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Local Generalized Morrey Spaces and Singular Integrals with Rough Kernel

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Abstract. Let $\Omega \in L_s(S^{n-1})$ be a homogeneous function of degree zero with $s > 1$ and have a mean value zero on S^{n-1}. In this paper, we will study the boundedness of homogeneous singular integrals with rough kernel on the local generalized Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$ for $s' \le p$ or $p < s$. We will also prove that the commutator operators formed by a local BMO function b and these rough operators are bounded on the local generalized Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$.

Key Words and Phrases: Homogeneous singular integrals; rough kernels; local generalized Morrey space; commutator; CBMO space

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball of radius r centered at x , ${}^{\complement}B(x,r)$ denote its complement and $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \ge 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^s(S^{n-1})$ be a homogeneous function of degree zero with $1 < s \le \infty$ and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The homogeneous singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy.$$

It is obvious that when $\Omega \equiv 1$, T_Ω is the singular integral operator T.

Theorem A ([13]) *Suppose that, $1 \le p < \infty$, $\Omega \in L_s(S^{n-1})$, $s > 1$, is a homogeneous function of degree zero and has a mean value zero on S^{n-1}. If $s' \le p$ or $p < s$, then the operator T_Ω is bounded on $L_p(\mathbb{R}^n)$. Also, the operator T_Ω is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

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Let b be a locally integrable function on $\mathbb{R}^n$. Then we shall define the commutators generated by singular integral operators with rough kernels and b as follows:

$$[b, T_\Omega]f(x) \equiv b(x)T_\Omega f_1(x) - T_\Omega(bf)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}[b(x)-b(y)]f(y)dy.$$

Theorem B ([13]) *Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is a homogeneous function of degree zero and has a mean value zero on S^{n-1}. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. If $s' \le p$ or $p < s$, then the commutator operator $[b, T_\Omega]$ is bounded on $L_p(\mathbb{R}^n)$.*

The classical Morrey spaces $M_{p,\lambda}$ were first introduced by Morrey in [27] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [1, 9, 29]. For the properties and applications of classical Morrey spaces see [10, 11, 15, 16] and references therein.

In this paper, we prove the boundedness of the operators T_Ω from one local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$, $1 < p < \infty$, and from the space $LM^{\{x_0\}}_{1,\varphi_1}$ to the weak space $WLM^{\{x_0\}}_{1,\varphi_2}$. In the case $b \in CBMO^{\{x_0\}}_{p_2}$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the commutator operators $[b, T_\Omega]$ from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$, $1 < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Local generalized Morrey spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^{\rm loc}_p(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, |B(x,r)|^{-\frac{1}{p}}\, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^{\rm loc}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, |B(x,r)|^{-\frac{1}{p}}\, \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the local generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(0,r))} < \infty.$$

Definition 3. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the local generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi}} < \infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WLM_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Wiener [30, 31] looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L_q spaces. Beurling [4] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q+1/q' = 1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L_q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [17] observed that the space B_q can be described by

$$\|f\|_{B_q} = \sup_{k\ge 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}, \tag{1}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$, $k = 1, 2, \ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called Beurling algebra now, can be described by

$$\|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-\frac{kn}{q'}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}. \tag{2}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (1) and (2) instead of $k \geq 0$ there.

If $\lambda < 0$ or $\lambda > n$, then $LM^{\{x_0\}}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$.

$$\dot{B}_{p,\mu} = LM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}}, \qquad W\dot{B}_{p,\mu} = WLM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}}.$$

Alvarez, Guzman-Partida and Lakey [3], in order to study the relationship between central BMO spaces and Morrey spaces, introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+np\mu}(\mathbb{R}^n)$, $\mu \in [-\frac{1}{p}, 0]$. If $\mu < -\frac{1}{p}$ or $\mu > 0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$. Note that $\dot{B}_{p,-\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$. Also define the weak central Morrey spaces $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

Inspired by this, we consider the boundedness of singular integral operator with rough kernel on generalized local Morrey spaces and give the central bounded mean oscillation estimates for their commutators.

3. Singular integral operators with rough kernels in the spaces $LM^{\{x_0\}}_{p,\varphi}$

In this section we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0, \infty)$.

Theorem 1. *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0, \infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H^*_w g(t) \leq C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{3}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, if C^* is the minimal value of C in (3), then $C^* = B$.

In [13], the following statement was proved for singular integral operators with rough kernels T_Ω, containing the results in [26, 28].

Theorem 2. *Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is a homogeneous function of degree zero and has a mean value zero on S^{n-1}. Let $1 \leq s' < p < \infty$ and $\varphi(x, r)$ satisfy conditions*

$$c^{-1}\varphi(x, r) \leq \varphi(x, t) \leq c\, \varphi(x, r) \tag{4}$$

whenever $r \le t \le 2r$, *where* $c\ (\ge 1)$ *does not depend on* t, r, $x \in \mathbb{R}^n$ *and*

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\,\varphi(x,r)^p, \tag{5}$$

where C *does not depend on* x *and* r. *Then the operator* T_Ω *is bounded on* $M_{p,\varphi}$.

The following statement, containing results obtained in [26], [28] was proved in [18, 20] (see also [2], [5]-[8], [19]).

Theorem 3. *Let* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \varphi_1(0,t) \frac{dt}{t} \le C\,\varphi_2(0,r), \tag{6}$$

where C *does not depend on* r. *Then the operator* T *is bounded from* LM_{p,φ_1} *to* LM_{p,φ_2} *for* $p > 1$ *and from* LM_{1,φ_1} *to* WLM_{1,φ_2} *for* $p = 1$.

Corollary 1. *Let* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty t^{\alpha-1} \varphi_1(x,t) dt \le C\,\varphi_2(x,r), \tag{7}$$

where C *does not depend on* x *and* r. *Then the operator* T *is bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{1,φ_2} *for* $p = 1$.

Lemma 1. *Let* $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, $\Omega \in L_s(S^{n-1})$, $s > 1$, *be a homogeneous function of degree zero. Then, for* $p > 1$ *and* $s' \le p$ *or* $p < s$ *the inequality*

$$\|T_\Omega f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ *and for all* $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, for $s > 1$ *the inequality*

$$\|T_\Omega f\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^\infty t^{-n-1} \|f\|_{L_1(B(x_0,t))} dt, \tag{8}$$

holds for any ball $B(x_0, r)$ *and for all* $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$ and $s' \le p$. Set $B = B(x_0, r)$ for the ball of radius r centered at x_0. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0, \tag{9}$$

and have

$$\|T_\Omega f\|_{L_(B)} \le \|T_\Omega f_1\|_{L_p(B)} + \|T_\Omega f_2\|_{L_p(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $T_\Omega f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of T_Ω on $L_p(\mathbb{R}^n)$ it follows that

$$\|T_\Omega f_1\|_{L_p(B)} \leq \|T_\Omega f_1\|_{L_p(\mathbb{R}^n)} \leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It's clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|T_\Omega f_2(x)| \leq 2^n c_1 \int_{{}^{\complement}(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{{}^{\complement}(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0 - y|^n} dy &\approx \int_{{}^{\complement}(2B)} |f(y)||\Omega(x-y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\approx \int_{2r}^{\infty} \int_{2r \leq |x_0 - y| \leq t} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}&\int_{{}^{\complement}(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0 - y|^n} dy \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} \, |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.\end{aligned} \tag{10}$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|T_\Omega f_2\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{11}$$

is valid. Thus

$$\|T_\Omega f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

On the other hand,

$$\|f\|_{L_p(2B)} \approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}} \leq r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{12}$$

Thus

$$\|T_\Omega f\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

When $1 < p < s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}\|T_\Omega f_2\|_{L_p(B)} &\le \Big(\int_B \Big(\int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)||\Omega(x-y)|dy \frac{dt}{t^{n+1}}\Big)^p dx\Big)^{\frac{1}{p}} \\ &\le \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\, \|\Omega(\cdot - y)\|_{L_p(B)} dy \frac{dt}{t^{n+1}} \\ &\le |B(x_0,r)|^{\frac{1}{p}-\frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\, \|\Omega(\cdot - y)\|_{L_s(B)} dy\, \frac{dt}{t^{n+1}} \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{13}$$

Let $p = 1 < s \le \infty$. From the weak $(1,1)$ boundedness of T_Ω and (12) it follows that:

$$\begin{aligned}\|T_\Omega f_1\|_{WL_1(B)} &\le \|T_\Omega f_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\ &= \|f\|_{L_1(2B)} \lesssim r^n \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}}. \end{aligned} \tag{14}$$

Then from (11) and (14) we get the inequality (8).◄

Theorem 4. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and $\Omega \in L_s(S^{n-1})$, $s > 1$, be a homogeneous function of degree zero. Let also, for $s' \le p$ or $p < s$ the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x_0,r), \tag{15}$$

where C does not depend on r.

Then the operator T_Ω is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{1,\varphi_2}$ for $p = 1$. Moreover, for $p > 1$

$$\|T_\Omega f\|_{LM^{\{x_0\}}_{p,\varphi_2}} \lesssim \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}},$$

and for $p = 1$

$$\|T_\Omega f\|_{WLM^{\{x_0\}}_{1,\varphi_2}} \lesssim \|f\|_{LM^{\{x_0\}}_{1,\varphi_1}}.$$

Proof. Let $s' \le p$ or $p < s$. By Lemma 1 and Theorem 1 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1(r) = \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}}$ and $w(r) = r^{-\frac{n}{p}}$ we have for $p > 1$

$$\begin{aligned}\|T_\Omega f\|_{LM^{\{x_0\}}_{p,\varphi_2}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{p}}\, \|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}} \end{aligned}$$

and for $p = 1$

$$
\begin{aligned}
\|T_\Omega f\|_{WLM^{\{x_0\}}_{1,\varphi_2}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^\infty \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-n}\, \|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{1,\varphi_1}}.
\end{aligned}
$$

Let $1 \le p < s$. By Lemma 1 and Theorem 1 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1(r) = \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}+\frac{n}{s}}$ and $w(r) = r^{-\frac{n}{p}+\frac{n}{s}}$ we have for $p > 1$

$$
\begin{aligned}
\|T_\Omega f\|_{LM^{\{x_0\}}_{p,\varphi_2}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} r^{-\frac{n}{s}} \int_r^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}-\frac{n}{s}+1}} \\
&\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{p}}\, \|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}
\end{aligned}
$$

and for $p = 1$

$$
\begin{aligned}
\|T_\Omega f\|_{WLM^{\{x_0\}}_{1,\varphi_2}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} r^{-\frac{n}{s}} \int_r^\infty \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n-\frac{n}{s}+1}} \\
&\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-n}\, \|f\|_{L_1(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{1,\varphi_1}}.
\end{aligned}
$$

◄

Corollary 2. *Let $1 \le p < \infty$ and $\Omega \in L_s(S^{n-1})$, $s > 1$, be a homogeneous function of degree zero. Let also, for $s' \le p$ or $p < s$ the pair (φ_1, φ_2) satisfy the condition*

$$
\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x,r),
$$

where C does not depend on x and r.

Then the operator T_Ω is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p = 1$. Moreover, for $p > 1$

$$
\|T_\Omega f\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},
$$

and for $p = 1$

$$
\|T_\Omega f\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.
$$

Corollary 3. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy condition (15). Then the operator T is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{1,\varphi_2}$ for $p = 1$.*

Remark 1. *Note that, in the case $s = \infty$ Corollary 2 was proved in [21]. The condition* (15) *in Theorem 4 is weaker than condition* (6) *in Theorem 3 (see [21]).*

4. Commutators of singular integral operators with rough kernels in the spaces $LM_{p,\varphi}^{\{x_0\}}$

Let T be a linear operator. For a function b, we define the commutator $[b,T]$ by

$$[b,T]f(x) = b(x)\,Tf(x) - T(bf)(x)$$

for any suitable function f. If $\widetilde{T}$ is a Calderón-Zygmund singular integral operator, a well known result of Coifman, Rochberg and Weiss [12] states that the commutator $[b,\widetilde{T}]f = b\widetilde{T}f - \widetilde{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1<p<\infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [10, 11, 15]).

The definition of local BMO space is as follows:

Definition 4. *Let $1 \le q < \infty$. A function $f \in L_q^{\rm loc}(\mathbb{R}^n)$ is said to belong to the $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ (central BMO space), if*

$$\|f\|_{CBMO_q^{\{x_0\}}} = \sup_{r>0}\Big(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(y)-f_{B(x_0,r)}|^q dy\Big)^{1/q} < \infty.$$

Define

$$CBMO_q^{\{x_0\}}(\mathbb{R}^n) = \{f \in L_q^{\rm loc}(\mathbb{R}^n) \ : \ \|f\|_{CBMO_q^{\{x_0\}}} < \infty\}.$$

In [22], Lu and Yang introduced the central BMO space $CBMO_q(\mathbb{R}^n) = CBMO_q^{\{0\}}(\mathbb{R}^n)$. Note that, $BMO(\mathbb{R}^n) \subset CBMO_q^{\{x_0\}}(\mathbb{R}^n)$, $1 \le q < \infty$. The space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \le q < \infty$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition

$$\sup_{r>0}\Big(\frac{1}{|B|}\int_B |f(y)-f_B|^q dy\Big)^{1/q} < \infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ depends on q. If $q_1<q_2$, then $CBMO_{q_2}^{\{x_0\}}(\mathbb{R}^n) \subsetneqq CBMO_{q_1}^{\{x_0\}}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

Lemma 2. *[24] Let b be a function in $CBMO_p^{\{x_0\}}(\mathbb{R}^n)$, $1 \le p < \infty$ and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|}\int_{B(x_0,r_1)} |b(y)-b_{B(x_0,r_2)}|^p dy\right)^{\frac{1}{p}} \le C\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_{CBMO_p^{\{x_0\}}},$$

where $C>0$ is independent of b, r_1 and r_2.

In [13], the following statement was proved for the commutators of fractional integral operators with rough kernels, containing the results in [26, 28].

Theorem 5. *Suppose that $\Omega \in L_s(S^{n-1})$, $s>1$, is a homogeneous function of degree zero and $b \in BMO(\mathbb{R}^n)$. Let $1 \le s' < p < \infty$ and $\varphi(x,r)$ satisfy the conditions (4) and (5). Then the operator $[b, T_\Omega]$ is bounded on $M_{p,\varphi}$.*

Lemma 3. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $s>1$, is a homogeneous function of degree zero. Let $1<p<\infty$, $b \in CBMO_{p_2}^{\{x_0\}}(\mathbb{R}^n)$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

Then, for $s' \le p_1$ or $p<s$ the inequality

$$\|[b,T_\Omega]f\|_{L_p(B(x_0,r))} \lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) t^{-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_{p_1}^{loc}(\mathbb{R}^n)$.

Proof. Let $1<p<\infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. As in the proof of Lemma 1, we represent function f in form (9) and have

$$\begin{aligned} &[b,T_\Omega]f(x) \equiv J_1+J_2+J_3+J_4 = \big(b(x)-b_B\big)T_\Omega f_1(x) \\ &- T_\Omega\Big(\big(b(\cdot)-b_B\big)f_1\Big)(x) + \big(b(x)-b_B\big)T_\Omega f_2(x) - T_\Omega\Big(\big(b(\cdot)-b_B\big)f_2\Big)(x). \end{aligned}$$

Hence we get

$$\|[b,T_\Omega]f\|_{L_p(B)} \le \|J_1\|_{L_p(B)} + \|J_2\|_{L_p(B)} + \|J_3\|_{L_p(B)} + \|J_4\|_{L_p(B)}.$$

From the boundedness of $[b,T_\Omega]$ on $L_{p_1}(\mathbb{R}^n)$ it follows that:

$$\begin{aligned} \|J_1\|_{L_p(B)} &\le \|\big(b(\cdot)-b_B\big)[b,T_\Omega]f_1(\cdot)\|_{L_p(\mathbb{R}^n)} \\ &\le \|\big(b(\cdot)-b_B\big)\|_{L_{p_2}(\mathbb{R}^n)} \|[b,T_\Omega]f_1(\cdot)\|_{L_{p_1}(\mathbb{R}^n)} \\ &\le C\|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p_2}} \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \\ &= C\|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p_2}+\frac{n}{p_1}} \|f\|_{L_{p_1}(2B)} \int_{2r}^{\infty} t^{-1-\frac{n}{p_1}} dt \\ &\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}} dt. \end{aligned}$$

For J_2 we have

$$\begin{aligned} \|J_2\|_{L_p(B)} &\le \|[b,T_\Omega]\big(b(\cdot)-b_B\big)f_1\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|(b(\cdot)-b_B)f_1\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|b(\cdot)-b_B\|_{L_{p_2}(\mathbb{R}^n)} \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p_2}+\frac{n}{p_1}} \|f\|_{L_{p_1}(2B)} \int_{2r}^{\infty} t^{-1-\frac{n}{p_1}} dt \end{aligned}$$

$$\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}} dt.$$

For J_3, it is known that $x \in B$, $y \in {}^{\complement}(2B)$, which implies $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$. When $s' \le p_1$, by Fubini's theorem and applying Hölder inequality we have

$$\begin{aligned}
|T_\Omega f_2(x)| &\le c_0 \int_{{}^{\complement}(2B)} |\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n} dy \\
&\approx \int_{2r}^{\infty} \int_{2r<|x_0-y|<t} |\Omega(x-y)||f(y)| dy\, t^{-1-n} dt \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)||f(y)| dy\, t^{-1-n} dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p_1}-\frac{1}{s}} t^{-1-\frac{n}{p_1}} dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}} dt.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\|J_3\|_{L_p(B)} &= \|\big(b(\cdot)-b_B\big) T_\Omega f_2(\cdot)\|_{L_p(\mathbb{R}^n)} \\
&\le \|\big(b(\cdot)-b_B\big)\|_{L_p(\mathbb{R}^n)} \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}} dt \\
&\le \|\big(b(\cdot)-b_B\big)\|_{L_{p_2}(\mathbb{R}^n)} r^{\frac{n}{p_1}} \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}} dt \\
&\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}} dt.
\end{aligned}$$

When $p < s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}
\|J_3\|_{L_p(B)} &\le \Big(\int_B \Big(\int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)||b(x)-b_B||\Omega(x-y)| dy \frac{dt}{t^{n+1}} \Big)^p \Big)^{\frac{1}{p}} \\
&\le \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\, \||b(\cdot)-b_B|\Omega(\cdot-y)\|_{L_p(B)} dy \frac{dt}{t^{n+1}} \\
&\le \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\, \|b(\cdot)-b_B\|_{L_{p_2}(B)}\, \|\Omega(\cdot-y)\|_{L_{p_1}(B)} dy\, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p_2}} |B|^{\frac{1}{p_1}-\frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\, \|\Omega(\cdot-y)\|_{L_s(B)} dy\, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}} \qquad (16) \\
&\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p_1}(B(x_0,t))} \frac{dt}{t^{\frac{n}{p_1}+1}}.
\end{aligned}$$

For $x \in B$ by Fubini's theorem and applying Hölder inequality we have

$$|T_\Omega\Big(\big(b(\cdot)-b_B\big)f_2\Big)(x)| \lesssim \int_{\complement_{(2B)}} |b(y)-b_B|\,|\Omega(x-y)|\,\frac{|f(y)|}{|x-y|^n}dy$$
$$\lesssim \int_{\complement_{(2B)}} |b(y)-b_B|\,|\Omega(x-y)|\,\frac{|f(y)|}{|x_0-y|^n}dy$$
$$\approx \int_{2r}^{\infty}\int_{2r<|x_0-y|<t} |b(y)-b_B|\,|\Omega(x-y)|\,|f(y)|dy\,\frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}||\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}}$$
$$+\int_{2r}^{\infty}|b_{B(x_0,r)}-b_{B(x_0,t)}|\int_{B(x_0,t)}|\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty}\|(b(\cdot)-b_{B(x_0,t)})f\|_{L_p(B(x_0,t))}\,\|\Omega(\cdot-y)\|_{L_s(B(x_0,t))}\,|B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}}\,\frac{dt}{t^{n+1}}$$
$$+\int_{2r}^{\infty}|b_{B(x_0,r)}-b_{B(x_0,t)}|\|f\|_{L_{p_1}(B(x_0,t))}\,\|\Omega(\cdot-y)\|_{L_s(B(x_0,t))}\,|B(x_0,t)|^{1-\frac{1}{p_1}-\frac{1}{s}}\,t^{-n-1}dt$$
$$\lesssim \int_{2r}^{\infty}\|b(\cdot)-b_{B(x_0,t)}\|_{L_{p_2}(B(x_0,t))}\|f\|_{L_{p_1}(B(x_0,t))}t^{-1-\frac{n}{p_1}}\,dt$$
$$+\|b\|_{CBMO_{p_2}^{\{x_0\}}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p_1}(B(x_0,t))}\,t^{-1-\frac{n}{p_1}}dt$$
$$\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p_1}(B(x_0,t))}\;t^{-1-\frac{n}{p_1}}dt.$$

Then for J_4 we have

$$\begin{aligned}\|J_4\|_{L_p(B)} &\le \|T_\Omega\big(b(\cdot)-b_B\big)f_2\|_{L_p(\mathbb{R}^n)}\\ &\lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}}\,r^{\frac{n}{p}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p_1}(B(x_0,t))}t^{-1-\frac{n}{p_1}}dt.\end{aligned}$$

When $p<s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}\|T_\Omega f_2\|_{L_p(B)} &\le \Big(\int_B\Big|\int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)||\Omega(x-y)|dy\frac{dt}{t^{n+1}}\Big|^p\Big)^{\frac{1}{p}}\\ &\le \int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|\,\|\Omega(\cdot-y)\|_{L_p(B)}dy\frac{dt}{t^{n+1}}\\ &\le |B|^{\frac{1}{p}-\frac{1}{s}}\int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|\,\|\Omega(\cdot-y)\|_{L_s(B)}dy\,\frac{dt}{t^{n+1}}\\ &\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L_1(B(x_0,t))}\frac{dt}{t^{n+1}} \qquad (17)\\ &\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L_{p_1}(B(x_0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}}.\end{aligned}$$

Now, combining by all the above estimates, we end the proof of Lemma 3.◄

The following theorem is true:

Theorem 6. *Suppose that* $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$ *with* $s > 1$, *is a homogeneous function of degree zero. Let* $1 < p < \infty$, $b \in CBMO_{p_2}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Let also, for* $s' \le p_1$ *or* $p < s$ *the pair* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x_0, r), \tag{18}$$

where C *does not depend on* r.

Then, the operator $[b, T_\Omega]$ *is bounded from* $LM_{p,\varphi_1}^{\{x_0\}}$ *to* $LM_{p,\varphi_2}^{\{x_0\}}$. *Moreover*

$$\|[b, T_\Omega]f\|_{LM_{p,\varphi_2}^{\{x_0\}}} \lesssim \|b\|_{CBMO_{p_2}^{\{x_0\}}} \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}.$$

Proof. The statement of Theorem 6 follows by Lemma 3 and Theorem 1 in the same manner as in the proof of Theorem 4.◄

Corollary 4. *Suppose that* $\Omega \in L_s(S^{n-1})$ *with* $s > 1$, *is a homogeneous function of degree zero. Let* $1 < p < \frac{n}{\alpha}$, $b \in BMO(\mathbb{R}^n)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Let also, for* $s' \le p_1$ *or* $p < s$ *the pair* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x, \tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x, r),$$

where C *does not depend on* x *and* r.

Then, the operator $[b, T_\Omega]$ *is bounded from* M_{p,φ_1} *to* M_{p,φ_2}. *Moreover*

$$\|[b, T_\Omega]f\|_{M_{q,\varphi_2}} \lesssim \|b\|_{BMO} \|f\|_{M_{p,\varphi_1}}.$$

Remark 2. *Note that in the case* $s = \infty$ *Corollary 4 was proved in [21].*

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References

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math. **42** (1975), 765-778.

[2] Ali Akbulut, V.S. Guliyev and R. Mustafayev, *Boundedness of the maximal operator and singular integral operator in generalized Morrey spaces*, Mathematica Bohemica, 137 (1) 2012, 27-43.

[3] J. Alvarez, M. Guzman-Partida, J. Lakey, *Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures*, Collect. Math., 51 (2000), 1-47.

[4] A. Beurling, *Construction and analysis of some convolution algebras*, Ann. Inst. Fourier (Grenoble), 14 (1964), 132.

[5] V.I. Burenkov, H.V. Guliyev, V.S. Guliyev, *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces,* J. Comput. Appl. Math. **208** (1) (2007), 280-301.

[6] V.I. Burenkov, V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces,* Potential Anal. **30** (3) (2009), 211-249.

[7] V. Burenkov, A. Gogatishvili, V.S. Guliyev, R. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces,* Complex Var. Elliptic Equ. **55** (8-10) (2010), 739-758.

[8] V. Burenkov, A. Gogatishvili, V.S. Guliyev, R. Mustafayev, *Boundedness of the Riesz potential in local Morrey-type spaces,* Potential Anal. **35** (1) (2011), 67-87.

[9] F. Chiarenza, M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat. **7** (1987), 273-279.

[10] F. Chiarenza, M. Frasca, P. Longo, *Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), 149-168.

[11] F. Chiarenza, M. Frasca, P. Longo, *$W^{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841-853.

[12] R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (2) (1976), 611-635.

[13] Y. Ding, D. Yang, Z. Zhou, *Boundedness of sublinear operators and commutators on* $L^{p,\omega}(\mathbb{R}^n)$, Yokohama Math. J. 46 (1998), 15-27.

[14] Y. Ding and S. Z. Lu, *Higher order commutators for a class of rough operators*, Ark. Mat. 37 (1999), 33-44.

[15] G. Di Fazio, M.A. Ragusa, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. 112 (1993), 241-256.

[16] G. Di Fazio, D. K. Palagachev and M. A. Ragusa, *Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients*, J. Funct. Anal, 166 (1999), 179-196.

[17] H. Feichtinger, *An elementary approach to Wieners third Tauberian theorem on Euclidean n-space*, Proceedings, Conference at Cortona 1984, Sympos. Math., 29, Academic Press 1987.

[18] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian)

[19] V.S. Guliyev, *Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications*, Cashioglu, Baku, 1999, 332 pp. (in Russian)

[20] V.S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009, Art. ID 503948, 20 pp.

[21] V.S. Guliyev, S.S. Aliyev, T. Karaman, P. S. Shukurov, *Boundedness of sublinear operators and commutators on generalized Morrey Space*, Int. Eq. Op. Theory. 71 (3) (2011), pp. 327-355.

[22] S.Z. Lu and D.C. Yang, *The central BMO spaces and Littlewood-Paley operators*, Approx. Theory Appl. (N.S.), 11 (1995), 72-94.

[23] G. Lu, S. Lu, D. Yang, *Singular integrals and commutators on homogeneous groups*, Analysis Mathematica, **28** (2002), 103-134.

[24] S.Z. Lu and Q. Wu, *CBMO estimates for commutators and multilinear singular integrals*, Math. Nachr., 276 (2004), 75-88.

[25] S. Lu, Y. Ding, D. Yan, *Singular integrals and related topics*, World Scientific Publishing, Singapore, 2006.

[26] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo (1991), 183-189.

[27] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126-166.

[28] E. Nakai, *Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95-103.

[29] J. Peetre, *On the theory of* $M_{p,\lambda}$, J. Funct. Anal. **4** (1969), 71-87.

[30] N. Wiener, *Generalized Harmonic Analysis*, Acta Math., 55 (1930), 117-258.

[31] N. Wiener, *Tauberian theorems*, Ann. Math., 33 (1932), 1-100.

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