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Geometric Issues in the Algebraic Theory of Many Valued Logics

L. P. Belluce*, A. Di Nola, G. Lenzi

Abstract. We apply to MV-algebras the theory of Universal Algebraic Geometry.

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1. Introduction

The present report is based on the joint paper *Diophantine Algebraic Geometry for MV-Algebras: Basic Issues,* by L. P. Belluce, A. Di Nola, G. Lenzi.

We present a preliminary study of applying the concepts of algebraic geometry over fields to the theory of MV-algebras. According to [2], rational polyhedra are the genuine algebraic varieties of the formulas of Łukasiewicz Logic, in a precise sense: zerosets of McNaughton functions coincide with rational polyhedra. Now, McNaughton functions are functions from $[0,1]^n$ $[0,1]$, so that in the theory of [2], the MV algebra $[0,1]$ plays a fundamental role. On the other hand, there are reasons to be interested in other MV algebras, because every MV algebra can be viewed as the Lindenbaum algebra of some many-valued logic, and as such, it has logical relevance. This is why we try in this paper to generalize somewhat the theory of [2] to MV algebras as general as possible.

We proceed along lines similar to Plotkin [4].

We note that in algebraic geometry the central notion is the one of polynomial. One has three possibilities:

- considering coefficient-free algebraic geometry; this allows one to evaluate polynomials in arbitrary fields;
- considering Diophantine algebraic geometry: this means that the field where coefficients are taken coincides with the field where polynomials are evaluated;
- considering general, non-Diophantine algebraic geometry, where polynomials take coefficients in a field K and are evaluated in an extension L of K.

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*Corresponding author.

It turns out that all these three possibilities can be extended to universal algebra, and this is done in [4]. Since universal algebra subsumes the equational theory of MV algebras, we can consider what happens in universal algebraic geometry (coefficient-free, Diophantine or non-Diophantine) over MV algebras.

2. MV Algebras and Polynomials

For all notions concerning MV algebras, we refer the readers to [1]. Here we just give the definition.

An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $\oplus$ is a binary operation, * is a unary operation and 0 is a constant such that the following axioms are satisfied for any a, $b \in A$:

i) $(A, \oplus, 0)$ is an abelian monoid,

ii) $(a^*)^* = a$,

iii) $0^* \oplus a = 0^*$

iv) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

On an MV-algebra A we define the constant 1 and the auxiliary operation $\odot$ as follows:

v) $1 := 0^*$

vi) $a \odot b := (a^* \oplus b^*)^*$

for any a, $b \in A$.

Recall from [1] that one can construct two functors Γ and Ξ from the category of MV algebras to the category of lattice ordered groups with strong unit (ℓu-groups) and conversely, so that the pair (Γ, Ξ) is an equivalence.

2.1. Truncated Functions and a Generalized McNaughton Theorem

The classical McNaughton Theorem [1] implies that a certain space of functions can be represented by two term algebras: truncated infima of suprema of affine functions from $[0,1]^n$ to R on one hand, and MV polynomials on $[0,1]$ on the other hand. This idea can be extended to any MV algebra A, so to relate truncated infima of suprema of affine functions from A^n to $\Xi(A)$ (where Ξ is the inverse Mundici functor; see [1]), and MV polynomial functions on A.

Let A be an MV algebra with associated ℓu-group (G, u). A (G, u)-affine term (with integer slopes) from A^n to G is a term (in the language of groups) of the form $f(x_1, \ldots, x_n) = g_0 + m_1 x_1 + \ldots + m_n x_n$, where $g_0 \in G$ and $m_1, \ldots, m_n \in Z$. Note that we identify a variable x_i with the corresponding projection.

Let (G, u) be an ℓu-group associated to an MV algebra A. For an element $g \in G$, we let $\rho(g) = (g \vee 0) \wedge u$. This defines a function $\rho : G \to A$.

A (G,u)-term is a term (in the language of ℓ-groups) of the form $\bigvee_i \bigwedge_j f_{ij}(x)$, where f_{ij} is affine, that is, a finite infimum of finite suprema of affine terms. A truncated (G,u) term is a (G,u)-term of the form $\rho \circ t$, where t is a (G,u)-term.

We let $TT_n(G,u)$ be the set of all truncated (G,u) terms in n variables.

A (G,u)-function is any function from A^n to the corresponding group G defined by a (G,u)-term. A (G,u) affine function is one defined by a (G,u)-affine term. A (G,u) truncated function is one defined by a truncated (G,u)-term.

We let $TF_n(G,u)$ be the set of all truncated (G,u) functions in n variables.

We note that the set $TF_n(G,u)$ of truncated (G,u) functions is an MV algebra. In fact, we can define $t \oplus u = \rho \circ (t+u)$ and $\neg t = u - t$. The set $TT_n(G,u)$ also becomes an MV algebra, but only modulo the axioms of MV algebra (or modulo larger congruences).

Since the inverse Mundici functor Ξ gives a bijection between MV algebras and ℓu-groups, we can write without ambiguity $TF_n(A)$ for $TF_n(\Xi(A))$. This notation will be useful in stating the next theorem, which clarifies the relation between (G,u)-terms and MV polynomials:

Theorem 1. *Let A be an MV algebra, with associated ℓu group (G,u). Then polynomials and truncated (G,u)-terms define the same functions from A^n to A.*

3. Algebraic Sets

In this section we focus on Diophantine algebraic geometry.

Let A be an MV algebra and n be a positive integer and let $A[x_1,\ldots,\ x_n]$ be the absolutely free term algebra over A and $\{x_1,\ldots,\ x_n\}$.

Definition 1. *Let A be an MV-algebra. Let $S \subseteq A[x_1,\ldots,\ x_n]$, $S \neq \emptyset$. Consider the set $\{(a_1,\ldots,\ a_n) \in A^n \mid p(a_1,\ldots.\ a_n) = 0,\ \forall p(x_1,\ \ldots,\ x_n) \in S\}$. Denote this set by $V(S)$, called the* algebraic set *determined by S.*

For a given k-tuple $(y_1.\ldots,\ y_k)$ we will often write $\bar{y}$ for $(y_1.\ldots,\ y_k)$, the arity to be understood in context.

Clearly if we let $I = id(S)$, the ideal of $A[x_1,\ldots,\ x_n]$ generated by S, then $V(I) = V(S)$. Thus algebraic sets are determined by ideals.

Note that for a given non-empty subset $S \subseteq A[x_1,\ldots,\ x_n]$ we may have $V(S) = \emptyset$. This would happen iff for each $\bar{a} \in A^n$ there is a $p \in S$ such that $p(\bar{a}) \neq 0$.

Definition 2. *Call an ideal $J \subseteq A[x_1,\ldots,\ x_n]$* singular *if $V(J) = \emptyset$. Otherwise call J* non-singular.

If $p(x_1,\ldots,\ x_n) \in A[x_1,\ldots,\ x_n]$, and $ord(p) < \infty$, then it cannot have a zero. Also, if $a \in A - \{0\}$ has infinite order, then $p(x_1,\ldots,\ x_n) = (a \wedge q(\bar{x})) + (a \wedge (q(\bar{x})^*)$ has infinite order but also has no zeros for any $q(\bar{x}) \in A[\bar{x}]$. Moreover, if $\hat{A}$ is any MV- extension of A, then it's still true that the above $p(\bar{x}) \in \hat{A}[\bar{x}]$ will have no zero in $\hat{A}$.

Definition 3. *Suppose we have a non-empty* $X \subseteq A^n$*. Let* $I(X) = \{p \in A[x_1, \ldots,\ x_n] \mid p(\bar{y}) = 0,\ \forall \bar{y} \in X\}$ *where* $\bar{y} = (y_1, \ldots,\ y_n)$*,* $y_i \in A$*. Then* $I(X)$ *is an ideal of* $A[x_1, \ldots,\ x_n]$*.*

For a given ideal $J \subseteq A[x_1, \ldots,\ x_n]$ we clearly have that $J \subseteq I(V(J))$ provided $V(J) \neq \emptyset$. Note that for $X \neq \emptyset$ we always have $0 \in I(X)$.

3.1. Point ideals and point radicals

Call an ideal $J \subseteq A[\bar{x}]$ a *point ideal* if for some $\bar{a} = (a_1, \ldots,\ a_n) \in A^n$ we have $J = I(\bar{a})$.

Note that if A is linearly ordered, then each $I(\bar{a})$ is a prime ideal; if A is simple, each $I(\bar{a})$ is a maximal ideal.

Lemma 1. *Each point ideal is non-singular, non-zero and proper.*

Proof. Given an $\bar{a} = (a_1, \ldots,\ a_n)$, let $p(x_1, \ldots,\ x_n) = a_1 x_1^* \oplus \cdots \oplus a_n x_n^*$. Then $p \neq 0$ and $p(a_1, \ldots,\ a_n) = 0$. Consequently, $\bar{a} \in V(I(\bar{a}))$ so $I(\bar{a})$ is non-singular. Clearly $1 \notin I(\bar{a})$, thus $I(\bar{a}) \neq A[\bar{x}]$.◄

We consider the fixpoints of the adjunction (I, V):

Proposition 1. *For a non-singular ideal* $J \subseteq A[x_1, \ldots,\ x_n]$*, we have* $I(V(J)) = \bigcap_{\bar{a} \in V(J)} I(\bar{a})$*.*

For an ideal $I \subseteq A[\bar{x}]$ let ${}_{pt}\sqrt{I} = \bigcap\{I(\bar{a}) \mid I \subseteq I(\bar{a})\}$. We call ${}_{pt}\sqrt{I}$ the *point radical* of I. Note it is an ideal as well.

Observe if J is non-singular so that $V(J) \neq \emptyset$ then there is an $\bar{a} \in V(J)$. Thus for all $p \in J$ we have $p(\bar{a}) = 0$. Hence $J \subseteq I(\bar{a})$. Thus $J \subseteq {}_{pt}\sqrt{J}$.

Corollary 1. *For a non-singular ideal* J*,* $I(V(J)) = {}_{pt}\sqrt{J}$*.*

The following Nullstellensatz theorem holds:

Theorem 2. *The ideals* J *such that* $I(V(J)) = J$ *are exactly the point-radical ideals.*

4. Coordinate algebras

Here again we are in Diophantine geometry.

Definition 4. *Let* $Z \subseteq A^n$ *be a non-empty algebraic set. By the* co-ordinate MV-algebra *of* Z *we mean the MV-algebra* $A[\bar{x}]/I(Z)$*.*

Now let $Z = V(J)$ for a non-singular ideal J. Thus $I = I(V(J)) = {}_{pt}\sqrt{J}$. Then,

Proposition 2. *For a non-singular ideal* J *the co-ordinate MV-algebra of* $V(J)$ *is* $A[\bar{x}]/{}_{pt}\sqrt{J}$*.*

Let $\mathcal{MV}_A = \{A[x_1, \ldots,\ x_n]/J \mid J = {}_{pt}\sqrt{J}, n = 0,\ 1,\ 2 \ldots\}$. We consider the set $\mathcal{MV}_A$ as a full subcategory of MV algebras.

Definition 5. *Let $Z_1 \subseteq A^n$, $Z_2 \subseteq A^m$ be algebraic sets. A mapping $\phi : Z_1 \to Z_2$ is called a* polynomial map *iff there are polynomials $p_1, \dots, p_m \in A[x_1, \dots, x_n]$ such that $\phi(a_1, \dots, a_n) = (p_1(a_1, \dots, a_n), \dots, p_m(a_1, \dots, a_n))$ for every $(a_1, \dots, a_n) \in Z_1$.*

Let $\mathcal{Z}(A)$ be the collection of all algebraic subsets of A^n. Then with polynomial maps as morphisms, $\mathcal{Z}(A)$ becomes a category.

We have the following duality:

Theorem 3. *The categories $\mathcal{MV}_A$ and $\mathcal{Z}(A)$ are dually isomorphic.*

5. Logic of polynomials

The completeness theorem of Łukasiewicz infinite valued logic can be phrased as follows: if the *function* $[\sigma]$ equals 1 on $[0, 1]^n$, then $[\sigma] = 1$ in the Lindenbaum algebra.

We can apply this idea to our context and we get what we call polynomial completeness.

We introduce the following notion:

Definition 6. *An MV algebra A is polynomially complete if for every n, the only polynomial in n variables inducing the zero function on A^n is the zero polynomial.*

We do not have a complete characterization of polynomially complete MV algebras, however in this paper we give one for MV chains.

Theorem 4. *Let C be an MV chain. The following are equivalent:*

1. *C is polynomially complete;*
2. *every polynomial $p \in C[x_1, \dots, x_n]$ which induces the zero function on C induces the zero function on $DH(C)$, where $DH(C)$ is the divisible hull of C.*

6. The finitely presented case

In [2], a study of finitely presented MV algebras is exposed, based on rational polyhedra in $[0,1]^n$. We would like to extend the results of [2] as far as possible in general MV algebras. To this aim we translate the framework of [2] into our more general situation, where:

- formulas ϕ are replaced by polynomials p,
- polynomials evaluating to zero are preferred to formulas evaluating to one (this convention is somewhat a mismatch between algebraic geometry and logic),
- theories Φ are replaced by ideals J,
- finitely axiomatizable theories are replaced by principal ideals,
- polynomials may have constants out of an arbitrary MV algebra C,

- the function Mod on theories is replaced by the function V on ideals of polynomials,
- the function Th on algebraic subsets of $[0,1]^n$ is replaced by the function I on algebraic subsets of C^n.

We can ask questions related to composed functions like $Th(Mod(T))$. Wójcicki's Theorem implies that if T is a finitely axiomatized theory in Lukasiewicz logic, then $Th(Mod(T))$ coincides with T. In algebraic terms, this corresponds to $I(V(p)) = id(p)$ for every polynomial p, which we called strong completeness.

Since Wójcicki's Theorem does not help us when polynomials may have constants, we could consider weakenings of strong completeness. For instance, for what algebras the ideal $I(V(p))$ is principal for every polynomial p? Logically, this corresponds to stating that for all finitely axiomatizable theory T, the theory $Th(Mod(T))$ is finitely axiomatized.

More generally, what are the ideals J such that $I(V(J))$ is principal? This corresponds to considering the theories T such that $Th(Mod(T))$ is finitely axiomatizable.

So let C be an MV algebra. If J is a nonsingular ideal of $C[x_1,\ldots,x_n]$, and p, q are elements of $C[x_1,\ldots,x_n]$, then we say $p \equiv_J q$ if for every zero v of J in C^n, $p(v) = q(v)$.

The Lindenbaum MV-algebra of J is $LIND_J = C[x_1,\ldots,x_n]/\equiv_J$.

We denote by $TF_n(C)$ the MV algebra of truncated funtions on $\Xi(C)$ as defined in Section 2, and by $TF_n(C)|_S$ the MV algebra of truncated functions restricted to S, where $S \subseteq C^n$.

Lemma 2. *Let $p \in C[x_1,\ldots,x_n]$ be a polynomial with at least one zero in C^n. Then the MV algebra $LIND_p$ is isomorphic to $TF_n(C)|_{V(p)}$.*

6.1. Łukasiewicz logic with constants

In this section we aim to show some results concerning the Łukasiewicz logic with constants in a fixed MV algebra A, denoted by $Ł_\infty(A)$. Indeed, we have:

Proposition 3. *For every MV algebra A, the MV algebras $Lind(A)$ and $A[x_1,x_2,\ldots]$ are isomorphic.*

We will say that a logic is complete if tautologies coincide with provable formulas. Clearly, for every A, every provable formula of $Ł_\infty(A)$ is a tautology. The converse implication does not hold in general, but we have a characterization in terms of polynomial completeness:

Proposition 4. *For every MV algebra A, the logic $Ł_\infty(A)$ is complete if and only if A is polynomially complete.*

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Lawrence P. Belluce
Department of Mathematics, University of British Columbia, Vancouver, B. C., Canada
E-mail: belluce@math.ubc.ca

Antonio Di Nola
Department of Mathematics, University of Salerno, 84084 Fisciano (SA), Italy
*E-mail:*adinola@unisa.it

Giacomo Lenzi
Department of Mathematics, University of Salerno, 84084 Fisciano (SA), Italy
*E-mail:*gilenzi@unisa.it

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