – Vladikavkaz. 2013. – 267 p.

[31] Guven Ali and Israfilov D. M. Trigonometric approximation in Generalized Lebesgue spaces $L^{p(x)}$// Journal of Mathematical Inequalities, Volume 4, Number 2 (2010), 285299

[32] Ramazan Akgun. Polynomial approximation of function in weigted Lebesgue and Smirnov spaces whith nonstandard growth// Georgian Math.J. 18(2011), 203-235.

[33] Ramazan Akgun. Trigonometric approximation of functions in generalized Lebesgue spaces whith variable exponent//Ukrainian Mathematical Journal, Vol. 63, No.1, June, 2011.

[34] Ramazan Akgun and Vakhtang Kokilashvili. On converse theorems of trigonometric approximation in weigted variable exponent Lebesgue spaces//Banach J.Math.Anal. 5(2011), No.1, 70-82

[35] Sharapudinov I.I. Approximative properties of Valle - Poussin means on Sobolev type classes with variable exponent// Herald of Daghestan scientific center of RAS. 45. 2012. p. 5-13.

[36] Sharapudinov I.I. Approximation of smooth functions in $L_{2\pi}^{p(x)}$ by Valle - Poussian means//Izv.Saratov. Univ. V.13. Issue 1. Part 1. P. 45 – 49.

[37] Zigmund A. Trigonometric seres. V. 1. – Moskov.: Mir. 1965.

Idris Sharapudinov
Daghestan Scientific Centre of the Russian Academy of Sciences, Makhachkala, Russian Federation
E-mail: sharapud@mail.ru

Received 04 June 2013
Accepted 15 September 2013