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Approximation by Modified Complex Szasz-Mirakjan Operators

N. Ispir

Abstract. In this paper, we give the order of approximation and Voronovskaja-type theorem with quantitative estimate for modified complex Szasz-Mirakjan operators attached to analytic functions in compact disks. We also obtain the exact orders of approximation with quantitative estimate for Stancu type generalization of these operators.

Key Words and Phrases: Complex Szasz-Stancu type operator, rate of convergence, Voronovskaja's theorem, exact order of approximation

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1. Introduction

The several approximation properties of the complex Szasz-Mirakjan (SM) operators defined by $S_n(f)(z) = e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} f\left(\frac{k}{n}\right), z \in \mathbb{C}$, were studied by Bohman [1], Deeba [2], Gergen et.al. [6] and Wood [13] in complex domains. Notice that in the mentioned works the convergence results without any quantitative estimates were obtained for the complex SM operators. Recently, the order of simultaneous approximation and Voronovskaja-type results with quantitative estimate for complex SM operators attached to analytic functions having different conditions in compact disks were obtained by Gal [3,4,5].

In the real case, some modifications of the SM operators are introduced and the approximation properties of these operators were investigated in [9,11,12].

We define the modified complex Szasz-Mirakjan operators as follows

$$S_{n,b_n}(f)(z) = e^{nz/b_n} \sum_{j=0}^{\infty} f\left(\frac{jb_n}{n}\right) \frac{(nz)^j}{j! b_n^j}, z \in \mathbb{C} \tag{1}$$

where $(b_n)_1^{\infty}$ is an increasing and unbounded numerical sequence such that

$$\lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{2}$$

Everywhere in the paper $f:[R,+\infty)\cup D_R\to\mathbb{C}$ is bounded on $[0,+\infty)$ and analytic in $D_R=\{z\in\mathbb{C}:|z|=R,R>2\}$. It is clear that if f is bounded on $[0,\infty)$ then $S_{n,b_n}(f)(z)$ is well defined for all $z\in\mathbb{C}$. Stancu type generalization of the operator (1) is defined by

$$S_n^{(\alpha,\beta;b_n)}(f)(z)=e^{-nz/b_n}\sum_{j=0}^{\infty}f\left(\frac{(j+\alpha)b_n}{n+\beta}\right)\frac{(nz)^j}{j!b_n^j},z\in\mathbb{C} \tag{3}$$

where $0\le\alpha\le\beta$, α,β are real numbers independent of n and $(b_n)_1^\infty$ is the sequence in definition of $S_{n,b_n}(f)(z)$.

In case $b_n=1$, we get the classical complex Stancu type SM operators $S_n^{(\alpha,\beta,1)}(f)(z)$.The convergence of $S_n^{(\alpha,\beta,1)}(f)(z)$ to $f(z)$ belonging to a class of analytic functions satisfying a suitable exponential-type growth condition in a parabolic domain, was investigated by Ghorbanalizadeh [7]. Notice that, the convergence results without any quantitative estimate were obtained for the operator $S_n^{(\alpha,\beta,1)}(f)(z)$ in [7]. In [8], Gupta and Verma studied the exact order of approximation of analytic functions without exponential growth conditions and obtained the Voronovskaja-type result with quantitative estimate for the operators $S_n^{(\alpha,\beta,1)}(f)(z)$. The recent works on this subject motivated us to study further on some different operators.

The goal of this paper is to obtain approximation results for the operators given by (3) on compact disks. For this aim first we give the order of approximation and the Voronovskaja type theorems with quantitative estimate for the operators $S_{n,b_n}(f)$ and $S_n^{(\alpha,\beta;b_n)}(f)$. These results allow us to obtain the exact order of approximation by the operators $S_n^{(\alpha,\beta;b_n)}(f)$ given by (3) .

2. Approximation by the operators $S_{n,b_n}(f)$

In order to establish the next results, we need the following auxiliary lemmas.

Lemma 1. *For all $n,k\in\mathbb{N}\cup\{0\}$, $0\le\alpha\le\beta$, $z\in\mathbb{C}$, we have*

$$S_n^{(\alpha,\beta;b_n)}(f)(z)=\sum_{j=0}^{\infty}\left(\frac{n}{b_n\,(n+\beta)}\right)^j\left[\frac{\alpha b_n}{n+\beta},\frac{(\alpha+1)b_n}{n+\beta},...,\frac{(\alpha+j)b_n}{n+\beta};f\right]z^j \tag{4}$$

where $[x_0,x_1,...,x_m;f]$ denotes the divided difference of the function f on the knots $x_0,x_1,...,x_m$.

Proof. The formula (4) is obtained by similar algebraic calculations in [10]. Note that such a formula was proved by Lupas in [10] for classical Szasz operators in real case, but the formula holds in complex setting too [see also 5].◄

We now give recurrence formula as follows:

Lemma 2. *Let $e_k(z)=z^k, S_n^{(\alpha,\beta;b_n)}(e_k)(z):=T_{n,k}^{(\alpha,\beta;b_n)}(z), k,n\in\mathbb{N}$ and (b_n) be an increasing sequence satisfying the condition (2). Then the recurrence relation*

$$T_{n,k+1}^{(\alpha,\beta;b_n)}(z)=\frac{b_nz}{n+\beta}\left(T_{n,k}^{(\alpha,\beta;b_n)}(z)\right)'+\left(\frac{\alpha b_n+nz}{n+\beta}\right)T_{n,k}^{(\alpha,\beta;b_n)}(z) \tag{5}$$

is satisfied.

Proof. Considering the equality (4) and applying the mean value theorem for divided difference for $f(z)=e_k(z)=z^k$, all $|z|\le r$ we have

$$\left|S_n^{(\alpha,\beta;b_n)}(e_k)(z)\right|\left|T_{n,k}^{(\alpha,\beta;b_n)}(z)\right|\le$$

$$\sum_{j=0}^{k}\left(\frac{n}{b_n\left(n+\beta\right)}\right)^{j}\left|\left[\frac{\alpha b_n}{n+\beta},\frac{(\alpha+1)b_n}{n+\beta},...,\frac{(\alpha+j)b_n}{n+\beta};e_k\right]\right|r^j$$

$$\sum_{j=0}^{k}\left(\frac{n}{b_n\left(n+\beta\right)}\right)^{j}\frac{k\left(k-1\right)...\left(k-j+1\right)}{j!}r^{k-j}r^j$$

$$\le r^k\sum_{j=0}^{k}\left(\frac{n}{b_n\left(n+\beta\right)}\right)^{j}\binom{k}{j}\le r^k\sum_{j=0}^{k}\frac{1}{b_n^j}\binom{k}{j}\le(2r)^k. \qquad (6)$$

Differentiating following the formula with respect to $z\neq 0$, we get

$$\frac{d}{dz}\left[e^{-nz/b_n}\sum_{j=0}^{\infty}\left((j+\alpha)b_n\right)^k\frac{(nz)^j}{j!b_n^j}\right]$$

$$=\sum_{j=0}^{\infty}\left((j+\alpha)b_n\right)^k\left[-\frac{n}{b_n}e^{-nz/b_n}\frac{(nz)^j}{j!b_n^j}+e^{-nz/b_n}jn\frac{(nz)^{j-1}}{j!b_n^j}\right],$$

and then dividing the formula by $(n+\beta)^{k+1}$ and by simple calculations, we get

$$\frac{b_n z}{n+\beta}\left(T_{n,k}^{(\alpha,\beta;b_n)}(z)\right)'=T_{n,k+1}^{(\alpha,\beta;b_n)}(z)-\left(\frac{\alpha b_n+nz}{n+\beta}\right)T_{n,k}^{(\alpha,\beta;b_n)}(z).$$

Hence we reach the recurrence formula given by (5).

In order to prove the main results of this section, we give the recurrence formula in the special case $\alpha=0=\beta$ for the operator (1). For this purpose, let $S_{n,b_n}(e_k)(z)=T_{n,b_n,k}(z)$ with $(e_k)(z)=z^k$. It is clear that $T_{n,b_n,k}(z)$ is a polynomial of degree $\le k,\ k=0,1,2,...,$ and $T_{n,b_n,0}(z)=1,T_{n,b_n,1}(z)=z$, for all $z\in\mathbb{C}$. Taking $\alpha=0=\beta$ in (5),we get

$$T_{n,b_n,k+1}(z)=\frac{b_n}{n}zT_{n,b_n,k}'(z)+zT_{n,b_n,k}(z)$$

for all $z\in\mathbb{C},\ k=0,1,2,...,\ n\in\mathbb{N}$. From this we can write the following formula

$$T_{n,b_n,k}(z)-z^k=\frac{b_n}{n}z\left[T_{n,b_n,k-1}(z)-z^{k-1}\right]'+z\left[T_{n,b_n,k-1}(z)-z^{k-1}\right]+\frac{b_n\left(k-1\right)z^{k-1}}{n} \qquad (7)$$

for all $z\in\mathbb{C},\ k,n\in\mathbb{N}$. ◄

Theorem 1. *Let $D_R=\{z\in\mathbb{C}:|z|<R,2<R<+\infty\}$ and $f:[R,+\infty)\cup D_R\to\mathbb{C}$ be bounded on $[0,+\infty)$ and analytic in D_R, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in D_R$. If $1<r<R/2$ is arbitrarily fixed number, then for all $|z|\le r$ and $n\in\mathbb{N}$*

$$\left|S_{n,b_n}(f)(z)-f\left(z\right)\right|\le\frac{b_n}{n}M_r\left(f\right),$$

where $M_r(f) = \sum_{k=2}^{\infty} 3|c_k|(k-1)(2r)^{k-1} < \infty$.

Proof. By using the equality (7), we can write

$$\left|T_{n,b_n,k}(z) - z^k\right| \leq \frac{b_n}{n} r \left|\left[T_{n,b_n,k-1}(z) - z^{k-1}\right]'\right| +$$

$$+r\left|T_{n,b_n,k-1}(z) - z^{k-1}\right| + \frac{b_n(k-1)r^{k-1}}{n}.$$

From the Bernstein's inequality for the polynomial $T_{n,b_n,k-1}(z)$ of degree $\leq k-1$, we get

$$\begin{aligned}\left|\left[T_{n,b_n,k-1}(z) - z^{k-1}\right]'\right| &\leq \frac{k-1}{r} \max\left\{\left|T_{n,b_n,k-1}(z) - z^{k-1}\right| : |z| \leq r\right\} \\ &= \frac{k-1}{r} \left\|T_{n,b_n,k-1} - e_{k-1}\right\|_r.\end{aligned}$$

Therefore, it follows

$$\begin{aligned}\left|T_{n,b_n,k}(z) - z^k\right| &\leq \frac{b_n}{n}(k-1)\left(\left\|T_{n,b_n,k-1}\right\|_r + \left\|e_{k-1}\right\|_r\right) \\ &\quad +r\left|T_{n,b_n,k-1}(z) - z^{k-1}\right| + \frac{b_n}{n}(k-1)r^{k-1} \\ &\leq 3(k-1)\frac{b_n}{n}(2r)^{k-1} + r\left|T_{n,b_n,k-1}(z) - z^{k-1}\right|.\end{aligned}$$

For $k=2$ and $k=3$, respectively, we get

$$\left|T_{n,b_n,2}(z) - z^2\right| \leq \frac{b_n}{n} 3r2^1 \text{ and } \left|T_{n,b_n,3}(z) - z^3\right| \leq \frac{b_n}{n} 3r^2\left(1.2^1 + 2.2^2\right).$$

Then for any $k \geq 2$, we finally get

$$\begin{aligned}\left|T_{n,b_n,k}(z) - z^k\right| &\leq \frac{b_n}{n} 3r^{k-1} \sum_{j=1}^{k-1} j2^j \leq \frac{b_n}{n} 3r^{k-1}(k-1)\sum_{j=0}^{k-2} 2^j \\ &\leq \frac{b_n}{n} 3r^{k-1}(k-1)\left(2^{k-1} - 1\right) \\ &\leq 3\frac{b_n}{n}(2r)^{k-1} k(k-1). \qquad (8)\end{aligned}$$

Now, as in the case of classical complex Szasz-Mirakjan operators (see [5], p. 115-116), we can write $S_{n,b_n}(f)(z) = \sum_{k=0}^{\infty} c_k T_{n,b_n,k}(z)$ which implies

$$\begin{aligned}\left|S_{n,b_n}(f)(z) - f(z)\right| &\leq \sum_{k=0}^{\infty} |c_k| \left|T_{n,b_n,k}(z) - z^k\right| \\ &\leq \frac{b_n}{n} \sum_{k=2}^{\infty} 3|c_k| k(k-1)(2r)^{k-1}.\end{aligned}$$

Note that $\sum_{k=2}^{\infty} 3\left|c_k\right| k(k-1)(2r)^{k-1} < \infty$ because of analyticity of f.

The following Voronovskaja type result will be used in the proof of Theorem 4.◀

Theorem 2. *Suppose that the hypothesis is same on f and R in the statement of Theorem 1. If $1 \leq r < R/2$ is arbitrarily fixed, then for all $|z| \leq r$ and $n \in \mathbb{N}$, we have*

$$\left|S_{n,b_n}(f)(z) - f(z) - \frac{b_n}{2n} z f''(z)\right| \leq \left(\frac{b_n}{n}\right)^2 28r \sum_{k=2}^{\infty} |c_k| (k-1)^2 (k-2) (2r)^{k-3} < \infty.$$

Proof. By the recurrence relationship in (7), denoting

$$E_{n,k}(z) = S_{n,b_n}(e_k)(z) - e_k(z) - \frac{b_n}{2n} k(k-1) z^{k-1}, \tag{9}$$

we obtain

$$E_{n,k}(z) = z\frac{b_n}{n} E'_{n,k-1}(z) + zE_{n,k-1}(z) + \frac{1}{2}\left(\frac{b_n}{n}\right)^2 z^{k-2}(k-1)(k-2)^2.$$

This implies, for all $|z| \leq r, n \in \mathbb{N}, k \geq 2$:

$$|E_{n,k}(z)| \leq |z| \frac{b_n}{2n}\left[2\left|E'_{n,k-1}(z)\right| + \frac{b_n}{n}|z|^{k-3}(k-1)(k-2)^2\right] + r\|E_{n,k-1}\|_r.$$

From the Bernstein inequality, using (8) and (9), we have

$$\begin{aligned} \left|E'_{n,k-1}(z)\right| &\leq \frac{k-1}{r}\|E_{n,k-1}\|_r \\ &\leq \frac{k-1}{r}\|T_{n,k-1} - e_{k-1}\|_r + \frac{b_n}{2rn}(k-1)^2(k-2) r^{k-2} \\ &\leq 6(k-1)^2(k-2)\frac{b_n}{n}(2r)^{k-3} + \frac{b_n}{n}(k-1)^2(k-2)(2r)^{k-3}. \end{aligned}$$

Hence

$$\begin{aligned} |E_{n,k}(z)| &\leq r\|E_{n,k-1}\|_r + |z|\frac{b_n}{n}\left[12(k-1)^2(k-2)\frac{b_n}{n}(2r)^{k-3}\right. \\ &\quad \left. + \frac{b_n}{n}(k-1)^2(k-2)(2r)^{k-3} + \frac{b_n}{n} r^{k-3}(k-1)(k-2)^2\right] \\ &= r\|E_{n,k-1}\|_r + 14|z|\left(\frac{b_n}{n}\right)^2(k-1)^2(k-2)(2r)^{k-3}. \end{aligned}$$

Since for $k = 1, 2$ we get $E_{n,k}(z) = 0$, for $k \geq 3$ in the latter relation, by similar calculations as in [6] page 117, we obtain that

$$E_{n,k}(z) \leq 28|z|\left(\frac{b_n}{n}\right)^2(k-1)^2(k-2)(2r)^{k-3}.$$

Since $\left|S_{n,b_n}(f)(z) - f(z) - \frac{b_n}{2n} z f''(z)\right| \leq \sum_{k=0}^{\infty} |c_k| |E_{n,k}(z)|$, by the above inequality we reach the desired result.

3. Results of approximation by the operators $S_n^{(\alpha,\beta;b_n)}(f)$

First we prove an upper estimate in the simultaneous approximation by $S_n^{(\alpha,\beta;b_n)}(f)$.
◄

Theorem 3. *Suppose that the hypotheses on the function f and the constant R in the statement of Theorem 1 hold.*

(a) If $1 \le r < R/2$ is arbitrarily fixed and $0 \le \alpha \le \beta$, then for all $|z| \le r$ and $n \in \mathbb{N}$ we have

$$\left|S_n^{(\alpha,\beta;b_n)}(f)(z) - f(z)\right| \le \frac{(\alpha+\beta r)\, b_n}{n+\beta} \sum_{k=1}^{\infty} |c_k|\, k(2r)^{k-1} + \frac{b_n r}{n+\beta} \sum_{k=2}^{\infty} |c_k|\, k(k-1)(2r)^{k-2}$$

$$= \frac{(\alpha+\beta r)\, b_n}{n+\beta} M_{1,r}(f) + \frac{b_n r}{n+\beta} M_{2,r}(f),$$

where $c_k, k \in \mathbb{N}$ is Taylor coefficient of f and the series $M_{1,r}(f)$, $M_{2,r}(f)$ are convergent in $|z| \le r$.

(b) Suppose that $1 \le r < r_1 < R/2$ and $0 \le \alpha \le \beta$. Then for all $|z| \le r$ and $p, n \in \mathbb{N}$ we have

$$\left|\left[S_n^{(\alpha,\beta;b_n)}(f)(z)\right]^{(p)} - f^{(p)}(z)\right| \le \frac{p!r_1}{(r_1-r)^{p+1}} \frac{b_n}{n+\beta} \left[M_{1,r_1}^{\alpha,\beta}(f) + M_{2,r_1}(f)\right],$$

where $M_{1,r_1}^{\alpha,\beta}(f) = 2\alpha \sum_{k=1}^{\infty} |c_k|\, k(2r_1)^{k-1} + 2\beta r_1 \sum_{k=1}^{\infty} |c_k|\, (2r_1)^{k-1} < \infty$, $M_{2,r_1}(f) = r_1 \sum_{k=2}^{\infty} |c_k|\, k(k-1)(2r_1)^{k-2} < \infty$.

Proof. (a) Clearly, from (5), we can write

$$T_{n,k}^{(\alpha,\beta;b_n)}(z) - z^k = \frac{b_n z}{n+\beta}\left(T_{n,k-1}^{(\alpha,\beta;b_n)}(z)\right)' + \frac{\alpha b_n + nz}{n+\beta} T_{n,k-1}^{(\alpha,\beta)}(z) - z^k$$

$$= \frac{b_n z}{n+\beta}\left(T_{n,k-1}^{(\alpha,\beta;b_n)}(z)\right)' + \frac{\alpha b_n + nz}{n+\beta}\left[T_{n,k-1}^{(\alpha,\beta;b_n)}(z) - z^{k-1}\right] + \frac{\alpha b_n + \beta z}{n+\beta} z^{k-1},$$

for all $z \in \mathbb{C}$, $k, n \in \mathbb{N}$.

From the above equality and using the Bernstein inequality, by (6), we have

$$\left|T_{n,k}^{(\alpha,\beta;b_n)}(z) - z^k\right| \le \frac{b_n r}{n+\beta}\left\|\left(T_{n,k-1}^{(\alpha,\beta;b_n)}(z)\right)\right\|_r \frac{k-1}{r} + \left(\frac{\alpha b_n}{n+\beta} + r\right) \left|T_{n,k-1}^{(\alpha,\beta;b_n)} - z^{k-1}\right| + \frac{(\alpha+\beta r)\, b_n}{n+\beta} r^{k-1}$$

$$\le \frac{b_n(k-1)}{n+\beta}(2r)^{k-1} + \left(\frac{\alpha b_n}{n+\beta} + r\right)\left|T_{n,k-1}^{(\alpha,\beta;b_n)}(z) - z^{k-1}\right|$$

$$+\frac{(\alpha+\beta r)\,b_n}{n+\beta}r^{k-1}. \tag{10}$$

Hence we get

$$\left|T_{n,k}^{(\alpha,\beta;b_n)}(z)-z^k\right|\leq 2r\left|T_{n,k-1}^{(\alpha,\beta;b_n)}(z)-z^{k-1}\right|$$
$$+\frac{b_n(k-1)}{n+\beta}(2r)^{k-1}+\frac{(\alpha+\beta r)\,b_n}{n+\beta}r^{k-1}.$$

Clearly $T_{n,0}^{(\alpha,\beta;b_n)}-e_0=0$ and

$$\left|T_{n,1}^{(\alpha,\beta;b_n)}-e_1\right|\leq\frac{(\alpha+\beta r)\,b_n}{n+\beta}.$$

Taking $k=2$ in (11), we obtain

$$\left|T_{n,2}^{(\alpha,\beta;b_n)}-e_2\right|\leq 2r\frac{(\alpha+\beta r)\,b_n}{n+\beta}+\frac{b_n}{n+\beta}1.(2r)+\frac{(\alpha+\beta r)\,b_n}{n+\beta}r.$$

By similar calculations we find

$$\left|T_{n,3}^{(\alpha,\beta;b_n)}-e_3\right|\leq(2r)^2\frac{(\alpha+\beta r)\,b_n}{n+\beta}+\frac{b_n}{n+\beta}\left[1.(2r)^2+2.(2r)^2\right]+2^2\frac{(\alpha+\beta r)\,b_n}{n+\beta}r^2,$$

and hence for any $k\geq 2$ we finally get

$$\begin{aligned}\left|T_{n,k}^{(\alpha,\beta;b_n)}(z)-z^k\right| &\leq (2r)^{k-1}\frac{(\alpha+\beta r)\,b_n}{n+\beta}+(2r)^{k-1}\frac{b_n}{n+\beta}\frac{k(k-1)}{2}+\\ 2^{k-1}\frac{(\alpha+\beta r)\,b_n}{n+\beta}r^{k-1} &\leq 2.(2r)^{k-1}\frac{(\alpha+\beta r)\,b_n}{n+\beta}+(2r)^{k-2}k(k-1)\frac{b_nr}{n+\beta}\\ &\leq k(2r)^{k-1}\frac{(\alpha+\beta r)\,b_n}{n+\beta}+(2r)^{k-2}k(k-1)\frac{b_nr}{n+\beta}. \end{aligned} \tag{11}$$

It is known that, for the complex Szasz operators

$$S_n(f)(z)=\sum_{k=0}^{\infty}c_kS_n(e_k)(z)=\sum_{k=0}^{\infty}c_kT_n(z)\text{ all }|z|\leq r,$$

(see [5], page 116-117). Therefore using the same idea we can write

$$S_n^{(\alpha,\beta;b_n)}(f)(z)=\sum_{k=0}^{\infty}c_kS_n^{(\alpha,\beta;b_n)}(e_k)(z)=\sum_{k=0}^{\infty}c_kT_n^{(\alpha,\beta;b_n)}(z).$$

Hence by (11) we obtain

$$\left|S_n^{(\alpha,\beta;b_n)}(f)(z)-f(z)\right| \leq \sum_{k=0}^{\infty}c_k\left|T_n^{(\alpha,\beta;b_n)}(z)-z^k\right|$$

$$\leq \quad \frac{(\alpha+\beta r)\, b_n}{n+\beta} M_{1,r}(f) + \frac{b_n r}{n+\beta} M_{2,r}(f).$$

Since by hypothesis, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is absolutely and uniform convergent in $|z| \leq r$, for any $1 \leq r < R/2$, it is clear that $M_{1,r}(f) = \sum_{k=1}^{\infty} |c_k|\, k(2r)^{k-1} < \infty$, and $M_{2,r}(f) = \sum_{k=2}^{\infty} |c_k|\, k(k-1)(2r)^{k-2} < \infty$.

(b) Denoting by γ the circle of radius $r_1 > r$ centered at 0, for any $|z| \leq r$ and $\nu \in \gamma$ we get $|\nu - z| \geq r_{1-}r$, and, by Cauchy's formula, for all $|z| \leq r$ it follows

$$\left| \left[S_n^{(\alpha,\beta;b_n)}(f)(z) \right]^{(p)} - f^{\ (p)}(z) \right| = \frac{p!}{2\pi} \left| \int_\gamma \frac{S_n^{(\alpha,\beta;b_n)}(f)(z) - f(z)}{(\nu - z)^{p+1}} dz \right|$$

$$\leq \frac{p! r_1}{(r_1 - r)^{p+1}} \left[\frac{(\alpha+\beta r_1)\, b_n}{n+\beta} \sum_{k=1}^{\infty} |c_k|\, k(2r_1)^{k-1} + \frac{b_n r_1}{n+\beta} \sum_{k=2}^{\infty} |c_k|\, k(k-1)(2r_1)^{k-2} \right],$$

which proves (b) and the theorem.

We present the following Voronovskaja's formula for $S_n^{(\alpha,\beta;b_n)}(f)$. ◄

Theorem 4. *Let the hypotheses on f and R in the statement of Theorem 1 hold. Then, for all $|z| \leq r$ with $1 \leq r < R/2$ and $n \in \mathbb{N}$, the following Voronovskaja type result holds:*

$$\left| S_n^{(\alpha,\beta;b_n)}(f)(z) - f(z) - \frac{\alpha b_n - \beta z}{n+\beta} f'(z) - \frac{b_n z}{2n} f''(z) \right|$$

$$\leq \left(\frac{b_n}{n} \right)^2 C_{1,r}(f) + \left(\frac{b_n}{n+\beta} \right)^2 \sum_{j=2}^{6} C_{j,r}(f),$$

where$C_{1,r}(f) = 28r \sum_{k=3}^{\infty} |c_k|\, (k-1)^2\, (k-2)\, (2r)^{k-3} < \infty$,
$C_{2,r}(f) = \frac{\alpha^2}{2} \sum_{k=2}^{\infty} |c_k|\, k(k-1)(2r)^{k-2} < \infty$,
$C_{3,r}(f) = \alpha r \sum_{k=2}^{\infty} |c_k|\, k(k-1)(k-2)(2r)^{k-3} < \infty$,
$C_{4,r}(f) = \left(\beta^2 r + \beta\right) \sum_{k=2}^{\infty} |c_k|\, k^2(k-1)(2r)^{k-2} < \infty$,
$C_{5,r}(f) = (\beta r)^2 \sum_{k=2}^{\infty} |c_k|\, k(k-1) r^{k-2} < \infty$,
$C_{6,r}(f) = \alpha\beta r \sum_{k=2}^{\infty} |c_k|\, k(k-1) r^{k-2} < \infty$.

Proof. For all $z \in D_R$, let us consider

$$\begin{aligned} & S_n^{(\alpha,\beta;b_n)}(f)(z) - f(z) - \frac{\alpha b_n - \beta z}{n+\beta} f'(z) - \frac{z b_n}{2n} f''(z) \\ = \quad & S_{n,b_n}(f)(z) - f(z) - \frac{z b_n}{2n} f''(z) + S_n^{(\alpha,\beta;b_n)}(f)(z) - S_{n,b_n}(f)(z) - \frac{\alpha b_n - \beta z}{n+\beta} f'(z). \end{aligned}$$

Since $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we get

$$S_n^{(\alpha,\beta;b_n)}(f)(z) - f(z) - \frac{\alpha b_n - \beta z}{n+\beta} f'(z) - \frac{z}{2n} f''(z)$$

$$= \sum_{k=2}^{\infty} c_k \left(S_{n,b_n}(e_k)(z) - z^k - \frac{zb_n}{2n} k(k-1) z^{k-2} \right)$$
$$+ \sum_{k=2}^{\infty} c_k \left(S_n^{(\alpha,\beta;b_n)}(e_k)(z) - S_n(e_k)(z) - \frac{\alpha b_n - \beta z}{n+\beta} k z^{k-1} \right).$$

First we use the Voronoskaja type result for the $S_{n,b_n}(f)(z)$ operators obtained in Theorem 2. Then the first sum is

$$\left| S_{n,b_n}(f)(z) - f(z) - \frac{zb_n}{2n} f''(z) \right| \leq 28r \left(\frac{b_n}{n} \right)^2 \sum_{k=3}^{\infty} |c_k| (k-1)^2 (k-2) (2r)^{k-2}.$$

We now estimate the second sum. Since $S_n^{(\alpha,\beta;b_n)}(e_k)(z) = T_{n,k}^{(\alpha,\beta;b_n)}(z)$ and considering that $S_n^{(\alpha,\beta;b_n)}(e_k)(z) = \sum_{j=0}^{k} \binom{k}{j} \frac{n^j (\alpha b_n)^{k-j}}{(n+\beta)^k} S_{n,b_n}(e_j)(z)$ with $S_n^{(0,0,b_n)}(e_k)(z) = S_{n,b_n}(e_j)(z)$ for $k, n \in \mathbb{N} \cup \{0\}$ we can write

$$S_n^{(\alpha,\beta;b_n)}(e_k)(z) - S_{n,b_n}(e_k)(z) - \frac{\alpha b_n - \beta z}{n+\beta} k z^{k-1}$$
$$= T_{n,k}^{(\alpha,\beta;b_n)}(z) - T_{n,b_n,k}(z) - \frac{\alpha b_n - \beta z}{n+\beta} k z^{k-1}$$

$$= \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j (\alpha b_n)^{k-j}}{(n+\beta)^k} T_{n,b_n,j}(z) + \left(\frac{n^k}{(n+\beta)^k} - 1 \right) T_{n,b_n,k}(z) - \frac{\alpha b_n - \beta z}{n+\beta} k z^{k-1}$$

$$= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j (\alpha b_n)^{k-j}}{(n+\beta)^k} T_{n,b_n,j}(z) + \frac{k n^{k-1} \alpha b_n}{(n+\beta)^k} T_{n,b_n,k-1}(z) -$$
$$- \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} T_{n,b_n,k}(z) - \frac{\alpha b_n - \beta z}{n+\beta} k z^{k-1}$$

$$= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j (\alpha b_n)^{k-j}}{(n+\beta)^k} T_{n,b_n,j}(z) + \frac{k n^{k-1} \alpha b_n}{(n+\beta)^k} \left[T_{n,b_n,k-1}(z) - z^{k-1} \right] -$$
$$- \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} T_{n,b_n,k}(z)$$
$$- \frac{k n^{k-1} \beta}{(n+\beta)^k} \left[T_{n,b_n,k}(z) - z^k \right] + \frac{k \alpha b_n}{(n+\beta)} \left(\frac{n^{k-1}}{(n+\beta)^{k-1}} - 1 \right) z^{k-1} +$$
$$+ \frac{k\beta}{(n+\beta)} \left(1 - \frac{n^{k-1}}{(n+\beta)^{k-1}} \right) z^k.$$

Considering the condition (2), we get

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j(\alpha b_n)^{k-2-j}}{(n+\beta)^{k-2}} = \sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j}{(n+\beta)^j}\frac{(\alpha b_n)^{k-2-j}}{(n+\beta)^{k-2-j}} = \left(\frac{n+\alpha b_n}{n+\beta}\right)^{k-2}\leq 2.$$

We also have the following inequalities:

$$1-\frac{n^k}{(n+\beta)^k}=1-\prod_{j=0}^{k}\frac{n}{n+\beta}\leq\sum_{j=1}^{k}\left(1-\frac{n}{n+\beta}\right)=\frac{k\beta}{n+\beta},$$

and from(6) $|T_{n,b_n,k}(z)|\leq(2r)^k$. Therefore we obtain

$$\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j(\alpha b_n)^{k-j}}{(n+\beta)^k}T_{n,b_n,j}(z)\right| \leq \sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j(\alpha b_n)^{k-j}}{(n+\beta)^k}\left|T_{n,b_n,j}(z)\right|$$

$$\leq\sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-j-1)}\binom{k-2}{j}\frac{n^j(\alpha b_n)^{k-j}}{(n+\beta)^k}\left|T_{n,b_n,j}(z)\right|$$

$$\leq k(k-1)\frac{(\alpha b_n)^2}{(n+\beta)^2}(2r)^{k-2}.$$

Consequently, by using (8):

$$\left|T_{n,k}^{(\alpha,\beta;b_n)}(z)-T_{n,b_n,k}(z)-\frac{\alpha b_n-\beta z}{n+\beta}kz^{k-1}\right|$$

$$\leq\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j(\alpha b_n)^{k-j}}{(n+\beta)^k}T_{n,b_n,j}(z)\right|+\frac{kn^{k-1}\alpha b_n}{(n+\beta)^k}\left|T_{n,b_n,k-1}(z)-z^{k-1}\right|+$$

$$\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}\left|T_{n,b_n,k}(z)\right|+\frac{kn^{k-1}\beta}{(n+\beta)^k}\left|T_{n,b_n,k}(z)-z^k\right|+$$

$$\frac{k\alpha b_n}{(n+\beta)}\left|\frac{n^{k-1}}{(n+\beta)^{k-1}}-1\right||z|^{k-1}+\frac{k\beta}{(n+\beta)}\left|1-\frac{n^{k-1}}{(n+\beta)^{k-1}}\right||z|^k$$

$$\leq k(k-1)\frac{(\alpha b_n)^2}{(n+\beta)^2}(2r)^{k-2}+\frac{kn^{k-1}\alpha b_n}{(n+\beta)^k}(2r)^{k-3}(k-1)(k-2)\frac{b_nr}{n}+(2r)^k\frac{k(k-1)\beta^2}{(n+\beta)^2}$$

$$+\frac{kn^{k-1}\beta}{(n+\beta)^k}(2r)^{k-2}k(k-1)\frac{rb_n}{n}+\frac{k(k-1)\alpha\beta b_n}{(n+\beta)^2}r^{k-1}+\frac{k(k-1)\beta^2}{(n+\beta)^2}r^k$$

$$\leq \alpha^2\left(\frac{b_n}{n+\beta}\right)^2 k\,(k-1)\,(2r)^{k-2}+\alpha r\left(\frac{b_n}{n+\beta}\right)^2 k(k-1)(k-2)\,(2r)^{k-3}$$
$$+\left(\frac{b_n}{n+\beta}\right)^2\left(r\beta^2+\beta\right)k^2(k-1)\,(2r)^{k-1}$$
$$+\alpha\beta\left(\frac{b_n}{n+\beta}\right)^2 k(k-1)r^{k-1}+\beta^2\left(\frac{b_n}{n+\beta}\right)^2 k(k-1)r^k,$$

which proves the theorem.◀

The following result will be useful to obtain the exact order of approximation by $S_n^{(\alpha,\beta;b_n)}(f)$.

Theorem 5. *Suppose that the hypotheses are same on f and R and also f is not a polynomial of degree ≤ 0. Then, for all $|z|\leq r$ and $n\in\mathbb{N}$, we have*

$$\left|S_n^{(\alpha,\beta;b_n)}(f)(z)-f(z)\right|\geq C_r(f)\frac{b_n}{n},$$

where the constant $C_r(f)$ depends only on f and r.

Proof. For all $|z|\leq r$ and $n\in\mathbb{N}$, we can write

$$S_n^{(\alpha,\beta;b_n)}(f)(z)-f(z)=\frac{b_n}{n}\left\{\frac{n}{b_n}\frac{(\alpha b_n-\beta z)}{n+\beta}f^{'}(z)+\frac{z}{2}f^{''}(z)+\frac{b_n}{n}\left(\frac{n}{b_n}\right)^2\right.$$
$$\left.\left(S_n^{(\alpha,\beta;b_n)}(f)(z)-f(z)-\frac{\alpha b_n-\beta z}{n+\beta}f^{'}(z)-\frac{zb_n}{2n}f^{''}(z)-\frac{b_n}{n}\frac{\beta\,(\alpha b_n-\beta z)\,f^{'}(z)}{(n+\beta)}\right)\right\}.$$

Applying the inequality $\|F+G\|\geq|\|F\|-\|G\||\geq\|F\|-\|G\|$ we obtain

$$\left\|S_n^{(\alpha,\beta;b_n)}(f)-f\right\|_r\geq\frac{b_n}{n}\left[(\alpha b_n-\beta e_1)\,f^{'}+\frac{e_1}{2}f^{''}\right.$$
$$\left.-\frac{b_n}{n}\left(\frac{n}{b_n}\right)^2\left\|S_n^{(\alpha,\beta;b_n)}(f)-f-\frac{\alpha b_n-\beta e_1}{n+\beta}f^{'}-\frac{b_n}{2n}e_1f^{''}-\frac{b_n}{n}\frac{\beta\,(\alpha b_n-\beta e_1)}{(n+\beta)}f^{'}\right\|_r\right].$$

Since f is not a polynomial of degree ≤ 0 in D_R, we get

$$\left\|(\alpha b_n-\beta e_1)\,f^{'}+\frac{e_1}{2}f^{''}\right\|_r\geq\left\|(\alpha b_1-\beta e_1)\,f^{'}+\frac{e_1}{2}f^{''}\right\|_r>0.$$

Indeed, supposing the contrary it follows that $(\alpha b_1-\beta z)\,f^{'}(z)+\frac{z}{2}f^{''}(z)=0$ for all $z\in\overline{D_R}$. Denoting $y(z)=f\,(z)$, seeking $y\,(z)$ in the form $y\,(z)=\sum_{k=0}^{\infty}\gamma_k z^k$ and replacing in the above differential equation, we easily get $\gamma_k=0$ for all $k=0,1,...$ (for similar reasoning, see [4] or [5] p.75-76). Thus we get that $f(z)$ is a constant function, which is a contradiction. Now, since by Theorem 4 it follows

$$\left(\frac{n}{b_n}\right)^2\left\|S_n^{(\alpha,\beta;b_n)}(f)-f-\frac{\alpha b_n-\beta e_1}{n+\beta}f^{'}-\frac{b_n}{2n}e_1f^{''}-\frac{b_n}{n}\frac{\beta\,(\alpha b_n-\beta e_1)}{(n+\beta)}f^{'}\right\|_r$$

$$\leq \left(\frac{n}{b_n}\right)^2 \left\| S_n^{(\alpha,\beta;b_n)}(f) - f - \frac{\alpha b_n - \beta e_1}{n+\beta} f' - \frac{b_n}{2n} e_1 f'' \right\|_r + \left\| \beta \left(\alpha b_n - \beta e_1\right) f' \right\|_r$$
$$\leq \sum_{j=1}^{6} C_{j,r}(f) + \beta\left(\alpha b_n - \beta r\right) \left\| f' \right\|_r ,$$

there exists $n_1 > n_0$ (depending on f, α, β and r only) such that for all $n > n_1$ we have

$$\left\| \left(\alpha b_n - \beta e_1\right) f' + \frac{e_1}{2} f'' \right\|_r - \frac{b_n}{n} \left(\frac{n}{b_n}\right)^2 \left\| S_n^{(\alpha,\beta;b_n)}(f) - f - \frac{\alpha b_n - \beta e_1}{n+\beta} f' - \frac{b_n}{2n} e_1 f'' \right\|_r$$
$$\geq \frac{1}{2} \left\| \left(\alpha b_n - \beta e_1\right) f' + \frac{e_1}{2} f'' \right\|_r ,$$

which implies

$$\left\| S_n^{(\alpha,\beta;b_n)}(f) - f \right\|_r \geq \frac{b_n}{2n} \left\| \left(\alpha b_n - \beta e_1\right) f' + \frac{e_1}{2} f'' \right\|_r ,$$

for all $n > n_1$. For $n \in \{n_0 + 1, ..., n_1\}$, we get $\left\| S_n^{(\alpha,\beta;b_n)}(f) - f \right\|_r \geq \frac{b_n}{n} A_r(f)$ with $A_r(f) = \frac{n}{b_n} \left\| S_n^{(\alpha,\beta;b_n)}(f) - f \right\|_r$ which implies $\left\| S_n^{(\alpha,\beta;b_n)}(f) - f \right\|_r \geq C_{r,f} \frac{b_n}{n}$ for all $n > n_0$, with

$$C_{r,f} = \min\left\{ A_{r,n_0+1}(f), ..., A_{r,n_1}(f), \frac{1}{2} \left\| \left(\alpha b_n - \beta e_1\right) f' + \frac{e_1}{2} f'' \right\|_r \right\},$$

which proves the theorem.◄

Now, we are ready to state the exact order of approximation for $S_n^{(\alpha,\beta;b_n)}(f)$.

4. Conclusions

1. By Theorems 5 and 3 (a), it easily follows that if f is not a constant function, then the exact order in the approximation by the operator $S_n^{(\alpha,\beta;b_n)}(f)$ is b_n/n.

2. Considering Theorem 3 (b) and by similar calculations it is seen that the exact order in the simultaneous approximation by the operator $\left(S_n^{(\alpha,\beta;b_n)}(f)\right)^{(p)}$ is b_n/n (see also [5] p. 119, for the case of classical complex Szasz-Mirakjan operators).

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Nurhayat Ispir
Gazi University,Department of Mathematics,Faculty of Science 06500, Teknikokullar, Ankara, Turkey
E-mail: nispir@gazi.edu.tr