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Marcinkiewicz Integrals with Rough Kernel Associated with Schrödinger Operator on Vanishing Generalized Morrey Spaces

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Abstract. Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential V belongs to the reverse Hölder class. In this paper, we study the boundedness of the Marcinkiewicz operator with rough kernels associated with Schrödinger operator $\mu^L_{j,\Omega}$ on vanishing generalized Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$. We find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operators $\mu^L_{j,\Omega}$ from one vanishing generalized Morrey space $VM_{p,\varphi_1}(\mathbb{R}^n)$ to another $VM_{p,\varphi_2}(\mathbb{R}^n)$, $1 < p < \infty$ and from the space $VM_{1,\varphi_1}(\mathbb{R}^n)$ to the weak space $WVM_{1,\varphi_2}(\mathbb{R}^n)$.

Key Words and Phrases: Marcinkiewicz operator, rough kernel, Schrödinger operator, vanishing generalized Morrey space
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1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [22] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [6, 9, 14, 22]. The classical version of Morrey spaces is equipped with the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}, \tag{1}$$

where $0 \le \lambda < n$ and $1 < p < \infty$. The generalized Morrey spaces are defined with r^λ replaced by a general non-negative function $\varphi(x, r)$ satisfying some assumptions.

The vanishing Morrey space $VM_{p,\lambda}$ in its classical version was introduced in [29], where applications to PDE were considered. We also refer to [4] and [23] for some properties of such spaces. This is a subspace of functions in $M_{p,\lambda}(\mathbb{R}^n)$, which satisfy the condition

$$\lim_{r\to 0} \sup_{\substack{x\in\mathbb{R}^n \\ 0<t<r}} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0. \tag{2}$$

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The main purpose of this paper is to study vanishing generalized Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$ (see Definition 2) and prove the boundedness of the Marcinkiewicz operator with rough kernel $\mu_{j,\Omega}^L$ on $VM_{p,\varphi}(\mathbb{R}^n)$ spaces..

Suppose that $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

In [27], Stein defined the Marcinkiewicz integral for higher dimensions. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{3}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has a mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{4}$$

where $x' = x/|x|$ for any $x \neq 0$.

(iii) $\Omega \in L_1(S^{n-1})$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Remark 1. *We easily see that the Marcinkiewicz integral operator of higher dimension μ_Ω can be regarded as a generalized version of the classical Marcinkiewicz integral in the one dimension case. Also, it is easy to see that μ_Ω is a special case of the Littlewood-Paley g-function if we take*

$$g(x) = \Omega(x')|x|^{-n+1} \chi_{|x| \leq 1}(|x|).$$

We say that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$ if there exists a constant $C > 0$ such that $|\Omega(x') - \Omega(y')| \leq C|x' - y'|^\alpha$ for all $x', y' \in S^{n-1}$.

In [27], Stein proved the following results.

Theorem 1. *Suppose that Ω satisfies* (3).

(a) *If $\Omega \in L_1(S^{n-1})$ and Ω is odd, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.*

(b) *If Ω satisfies* (4) *and $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, then μ_Ω is of weak type $(1,1)$. That is, there exists a constant C such that for any $t > 0$ and $f \in L_1(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : \ \mu_\Omega(f)(x) > t\}| \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| dx.$$

(c) *If Ω satisfies* (4) *and $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, then μ_Ω is of type (p,p) for $1 < p \leq 2$. That is, there exists a constant A_p such that for any $f \in L_p(\mathbb{R}^n)$,*

$$\|\mu_\Omega(f)\|_{L_p} \leq A_p \|f\|_{L_p}.$$

The L_p boundedness of μ_Ω has been studied extensively. See [3, 20, 27, 28], among others. A survey of past studies can be found in [7]. Recently the following result was obtained in [2] and [10].

Theorem 2. *Suppose that* Ω *satisfies* (3) *and* (4). *If*

$$\Omega \in L(\log^+ L)^{1/2}(S^{n-1}), \tag{5}$$

then μ_Ω *is bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$ *and if*

$$\Omega \in L(\log^+ L)(S^{n-1}), \tag{6}$$

then μ_Ω *is bounded from* $L_1(\mathbb{R}^n)$ *to* $WL_1(\mathbb{R}^n)$. *The exponent* $1/2$ *is the best possible.*

The following theorem was proved in [5] for $p = 1$ and in [21] for $1 < p \leq \infty$.

Theorem 3. *Suppose that* Ω *satisfies* (3). *If* $\Omega \in L_1(S^{n-1})$, *then* M_Ω *is bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p \leq \infty$ *and if* Ω *satisfies the condition* (6), *then* M_Ω *is bounded from* $L_1(\mathbb{R}^n)$ *to* $WL_1(\mathbb{R}^n)$.

Corollary 1. *Let* $1 \leq p < \infty$ *and* $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$. *Then, for* $p > 1$ M_Ω *is bounded on* $L_p(\mathbb{R}^n)$ *and for* $p = 1$ *from* $L_1(\mathbb{R}^n)$ *to* $WL_1(\mathbb{R}^n)$.

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [25] considered L_p estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R_j^L = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \dots, n$. Then, Dziubański and Zienkiewicz [8] introduced the Hardy type space $H_L^1(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$.

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu_{j,\Omega}^L f(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} |\Omega(x-y)| K_j^L(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $K_j^L(x,y) = \widetilde{K_j^L}(x,y)|x-y|$ and $\widetilde{K_j^L}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \dots, n$. In particular, when $V = 0$, $K_j^\Delta(x,y) = \widetilde{K_j^\Delta}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K_j^\Delta}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} \Delta^{-\frac{1}{2}}$, $j = 1, \dots, n$. In this paper, we write $K_j(x,y) = K_j^\Delta(x,y)$ and

$$\mu_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} |\Omega(x-y)| K_j(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Obviously, μ_j are classical Marcinkiewicz functions. Therefore, it will be an interesting thing to study the properties of $\mu_{j,\Omega}^L$. The main purpose of this paper is to show that Marcinkiewicz integrals with rough kernel associated with Schrödinger operators are bounded from one vanishing generalized Morrey space VM_{p,φ_1} to another VM_{p,φ_2}, $1 < p < \infty$, and from the space VM_{1,φ_1} to the weak space WVM_{1,φ_2}.

Note that a nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \leq C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \tag{7}$$

holds for every ball $x \in \mathbb{R}^n$ and $r > 0$, where $B(x,r)$ denotes the open ball centered at x with radius r; see [25]. It is worth pointing out that the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (7), such that $V \in B_{q+\varepsilon}$. Throughout this paper, we always assume that $0 \neq V \in B_n$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Vanishing generalized Morrey spaces

Definition 1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, i.e. the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|f\|_{WL_p(B(x,r))} < \infty.$$

where $WL_p(B(x,r))$ denotes the weak L_p-space consisting of all measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} < \infty.$$

Also the spaces $L_p^{\rm loc}(\mathbb{R}^n)$ and $WL_p^{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology are defined as the sets of all functions f such that $f\chi_B \in L_p(\mathbb{R}^n)$ and $f\chi_B \in WL_p(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$, respectively.

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda}{p}}}.$$

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator with rough kernel M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)||f(y)|dy.$$

Suppose that T_Ω represents a linear or a sublinear operator, such that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_\Omega f(x)| \le c_0 \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} \, |f(y)|dy, \tag{8}$$

where c_0 is independent of f and x.

We point out that the condition (8) was first introduced by Soria and Weiss in [26]. The condition (8) are satisfied by many interesting operators in harmonic analysis, such as the Calderón–Zygmund operators, Carleson's maximal operator, Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci–Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [21], [26] for details).

The following statements, were proved in [19] (see also [16] and for $\Omega \equiv 1$ [12, 13, 14, 17]).

Lemma 1. *Let $1 \le p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$ satisfy (3). Let also T_Ω be a sublinear operator satisfying the condition (8), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. If $p > 1$, then for $q' \le p$ or $p < q$ the inequality*

$$\|T_\Omega f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt \tag{9}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

If $p = 1$, then the inequality

$$\|T_\Omega f\|_{WL_1(B(x_0,r))} \lesssim r^{n} \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))} dt \tag{10}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Theorem 4. *Suppose that Ω is homogeneous of degree zero. Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)}{t^{\frac{n}{p}+1}} dt \le C \, \frac{\varphi_2(x,r)}{r^{\frac{n}{p}}}, \tag{11}$$

where C does not depend on x and r. Let T_Ω be a sublinear operator satisfying the condition (8), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Then the operator T_Ω is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$

$$\|T_\Omega f\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|T_\Omega f\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

From Theorems 2 and 4 it follows

Corollary 2. *Let $1 \le p < \infty$, Ω satisfy the conditions* (3), (4) *and* (φ_1, φ_2) *satisfy the condition* (11). *If Ω satisfies the condition* (5), *then the operators μ_Ω, $\mu_{j,\Omega}$ are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and if Ω satisfies the condition* (6), *the operators μ_Ω, $\mu_{j,\Omega}$ are bounded from M_{1,φ_1} to WM_{1,φ_2}.*

From Theorems 3 and 4 it follows

Corollary 3. *Let $1 \le p < \infty$, Ω satisfy the condition* (3) *and* (φ_1, φ_2) *satisfy the condition* (11). *If $\Omega \in L_1(S^{n-1})$, then M_Ω is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and if Ω satisfies the condition* (6), *then M_Ω is bounded from M_{1,φ_1} to WM_{1,φ_2}.*

Corollary 4. *Let $1 \le p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$ and (φ_1, φ_2) satisfy the condition* (11). *Then the operators μ_Ω, $\mu_{j,\Omega}$ and M_Ω are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.*

Definition 2. *The vanishing generalized Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$ are defined as the spaces of functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))} = 0. \tag{12}$$

Everywhere in the sequel we assume that

$$\lim_{t\to 0} \frac{t^{\frac{n}{p}}}{\varphi(x,t)} = 0, \tag{13}$$

and

$$\sup_{0<t<\infty} \frac{t^{\frac{n}{p}}}{\varphi(x,t)} < \infty, \tag{14}$$

which makes the space $VM_{p,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong then to this space. The vanishing spaces $VM_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm (see, for example [24])

$$\|f\|_{VM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))}. \tag{15}$$

3. Marcinkiewicz operator $\mu_{j,\Omega}^L$ in the spaces $VM_{p,\varphi}$

In this section, we prove the boundedness of the Marcinkiewicz operator $\mu_{j,\Omega}^L$ on $VM_{p,\varphi}(\mathbb{R}^n)$ spaces. For $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V\ (y)dy\ \le 1 \right\}.$$

Lemma 2. *[25] Let $V \in B_q$ with $q \geq n/2$. Then there exists $l_0 > 0$ such that*

$$\frac{l}{C}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\left(1+\frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x-y| < C\rho(x)$.

Lemma 3. *[25] Let $V \in B_q$ with $q \geq n/2$. For any $l > 0$, there exists $C_l > 0$ such that*

$$\left|K_j^L(x,y)\right| \leq \frac{C_l}{\left(1+\frac{|x-y|}{\rho(x)}\right)^l} \frac{1}{|x-y|^{n-1}},$$

and

$$\left|K_j^L(x,y) - K_j(x-y)\right| \leq \ C\frac{\rho(x)}{|x-y|^{n-2}}.$$

Theorem 5. *Suppose that Ω satisfies (3), (4) and $V \in B_n$. If Ω satisfies the condition (5), then the operators $\mu_{j,\Omega}^L$, $j = 1, \ldots, n$ are bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$, and if Ω satisfies the condition (6), then these operators are bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

Proof. In the proof we used the idea in [11]. It suffices to show that

$$\mu_{j,\Omega}^L f(x) \leq \mu_{j,\Omega} f(x) + CM_\Omega f(x), \ \textit{a.e. } x \in \mathbb{R}^n, \tag{16}$$

where M_Ω denotes the Hardy-Littlewood operator with rough kernel.

Fix $x \in \mathbb{R}^n$ and let $r = \rho(x)$.

$$\begin{aligned}
\mu_{j,\Omega}^L f(x) &\leq \left(\int_0^r \left|\int_{|x-y|\leq t} |\Omega(x-y)| K_j^L(x,y) f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&+ \left(\int_r^\infty \left|\int_{|x-y|\leq r} |\Omega(x-y)| K_j^L(x,y) f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&+ \left(\int_r^\infty \left|\int_{r<|x-y|\leq t} |\Omega(x-y)| K_j^L(x,y) f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^r \left|\int_{|x-y|\leq t} |\Omega(x-y)| [K_j^L(x,y) - K_j(x,y)] f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&+ \left(\int_0^r \left|\int_{|x-y|\leq t} |\Omega(x-y)| K_j(x,y) f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
& + \left(\int_r^\infty \left| \int_{|x-y|\le r} |\Omega(x-y)| K_j^L(x,y) f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left| \int_{r<|x-y|\le t} |\Omega(x-y)| K_j^L(x,y) f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
: & = E_1 + E_2 + E_3 + E_4.
\end{aligned}
$$

For E_1, by Lemma 3, we have

$$
E_1 \le C \left(\int_0^r \left| \frac{1}{r} \int_{|x-y|\le t} \frac{|\Omega(x-y)|}{|x-y|^{n-2}} \ |f(y)|dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \le C M_\Omega f(x).
$$

Obviously,

$$
E_2 \le \mu_{j,\Omega} f(x).
$$

For E_3, using Lemma 3 again, we get

$$
E_3 \le \left(\int_r^\infty \left| \frac{1}{r} \int_{|x-y|\le r} \frac{|\Omega(x-y)|}{|x-y|^{n-2}} \ |f(y)|dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \le C M_\Omega f(x).
$$

It remains to estimate E_4. From Lemma 3, we obtain

$$
\begin{aligned}
E_4 & \le C \left(\int_r^\infty \left| r \int_{r<|x-y|\le t} \frac{|\Omega(x-y)|}{|x-y|^n} \ |f(y)|dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \le C_r \left(\int_r^\infty \left| \sum_{k=0}^{[\log_2 t/r]+1} (2^k r)^n \int_{|x-y|\le 2^k r} |\Omega(x-y)||f(y)|dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \le C_r \left(\int_r^\infty \left| \left(\left[\log_2 \frac{t}{r} \right] + 1 \right) M_\Omega f(x) \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \le C_r \left(\int_r^\infty \frac{t}{r} M_\Omega f(x)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \le C M_\Omega f(x).
\end{aligned}
$$

Thus, Theorem 5 is proved.◀

Corollary 5. *Suppose that Ω satisfies (3), (4) and $V \in B_n$. If $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$ then the operators $\mu^L_{j,\Omega}$, $j = 1, \dots, n$ are bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$, and from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

Lemma 4. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$ satisfy (3), (4) and $V \in B_n$. If $p > 1$, then for $q' \le p$ or $p < q$ the inequality*

$$\|\mu^L_{j,\Omega} f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt \tag{17}$$

holds for any ball $B(x_0, r)$, and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

If $p = 1$, then the inequality

$$\|\mu^L_{j,\Omega} f\|_{WL_1(B(x_0,r))} \lesssim r^{n} \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))} dt \tag{18}$$

holds for any ball $B(x_0, r)$, and for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. In the proof we use the idea and technique of Guliyev (see [14], Theorem 6.1). Note that

$$\begin{aligned}
\|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} &= \Big(\int_{B(0,t)} |\Omega(y)|^q \, dy \Big)^{1/q} \\
&= \Big(\int_0^t r^{n-1} \, dr \int_{S^{n-1}} |\Omega(y')|^q \, d\sigma(y') \Big)^{1/q} \\
&\approx c_0 \|\Omega\|_{L_q(S^{n-1})} \, |B(x_0,t)|^{1/q}.
\end{aligned}$$

Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 with radius r, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0,$$

and have

$$\|\mu^L_{j,\Omega} f\|_{L_p(B)} \le \|\mu^L_{j,\Omega} f_1\|_{L_p(B)} + \|\mu^L_{j,\Omega} f_2\|_{L_p(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $\mu^L_{j,\Omega} f_1 \in L_p(\mathbb{R}^n)$, from the boundedness of $\mu^L_{j,\Omega}$ in $L_p(\mathbb{R}^n)$ (see Corollary 5) it follows that:

$$\|\mu^L_{j,\Omega} f_1\|_{L_p(B)} \le \|\mu^L_{j,\Omega} f_1\|_{L_p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_q(S^{n-1})} \|f_1\|_{L_p(\mathbb{R}^n)} \approx \|\Omega\|_{L_q(S^{n-1})} \|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It's clear that $x \in B$, $y \in {}^{\complement}(2B)$ imply $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$|\mu^L_{j,\Omega} f_2(x)| \lesssim \int_{\complement_{(2B)}} \frac{\Omega(x-y)\, |f(y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy &\approx \int_{\complement_{(2B)}} |\Omega(x-y)|\,|f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\approx \int_{2r}^{\infty} \int_{2r\le |x_0-y|<t} |\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}}\ .\end{aligned}$$

If $q' \le p$, then by applying Hölder's inequality we get

$$\begin{aligned}&\int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n} dy \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))}\, \|f\|_{L_{q'}(B(x_0,t))}\, \frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{q}}\, |B(x_0,t)|^{\frac{1}{q}}\, \frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt.\end{aligned}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|\mu^L_{j,\Omega} f_2\|_{L_p(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\, r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt \tag{19}$$

is valid. Thus

$$\|\mu^L_{j,\Omega} f\|_{L_p(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})} \Big(\|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt \Big)$$

On the other hand,

$$\begin{aligned}\|f\|_{L_p(2B)} &\approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt.\end{aligned} \tag{20}$$

Thus

$$\begin{aligned}\|\mu^L_{j,\Omega} f\|_{L_p(B)} &\lesssim \|\Omega\|_{L_q(S^{n-1})}\, r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt.\end{aligned}$$

If $1 < p < q$, then by Minkowski theorem and the Hölder inequality we get

$$\|\mu^L_{\Omega} f_2\|_{L_p(B)} \le \left(\int_B \left(\int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)||f(y)| dy\, \frac{dt}{t^{n+1}} \right)^p dx \right)^{\frac{1}{p}}$$

$$\le \int_{2r}^{\infty}\int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_p(B)}|f(y)|\,dy\,\frac{dt}{t^{n+1}}$$
$$\lesssim |B(x_0,r)|^{\frac{1}{p}-\frac{1}{q}}\int_{2r}^{\infty}\int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_q(B)}\,|f(y)|\,dy\,\frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\,r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L_1(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}\,dt.$$

Thus

$$\|\mu_{j,\Omega}^{L}f\|_{L_p(B)} \lesssim r^{\frac{n}{p}}\int_{2r}^{\infty} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}dt.$$

Let $p = 1 < s < \infty$. From the weak $(1,1)$ boundedness of $\mu_{j,\Omega}^{L}$ and (20) it follows that

$$\begin{aligned}\|\mu_{j,\Omega}^{L}f_1\|_{WL_1(B)} &\le \|\mu_{j,\Omega}^{L}f_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} = \|f\|_{L_1(2B)}\\ &\lesssim r^n\int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|dy\,\frac{dt}{t^{n+1}}\\ &\le r^n\int_{2r}^{\infty} t^{-n-1}\|f\|_{L_1(B(x_0,t))}\,dt. \qquad (21)\end{aligned}$$

Then by (19) and (21) we get the inequality (18).◀

Theorem 6. *Let $1 < p < \infty$, $1 < q \le \infty$ and $q' \le p$ or $p < q$. Let also $\Omega \in L_q(S^{n-1})$ satisfy (3), (4) and $V \in B_n$. Then the operators $\mu_{j,\Omega}^{L}$, $j = 1, \ldots, n$ are bounded from VM_{p,φ_1} to VM_{p,φ_2}, if (φ_1, φ_2) satisfies the condition (13)-(14) and*

$$c_\delta := \int_{\delta}^{\infty}\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)t^{-\frac{n}{p}-1}dt < \infty \qquad (22)$$

for every $\delta > 0$, and

$$\int_{r}^{\infty}\frac{\varphi_1(x,t)}{t^{\frac{n}{p}+1}}dt \le C_0\frac{\varphi_2(x,r)}{r^{\frac{n}{p}}} \qquad (23)$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Remark 2. *The condition (22) is not needed in the case when $\varphi(x,r)$ does not depend on x, since (22) follows from (23) in this case.*

Proof. The statement is derived from the estimates (17) and (18). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space follows from Lemma 4 and condition (23)

$$\|\mu_{j,\Omega}^{L}f\|_{VM_{p,\varphi_2}} = \sup_{x\in\mathbb{R}^n, r>0}\varphi_2(x,r)^{-1}\,\|\mu_{j,\Omega}^{L}\|_{L_p(B(x,r))}$$

$$
\begin{aligned}
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\, r^{\frac{n}{p}} \int_r^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\, r^{\frac{n}{p}} \int_r^\infty \varphi_1(x,t) \left[\varphi_1(x,t)^{-1} \|f\|_{L_p(B(x_0,t))}\right] \frac{dt}{t^{\frac{n}{p}+1}} \\
&\lesssim \|f\|_{VM_{p,\varphi_1}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\, r^{\frac{n}{p}} \int_r^\infty \varphi_1(x,t) \frac{dt}{t^{\frac{n}{p}+1}} \\
&\lesssim \|f\|_{VM_{p,\varphi_1}}.
\end{aligned}
$$

So we only have to prove that

$$
\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_1(x,r)^{-1}\|f\|_{L_p(B(x,r))} = 0 \quad \Rightarrow \quad \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_2(x,r)^{-1}\|\mu^L_{j,\Omega}\|_{L_p(B(x,r))} = 0 \tag{24}
$$

To show that $\sup\limits_{x\in\mathbb{R}^n} \varphi_2(x,r)^{-1}\|\mu^L_{j,\Omega}\|_{L_p(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (17):

$$
\varphi_2(x,r)^{-1}\|\mu^L_{j,\Omega}\|_{L_p(B(x,r))} \le C[I_\delta(x,r) + J_\delta(x,r)], \tag{25}
$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$
I_\delta(x,r) := \frac{r^{\frac{n}{p}}}{\varphi_2(x,r)} \left(\int_r^{\delta_0} \varphi_1(x,t) t^{-\frac{n}{p}-1} \left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right) dt \right),
$$

$$
J_\delta(x,r) := \frac{r^{\frac{n}{p}}}{\varphi_2(x,r)} \left(\int_{\delta_0}^\infty \varphi_1(x,t) t^{-\frac{n}{p}-1} \left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right) dt \right)
$$

and it is supposed that $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$
\sup_{x\in\mathbb{R}^n} \varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0}
$$

where C and C_0 are constants from (23) and (25). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$
\sup_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.
$$

The estimation of the second term now may be made by the choice of r sufficiently small. Indeed, thanks to the condition (13) we have

$$
J_\delta(x,r) \le c_{\delta_0} \|f\|_{VM_{p,\varphi}} \frac{r^{\frac{n}{p}}}{\varphi(x,r)},
$$

where c_{δ_0} is the constant from (22). Then, by (13) it suffices to choose r small enough such that

$$
\sup_{x\in\mathbb{R}^n} \frac{r^{\frac{n}{p}}}{\varphi(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{VM_{p,\varphi}}},
$$

which completes the proof.◀

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