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On the Key Estimate for Variable Exponent Spaces

L. Diening, S. Schwarzacher

Abstract. The so-called *key estimate* is a fundamental tool for variable exponent spaces. Among other things it implies the boundedness of the Hardy-Littlewood maximal operator, which opens the door to the tools of harmonic analysis. We give a survey on the key estimate and present an improved version, which allows to apply the key estimate to a larger class of functions and provides better error estimates.

Key Words and Phrases: Variable exponent spaces, maximal operator

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1. History of the key estimate

In recent years there has been an extensive growth in the field of variable exponent spaces $L^{p(\cdot)}$. Different from the classical Lebesgue spaces L^p, the exponent is not a constant but a function $p : \mathbb{R}^n \to [1, \infty]$ depending on the space-variable. The introduction of these spaces goes already back to Orlicz [11]. We refer to the recent books [3, 6] for a detailed study of the variable exponent spaces.

A major breakthrough in the theory of variable exponent spaces was the fact that the right condition on the exponent was found: the log-*Hölder continuity*. This condition, which consists of a local and a decay condition, ensures the important boundedness for the Hardy-Littlewood maximal operator M on $L^{p(\cdot)}$, see [4, 8]. In fact for such exponents (which are bounded away from one) the boundedness of M is a consequence of the *key estimate for variable exponent spaces*, which roughly reads

$$\left(\fint_Q |f| \, dx \right)^{p(y)} \leq c \fint_Q |f|^{p(x)} \, dx + \text{error} \tag{1.1}$$

where Q is a ball or cube, $y \in Q$ and the "error" denotes an appropriate error term, which is essentially independent of f. See Theorem 1 for the precise statement.

Since both sides of the key estimate have a different scaling behavior, it cannot hold for all functions f, but we have to require that a certain norm of f is bounded. The original assumption form [8] was $\|f\|_{p(\cdot)} \leq 1$. To conclude from this the boundedness of M it was necessary to prove the key estimate with $p(\cdot)$ replaced by $p(\cdot)/p^-$, where $p^- = \inf p$. Note that [8] was before the discovery of the log-Hölder decay condition, so it was additionally assumed that p is constant outside a large, compact set. This restriction was overcome in [4] by introducing the log-Hölder decay condition. In that article the key estimate is not used exactly in the form above, but the crucial estimates in their Lemma 2.3 and Lemma 2.5 are very similar to the key estimate. The only difference is that the error was not independent of f but contained an additional term depending on the Hardy operator of f. However, the crucial condition for the validity of the estimates was again $\|f\|_{p(\cdot)} \leq 1$; their estimates involved used the exponent $p(\cdot)/p^-$ as well.

In [9, Lemma 3.1] and [5, Lemma 3.3] it has been discovered that it is possible to derive the key estimate for all log-Hölder continuous exponents which satisfy the decay condition. Moreover, this key estimate holds for the full range $1 \leq p^- \leq p^+ \leq \infty$, where $p^+ = \sup p$, while the ones of [8, 4] were restricted to the case $p^+ < \infty$. Again the boundedness of M is an immediate consequence for $p^- > 1$.

It was also discovered in [9, 5] that it is not necessary to prove the key estimate for the exponent $p(\cdot)/p^-$. Instead it suffices to prove the key estimate for the exponent $p(\cdot)$ for a larger class of functions, namely $\|f\|_{L^{p(\cdot)}+L^\infty} \leq 1$. Indeed, the embedding $L^{p(\cdot)} \hookrightarrow L^{p(\cdot)/p^-} + L^\infty$ allows to apply the key estimate then directly to the exponent $p(\cdot)/p^-$. The same key estimate also appears in [6, Theorem 4.2.4] with a slight improvement (an additional indicator function $\chi_{\{|f| \leq 1\}}$ appears in the error term), which is needed for a suitable weak-type estimate of the Riesz potential operator, see [6, Theorem 6.1.11].

From the point of application, it is important that we can apply the key estimate even to a larger class of functions. For example, in the study of higher integrability of weak solutions to the $p(\cdot)$-Laplacian system [1, 12], i.e.

$$-\mathrm{div}(|\nabla u|^{p(\cdot)-2}\nabla u) = -\mathrm{div}(|F|^{p(\cdot)-2}F)$$

for certain F, it is very useful to apply the key estimate to functions in the unit ball of $L^{p_Q^-} + L^\infty$, where $p_Q^- = \inf_Q p(\cdot)$. We will use this approach for example in [10]. It is the aim of this paper to provide the necessary extended version of the key estimate. In particular, we will show that the key estimate holds for all functions from the unit ball of $L^1 + L^\infty$, which includes all of the cases mentioned above. Actually, we will allow an even larger class of functions. Unfortunately, this improvement requires us to restrict the key estimate again to $p^+ < \infty$. This is however sufficient for the applications that we have in mind.

Another application that we have in mind comes from the finite element approximation of a $p(\cdot)$-Laplacian system, see [2]. To derive the a priori error estimates for the discrete weak solution it is necessary to control the Scott-Zhang [13] interpolation operator in the variable exponent context. These estimates are again based on the key estimate. For the applications above it was sufficient that the error of the key estimate is controlled in $L^{1,\infty} \cap L^\infty$, where $L^{1,\infty}$ is the Marcinkiewicz space. However, in the finite element context, we also need that the error is small for small Q. Therefore, we present a key estimate that additionally has this feature. This kind of smallness was first introduced in the key estimate of [5] (in the case of bounded $L^{p(\cdot)} + L^\infty$ norm) and used in the study of spaces with variable smoothness and integrability [7].

It is the goal of this paper to provide a key estimate that combines the advantages of all the approaches mentioned above.

2. The key estimate

For a measurable set $E \subset \mathbb{R}^n$ let $|E|$ be the Lebesgue measure of E and χ_E its characteristic function. For $0 < |E| < \infty$ and $f \in L^1(E)$ we define the mean value of f over E by

$$\langle f \rangle_E := \fint_E f\,dx := \frac{1}{|E|}\int_E f\,dx.$$

For an open set $\Omega \subset \mathbb{R}^n$ let $L^0(\Omega)$ denote the set of measurable functions.

Let us introduce the spaces of variable exponents $L^{p(\cdot)}$. We use the notation of the recent book [6]. We define $\mathcal{P}$ to consist of all $p \in L^0(\mathbb{R}^n)$ with $p : \mathbb{R}^n \to [1, \infty]$ (called variable exponents). For $p \in \mathcal{P}$ we define $p_\Omega^- := \operatorname{ess\,inf}_\Omega p$ and $p_\Omega^+ := \operatorname{ess\,sup}_\Omega p$. Moreover, let $p^+ := p^+_{\mathbb{R}^n}$ and $p^- := p^-_{\mathbb{R}^n}$.

For $t \geq 0$ and $q \in [1, \infty)$ we define

$$\widetilde{\varphi}_q(t) := \frac{1}{q}t^q, \qquad \bar{\varphi}_q(t) := t^q$$

and

$$\bar{\varphi}_\infty(t) := \widetilde{\varphi}_\infty(t) := \infty \cdot \chi_{(1,\infty)}(t) = \begin{cases} 0 & \text{if } t \in [0,1], \\ \infty & \text{if } t \in (1,\infty). \end{cases}$$

Moreover, by φ_q we denote in the following either $\widetilde{\varphi}_q$ or $\bar{\varphi}_q$.

For $p \in \mathcal{P}$ the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) := \big\{ f \in L^0(\Omega) \,:\, \|f\|_{L^{p(\cdot)}(\Omega)} < \infty \big\},$$

where

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{ \lambda > 0 \,:\, \int_{\mathbb{R}^n} \varphi_{p(x)}\left(\left| \frac{f(x)}{\lambda} \right| \right) dx \leq 1 \right\}.$$

Note that both choices $\widetilde{\varphi}_{p(\cdot)}$ and $\bar{\varphi}_{p(\cdot)}$ produce the same space. The induced norms are different but equivalent, see [6, (3.2.2)]. The advantage of $\bar{\varphi}_{p(\cdot)}$ is that the norm is just the classical L^p norm if $p(\cdot)$ is constant. The advantage of $\widetilde{\varphi}_{p(\cdot)}$ is that it behaves better under duality. The key estimate that we present will be valid for both versions of $\varphi_{p(\cdot)}$.

We say that a function $\alpha \colon \mathbb{R}^n \to \mathbb{R}$ is log-*Hölder continuous* on Ω if there exists a constant $c \geq 0$ and $\alpha_\infty \in \mathbb{R}$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{c}{\log(e + 1/|x-y|)} \qquad \text{and} \qquad |\alpha(x) - \alpha_\infty| \leq \frac{c}{\log(e + |x|)}$$

for all $x, y \in \mathbb{R}^n$. The first condition describes the so called local log-Hölder continuity and the second the decay condition. The smallest such constant c is the log-Hölder constant of α. We define $\mathcal{P}^{\log}$ to consist of those exponents $p \in \mathcal{P}$ for which $\frac{1}{p} : \mathbb{R}^n \to [0,1]$ is log-Hölder continuous. By p_∞ we denote the limit of p at infinity, which exists for $p \in \mathcal{P}^{\log}$. If $p \in \mathcal{P}$ is bounded, then $p \in \mathcal{P}^{\log}$ is equivalent to the log-Hölder continuity of p. However, working with $\frac{1}{p}$ gives better control of the constants especially in the context of averages and maximal functions. Therefore, we define $c_{\log}(p)$ as the log-Hölder constant of $1/p$. Expressed in p we have for all $x, y \in \mathbb{R}^n$

$$|p(x) - p(y)| \leq \frac{(p^+)^2 c_{\log}(p)}{\log(e + 1/|x-y|)} \qquad \text{and } |p(x) - p_\infty| \qquad \leq \frac{(p^+)^2 c_{\log}(p)}{\log(e + |x|)}.$$

Remark 1. *It is also possible to consider* log-*Hölder continuous exponents on a domain* $\Omega \subset \mathbb{R}^n$*. However, due to [6, Proposition 4.17] it is always possible to extend such exponents to* $\mathbb{R}^n$ *while preserving the* log-*Hölder constants.*

We are now able to present our *key estimate*.

Theorem 1 (Key estimate)**.** *Let* $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $p^+ < \infty$*. Then for every* $m > 0$ *there exists* $\beta \in (0,1)$ *only depending on* m *and* $c_{\log}(p)$ *and* p^+ *such that*

$$\varphi_{p(x)}\Bigg(\beta \fint_Q |f(y)|\,dy\Bigg) \leq \fint_Q \varphi_{p(y)}(|f(y)|)\,dy + e_Q(x),$$

with

$$e_Q(x) = \frac{1}{2}\min\{1, |Q|^m\} \fint_Q \big((e+|x|)^{-m} + (e+|y|)^{-m}\big)\chi_{\{0<|f(y)|\leq 1\}}\,dy$$

for every cube (or ball) $Q \subset \mathbb{R}^n$*, all* $x \in Q$*, and all* $f \in L^1(Q)$ *with*

$$\fint_Q |f|\,dy \leq \max\{1, |Q|^{-m}\}.$$

We will prove Theorem 1 below on page 72.

Remark 2. *Let us point out that Theorem 1 is in particular valid for all functions* f *with* $\|f\|_{L^1+L^\infty} \leq 1$*, since*

$$\fint_Q |f|\,dx \leq 2\,\|f\|_{L^1+L^\infty} \max\{1, |Q|^{-1}\}.$$

The additional constant 2 can be removed by adapting β *accordingly. The same conclusion also holds for the other lemmas and corollaries of this paper.*

Before we get to the proof of Theorem 1 we need a few auxiliary results.

Lemma 1. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $m > 0$. Then there exists $\beta \in (0,1)$ which only depends on $c_{\log}(p)$ and m such that*

$$\varphi_{p(x)}\Big(\beta\varphi^{-1}_{p_Q^-}\big(\lambda \max\{1,|Q|^{-m}\}\big)\Big) \leq \lambda \max\{1,|Q|^{-m}\},$$

for all $\lambda \in [0,1]$, any cube (or ball) $Q \subset \mathbb{R}^n$ and any $x \in Q$.

Proof. The case $\max\{1,|Q|^{-m}\} = 1$ is obvious. The case $\max\{1,|Q|^{-m}\} = |Q|^{-m}$ follows from [6, Lemma 4.2.1] raised to the power m.◀

Lemma 2. *Let $p \in \mathcal{P}(\mathbb{R}^n)$ and let $\frac{1}{p}$ be locally* log*-Hölder continuous with $p^+ < \infty$. Define $q \in \mathcal{P}^{\log}(\mathbb{R}^n \times \mathbb{R}^n)$ by*

$$\frac{1}{q(x,y)} := \max\left\{\frac{1}{p(x)} - \frac{1}{p(y)}, 0\right\}.$$

Then for any $\gamma \in (0,1)$ there exists $\beta \in (0,1)$ only depending on γ, $c_{\log}(p)$ and p^+ such that

$$\varphi_{p(x)}\left(\beta \fint_Q |f(y)|\,dy\right) \leq \fint_Q \varphi_{p(y)}(|f(y)|)\,dy + \fint_Q \varphi_{q(x,y)}(\gamma)\,\chi_{\{0<|f(y)|\leq 1\}}\,dy$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, $x \in Q$, and $f \in L^1(Q)$ with

$$\fint_Q |f|\,dy \leq \max\{1,|Q|^{-m}\}.$$

Proof. The proof is in most parts the same as in [6], so we only point out the differences. As there, we prove it with no loss of generality for $\bar{\varphi}$. We split f into three parts

$$\begin{aligned} f_1(y) &:= f(y)\,\chi_{\{y\in Q:\,|f(y)|>1\}},\\ f_2(y) &:= f(y)\,\chi_{\{y\in Q:\,|f(y)|\leq 1, p(y)\leq p(x)\}},\\ f_3(y) &:= f(y)\,\chi_{\{y\in Q:\,|f(y)|\leq 1, p(y)>p(x)\}}. \end{aligned}$$

The estimates for f_2 and f_3 are just as in [6] so we need to adapt the estimate for f_1. Let $A := \fint_Q f_1\,dy$. We can assume $A \neq 0$. Then by the assumptions on f we have $A \leq \max\{1,|Q|^{-m}\}$. So by Lemma 1 we get for some $\tilde{\beta} > 0$

$$\bar{\varphi}_{p(x)}\Big(\tilde{\beta}\bar{\varphi}^{-1}_{p_Q^-}(A)\Big) \leq A.$$

Hence, for $\beta := \tilde{\beta}^{p+}$

$$\bar{\varphi}_{p(x)}(\beta A) \leq \bar{\varphi}_{p_Q^-}\Big(\bar{\varphi}_{p(x)}\big(\tilde{\beta}\bar{\varphi}^{-1}_{p_Q^-}(A)\big)\frac{1}{A}\Big)\bar{\varphi}_{p_Q^-}(A) \leq \bar{\varphi}_{p_Q^-}(A).$$

This and Jensen's inequality imply

$$\bar{\varphi}_{p(x)}\bigg(\beta \fint_Q f_1 dy\bigg) \le \bar{\varphi}_{p_Q^-}\bigg(\fint_Q f_1 dy\bigg) \le \fint_Q \bar{\varphi}_{p_Q^-}(|f_1|)dy$$
$$= \fint_Q \bar{\varphi}_{p_Q^-}(|f|)\chi_{\{|f|\le 1\}}dy \le \fint_Q \bar{\varphi}_{p(y)}(|f|)dy.$$

This proves the estimate for f_1. The rest of the proof is as in [6]. ◄

We can now prove our key estimate.

Proof of Theorem 1. In view of Lemma 2 it suffices to prove

$$\fint_Q \varphi_{q(x,y)}(\gamma)\,\chi_{\{0<|f(y)|\le 1\}}\,dy \le e_Q(y).$$

This follows exactly from the proof of [5, Lemma 3.3] while keeping the indicator function $\chi_{\{0<|f(y)|\le 1\}}$ at all steps. The main idea is to estimate $\varphi_{q(x,y)}(\gamma)$ by the square of $\varphi_{q(x,y)}(\gamma^{1/2})$. One factor is used to produce the $\min\{1,|Q|^m\}$ part of e_Q. The other factor is used to produce the mean value integral part of e_Q.

3. A few consequences of the key estimate

Let us state a few direct consequences of our improved key estimate. We begin with an integral version of the key estimate that we will need in the finite element analysis of [2].

Corollary 1. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty$. Then for every $m > 0$ there exists $\beta \in (0,1)$ only depending on m and $c_{\log}(p)$ such that*

$$\int_Q \varphi_{p(x)}\bigg(\beta \fint_Q |f(y)|\,dy\bigg)\,dx \le \int_Q \varphi_{p(y)}(|f(y)|)\,dy$$
$$+ \min\{1,|Q|^m\}\int_Q (e+|y|)^{-m}\,dy,$$

for every cube (or ball) $Q \subset \mathbb{R}^n$ and all $f \in L^1(Q)$ with

$$\fint_Q |f|\,dy \le \max\{1,|Q|^{-m}\}.$$

The following lemma is a Jensen's type inequality with singular measure like [6, Lemma 6.1.12].

Corollary 2 (Jensen inequality with singular measure)**.** *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty$. For every $m > 0$ there exists $\beta \in (0,1)$ only depending on m and $c_{\log}(p)$ such that*

$$\varphi_{p(x)}\bigg(\beta \int_B \frac{|f(y)|}{r\,|x-y|^{n-1}}\,dy\bigg) \leq \int_B \frac{\varphi_{p(y)}(|f(y)|)}{r\,|x-y|^{n-1}}\,dy + \min\{1, |B|^m\}\, M\big((e+|\cdot|)^{-m}\big)(x)$$

for every ball B with radius r and all $f \in L^1(B)$ with

$$\fint_Q |f|\,dy \leq \max\{1, |B|^{-m}\}.$$

As in [6, Proposition 8.2.11] (which is based on [12]) this immediately implies

Corollary 3. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$ and let $s \leq p^-$ satisfy $s \in [1, \frac{n}{n-1})$. Then for every $m > 0$ there exists a constant c depending on n, $c_{\log}(p)$, m, and s such that*

$$\fint_{B_R} \bigg(\frac{|v - \langle v\rangle_{B_R}|}{R}\bigg)^{p(x)} dx \leq c\bigg(\fint_{B_R} |\nabla v|^{\frac{p(\cdot)}{s}}\,dx\bigg)^s + c\min\{1, |B_R|^m\} \fint_{B_R} (e+|x|)^{-ms}\,dx$$

for every ball B_R with radius R, and every $v \in W^{1,\frac{p(\cdot)}{s}}(B_R)$ with

$$\fint_Q |\nabla v|\,dy \leq \max\{1, |Q|^{-m}\}.$$

This corollary is very important for [10].

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L. Diening
LMU Munich, Institute of Mathematics, Theresienstr. 39, 80333-Munich, Germany
E-mail: diening@math.lmu.de

S. Schwarzacher
LMU Munich, Institute of Mathematics, Theresienstr. 39, 80333-Munich, Germany
E-mail: schwarz@math.lmu.de