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Tikhonov-Lavrentev type inverse problem including Cauchy-Riemann equation

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Abstract. In this paper, we consider an inverse problem which contains the Cauchy-Riemann equation with two non-local boundary conditions. Besides the sought function, the right-hand side of second boundary condition is unknown. To solve this problem, we first obtain necessary conditions for the fundamental solution of Cauchy-Riemann equation and then provide sufficient conditions for reducing this problem to a Fredholm integral equation of the second kind. Finally, we regularize the singularities in the kernels of integrals.

Key Words and Phrases: inverse problem, fundamental solution, singularities, weak singularities.

2000 Mathematics Subject Classifications: 45Q05; 35R30

1. Introduction

The mathematical models of some physical and geophysical phenomena are described by inverse problems of Tikhonov-Lavrentev type [1]. In these problems, along with the sought function, one of the coefficients or the right-hand side of boundary conditions is also unknown [2, 6]. In this case, the number of boundary conditions is more than in the classical cases of boundary value problems. For the first time, these problems have been investigated by Lavrentev in [5]. In this paper, we consider a Cauchy-Riemann equation with two non-local boundary conditions. Besides the sought function, the right-hand side of second boundary condition is unknown. We first obtain necessary conditions for the solution of Cauchy-Riemann equation and then, using these necessary conditions and boundary values of solution, we arrive to a Fredholm integral equation of the second kind with respect to the unknown function ϕ_2. Next, using the obtained conditions, the singularities of the integral relations are regularized. And, finally, solving the integral equation with respect to ϕ_2, we calculate the solution of the problem.

2. Mathematical statement of problem

We consider the following inverse problem:

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = 0 \qquad x = (x_1, x_2),\ x_1 \in \mathbb{R},\ x_2 \in (0,1), \tag{1}$$

$$\alpha_j(x_1)\, u(x_1, 0) + \beta_j(x_1)\, u(x_1, 1) = \phi_j(x_1) \qquad j = 1, 2,\ x_1 \in \mathbb{R}, \tag{2}$$

where $i = \sqrt{-1}$, $u(x)$ and $\phi_2(x_1)$ are unknown functions and $\phi_1(x_1),\ \alpha_j(x_1)$ and $\beta_j(x_1);\ j = 1, 2,$ are given continuous functions.

3. Necessary conditions

We know that the fundamental solution of equation (1) is given in the following form [8]:

$$U(x-\xi) = \frac{1}{2\pi(x_2 - \xi_2 + i(x_1 - \xi_1))}\,. \tag{3}$$

According to the definition of the fundamental solution, we have

$$\frac{\partial U(x-\xi)}{\partial x_2} + i\,\frac{\partial U(x-\xi)}{\partial x_1} \;=\; \delta(x-\xi),$$

where $\delta(x-\xi)$ is the Dirac delta function.
If we suppose

$$\lim_{x_1 \to \pm\infty} [\, u(x)\, U(x-\xi)\,] = 0\,, \tag{4}$$

then, multiplying (3) by both sides of equation (1), integrating over the domain $D = \{\,(x_1, x_2)\,;\, x_1 \in \mathbb{R},\ x_2 \in (0,1)\}$ and using the Ostrogradsky-Gauss formula [3], we obtain

$$0 = \int_D (\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1})\, U(x-\xi)\, dx = \int_D \frac{\partial u(x)}{\partial x_2}\, U(x-\xi)\, dx + i\int_D \frac{\partial u(x)}{\partial x_1}\, U(x-\xi)\, dx$$

$$= \int_\Gamma u(x)\, U(x-\xi) \cos(v, x_2)\, dx - \int_D u(x)\, \frac{\partial U(x-\xi)}{\partial x_2}\, dx + i\int_\Gamma u(x)\, U(x-\xi) \cos(v, x_1)\, dx$$

$$-i\int_D u(x) \frac{\partial U(x-\xi)}{\partial x_1}\, dx,$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 = \{(x_1, 0);\ x_1 \in \mathbb{R}\}$ and $\Gamma_2 = \{(x_1, 1);\ x_1 \in \mathbb{R}\}$ is the boundary of D and ν is a unit normal vector to the boundary of D. Note that $\cos(\nu, x_2)|_{x_2=1} = 1$ and $\cos(\nu, x_2)|_{x_2=0} = -1$.
The method of integration by parts, the definition of fundamental solution and the property of Dirac delta function $\delta(x)$ yield

$$\begin{aligned}
&\int_R u(x_1,1)\,U(x_1-\xi_1,1-\xi_2)\,dx_1 - \int_R u(x_1,0)\,U(x_1-\xi_1,-\xi_2)\,dx_1 \\
&= \int_D u(x)\,\big(\frac{\partial U(x-\xi)}{\partial x_2} + i\frac{\partial U(x-\xi)}{\partial x_1}\big)\,dx = \int_D u(x)\,\delta(x-\xi)\,dx \\
&= \begin{cases} u(\xi) & \xi_1\in\mathbb{R},\ \xi_2\in(0,1), \\ 1/2\,u(\xi) & \xi_1\in\mathbb{R},\ \xi_2=0 \ \text{ or } \ \xi_2=1. \end{cases}
\end{aligned} \tag{5}$$

Note that from (4) we have

$$\int_0^1 \lim_{x_1\to\infty} [\,u(x_1,x_2)\,U(x_1-\xi_1,x_2-\xi_2) - u(-x_1,x_2)\,U(-x_1-\xi_1,x_2-\xi_2)\,]\,dx_2 = 0.$$

From (5), for the boundary values $\xi_2 = 0$ and $\xi_2 = 1$, we obtain the following relations which we call necessary condition:

$$\begin{aligned}
1/2\,u(\xi_1,0) &= \int_R [\,u(x_1,1)\,U(x_1-\xi_1,1) - u(x_1,0)\,U(x_1-\xi_1,0)\,]\,dx_1, \\
1/2\,u(\xi_1,1) &= \int_R [\,u(x_1,1)\,U(x_1-\xi_1,0) - u(x_1,0)\,U(x_1-\xi_1,-1)\,]\,dx_1.
\end{aligned} \tag{6}$$

To sum up, we get the following theorem:

Theorem 1. *Under condition (4), every solution of equation (1) satisfies the necessary conditions (6) in the domain* $D = \{\,(x_1,x_2);\ x_1\in R,\ x_2\in(0,1)\}$.

4. Fredholm integral equation with respect to ϕ_2

From the second of boundary conditions (2), using (6), we have:

$$\begin{aligned}
&-\alpha_1(\xi_1)\,u(\xi_1,0) + \beta_1(\xi_1)\,u(\xi_1,1) \\
&= -2\alpha_1(\xi_1)\int_R [\,u(x_1,1)\,U(x_1-\xi_1,1) - u(x_1,0)\,U(x_1-\xi_1,0)\,]\,dx_1 \\
&\quad +2\beta_1(\xi_1)\int_R [\,u(x_1,1)\,U(x_1-\xi_1,0) - u(x_1,0)\,U(x_1-\xi_1,-1)\,]\,dx_1 \\
&= -2\alpha_1(\xi_1)\int_R u(x_1,1)\,U(x_1-\xi_1,1)\,dx_1 - 2\beta_1(\xi_1)\int_R u(x_1,0)\,U(x_1-\xi_1,-1)\,dx_1 \\
&\quad +2\int_R U(x_1-\xi_1,0)[\,(\alpha_1(\xi_1)-\alpha_1(x_1))\,u(x_1,0) + (\beta_1(\xi_1)-\beta_1(x_1))\,u(x_1,1)\,]\,dx_1 \\
&\quad +2\int_R U(x_1-\xi_1,0)[\,\alpha_1(x_1)\,u(x_1,0) + \beta_1(x_1)\,u(x_1,1)\,]\,dx_1\,.
\end{aligned} \tag{7}$$

Subsituting (6) and the fundamental solution (3) into (7) we get

$$
\begin{aligned}
&-\alpha_1(\xi_1)\, u(\xi_1,0) + \beta_1(\xi_1)\, u(\xi_1,1) \\
&= -1/\pi\, \alpha_1(\xi_1) \int_R \frac{u(x_1,1)}{1+i(x_1-\xi_1)}\, dx_1 - 1/\pi\, \beta_1(\xi_1) \int_R \frac{u(x_1,0)}{-1+i(x_1-\xi_1)}\, dx_1 \\
&\quad +1/(\pi i) \int_R \frac{1}{x_1-\xi_1} \cdot [\, (\alpha_1(\xi_1)-\alpha_1(x_1))\, u(x_1,0) + (\beta_1(\xi_1)-\beta_1(x_1))\, u(x_1,1)\,]\, dx_1 \\
&\quad +1/(\pi i) \int_R \frac{\phi_1(x_1)}{x_1-\xi_1}\, dx_1.
\end{aligned}
\tag{8}
$$

Note that using the first of boundary conditions (2) and substituting the function $\phi_1(x_1)$ into the last integral will remove its singularity because $\phi_1(x_1)$ is a known function. Now we solve the two boundary conditions (2) for $u(x_1,0)$ and $u(x_1,1)$ applying Cramer's rule. For this we need to suppose

$$
W(x_1) = \begin{vmatrix} \alpha_1(x_1) & \beta_1(x_1) \\ \alpha_2(x_1) & \beta_2(x_1) \end{vmatrix} \neq 0. \tag{9}
$$

Then we have

$$
\begin{aligned}
u(x_1,0) &= \frac{\phi_1(x_1)\,\beta_2(x_1) - \phi_2(x_1)\,\beta_1(x_1)}{W(x_1)}\,, \\
u(x_1,1) &= \frac{\phi_2(x_1)\,\alpha_1(x_1) - \phi_1(x_1)\,\alpha_2(x_1)}{W(x_1)}\,.
\end{aligned}
\tag{10}
$$

Substituting (10) into (8), we obtain

$$
\begin{aligned}
&-\alpha_1(\xi_1)\, u(\xi_1,0) + \beta_1(\xi_1)\, u(\xi_1,1) \\
&= -\alpha_1(\xi_1)[\, \frac{\phi_1(\xi_1)\,\beta_2(\xi_1) - \phi_2(\xi_1)\,\beta_1(\xi_1)}{W(\xi_1)}\,] + \beta_1(\xi_1)[\, \frac{\phi_2(\xi_1)\,\alpha_1(\xi_1) - \phi_1(\xi_1)\,\alpha_2(\xi_1)}{W(\xi_1)}\,] \\
&= -1/\pi\, \alpha_1(\xi_1) \int_R \frac{1}{1+i(x_1-\xi_1)} \cdot [\, \frac{\phi_2(x_1)\,\alpha_1(x_1) - \phi_1(x_1)\,\alpha_2(x_1)}{W(x_1)}\,]\, dx_1 \\
&\quad -1/\pi\, \beta_1(\xi_1) \int_R \frac{1}{-1+i(x_1-\xi_1)} \cdot [\, \frac{\phi_1(x_1)\,\beta_2(x_1) - \phi_2(x_1)\,\beta_1(x_1)}{W(x_1)}\,]\, dx_1 \\
&\quad +1/(\pi i) \int_R \frac{\alpha_1(\xi_1)-\alpha_1(x_1)}{x_1-\xi_1} \cdot [\, \frac{\phi_1(x_1)\,\beta_2(x_1) - \phi_2(x_1)\,\beta_1(x_1)}{W(x_1)}\,]\, dx_1 \\
&\quad +1/(\pi i) \int_R \frac{\beta_1(\xi_1)-\beta_1(x_1)}{x_1-\xi_1} \cdot [\, \frac{\phi_2(x_1)\,\alpha_1(x_1) - \phi_1(x_1)\,\alpha_2(x_1)}{W(x_1)}\,]\, dx_1 \\
&\quad +1/(\pi i) \int_R \frac{\phi_1(x_1)}{x_1-\xi_1}\, dx_1.
\end{aligned}
\tag{11}
$$

Finally, from (11) for the unknown function $\phi_2(x_1)$ we obtain the following Fredholm integral equation of the second kind:

$$\phi_2(\xi_1) = \int_R K(\xi_1, x_1)\, \phi_2(x_1)\, dx_1 + f(\xi_1)\,, \quad \xi_1 \in R, \tag{12}$$

where

$$\begin{aligned} K(\xi_1, x_1) \ =\ & \frac{W(\xi_1)}{2\pi W(x_1)} [-\frac{\alpha_1(x_1)}{\beta_1(\xi_1)(1+i(x_1-\xi_1))} + \frac{\beta_1(x_1)}{\alpha_1(\xi_1)(-1+i(x_1-\xi_1))} \\ & -\frac{\beta_1(x_1)(\alpha_1(\xi_1)-\alpha_1(x_1))}{i(x_1-\xi_1)\,\alpha_1(\xi_1)\beta_1(\xi_1)} + \frac{\alpha_1(x_1)(\beta_1(\xi_1)-\beta_1(x_1))}{i(x_1-\xi_1)\,\alpha_1(\xi_1)\,\beta_1(\xi_1)}\ ], \end{aligned}$$

$$\begin{aligned} f(\xi_1) =\ & 1/2\,\phi_1(\xi_1)[\frac{\beta_2(\xi_1)}{\beta_1(\xi_1)} + \frac{\alpha_2(\xi_1)}{\alpha_1(\xi_1)}] + 1/(2\pi)\int_R \frac{W(\xi_1)\phi_1(x_1)}{\alpha_1(\xi_1)\beta_1(\xi_1)(x_1-\xi_1)}\, dx_1 \\ & +1/(2\pi)\int_R \frac{\phi_1(x_1)W(\xi_1)}{W(x_1)} [\frac{\alpha_2(x_1)}{\beta_1(\xi_1)(1+i(x_1-\xi_1))} - \frac{\beta_2(x_1)}{\alpha_1(\xi_1)(-1+i(x_1-\xi_1))} \\ & + \frac{\beta_2(x_1)(\alpha_1(\xi_1)-\alpha_1(x_1))}{i(x_1-\xi_1)\alpha_1(\xi_1)\beta_1(\xi_1)} - \frac{\alpha_2(x_1)(\beta_1(\xi_1)-\beta_1(x_1))}{i(x_1-\xi_1)\beta_1(\xi_1)\alpha_1(\xi_1)}]\, dx_1. \end{aligned}$$

5. Removing Singularities

The first and second integrals in (8) do not have singularities. To remove singularities in the third and fourth integrals, we suppose that the functions $\alpha_1(x_1)$, $\beta(x_1)$ satisfy a Hölder condition. By considering the limit conditions $\lim_{x_1\to\pm\infty} \phi_1(x_1) = 0$, we have

$$\begin{aligned} & -\alpha_1(\xi_1)\, u(\xi_1, 0) + \beta_1(\xi_1)\, u(\xi_1, 1) \\ & = 1/(\pi i)\int_R \frac{\phi_1(x_1)}{x_1-\xi_1}\, dx_1 - 1/(\pi i)\int_R [\alpha_1'(\sigma_1(\xi_1,x_1))\, u(x_1,0) + \beta_1'(\sigma_2(\xi_1,x_1))\, u(x_1,1)]\, dx_1 + \cdots \\ & = 1/(\pi i)\lim_{t\to+\infty} [\phi_1(x_1)\, \ln|x_1-\xi_1|]_{x_1=-t}^{t} - 1/(\pi i)\int_R \phi_1'(x_1)\, \ln|x_1-\xi_1|\, dx_1 + \cdots\,. \end{aligned} \tag{13}$$

Note that dots on the right-hand sides of the above relation represent terms without singularities.

6. Main results

To sum up, we get the following theorems:

Theorem 2. *Let the conditions of Theorem 1 be satisfied. If the functions* $\alpha_1(x_1)$, $\beta(x_1)$ *satisfy a Hölder condition,* $\phi_1(x_1)$ *is a continuously differentiable function and* $\lim_{x\to\pm\infty}\phi_1(x_1)=0$*, then the singularities in (8) are regularized.*

Theorem 3. *Let the conditions of Theorem 2, assumption (9) and relation (11) be satisfied. Then the kernel of integral equation (12) has weak singularities.*

7. Conclusion

Finding $\phi_2(x_1)$ from the integral equation (12) and inserting it into (10), we calculate the boundary values of $u(x_1, x_2)$.

Finally, using the first case of (5), we obtain the analytic solution of the inverse problem (1)-(2).

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