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Some Properties for Certain Subclass of p-Valent Analytic Functions Defined Using Convolution

M. K. Aouf*, A. A. Shamandy, A. O. Mostafa, A. K. Wagdy

Abstract. In this paper, we introduce a new subclass of analytic p-valent functions defined using convolution and investigate many properties of functions of this class using Jack's lemma.

Key Words and Phrases: Analytic, p-valent, Hadamard product, Jack's lemma, convolution

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1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ \ (p \in \mathbb{N};\ \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions f and ϕ, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $\phi(z)$ in $\mathbb{U}$, written $f(z) \prec \phi(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = \phi\,(w(z))$. Indeed, it is known that

$$f(z) \prec \phi(z) \Rightarrow f(0) = \phi(0) \text{ and } \ f(\mathbb{U}) \subset \phi(\mathbb{U}\ ).$$

Furthermore, if the function ϕ is univalent in $\mathbb{U}$, then we have the following equivalence (see [1] and [6]):

$$f(z) \prec \phi(z) \Leftrightarrow f(0) = \phi(0) \text{ and } \ f(\mathbb{U}) \subset \phi(\mathbb{U}\ ). \tag{1.2}$$

Let $g, h \in A(p)$ be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{1.3}$$

*Corresponding author.

and

$$h(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k. \tag{1.4}$$

Given two functions f and g in the class $A(p)$, where $f(z)$ is given by (1.1) and $g(z)$ is given by (1.3), the Hadamard product (or convolution) $f* g$ of f and g is defined (as usual) by

$$(f*g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g*f)(z). \tag{1.5}$$

A function $f(z) \in A(p)$ is said to be in the class of p-valently strongly starlike functions of order β, denoted by $\overline{S_p^*}(\beta)$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2p} \quad (0 \leq \beta < p;\ p \in \mathbb{N};\ z \in \mathbb{U}).$$

A function $f(z) \in A(p)$ is said to be in the class of p-valently strongly convex functions of order β, denoted by $\overline{K_p}(\beta)$, if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\beta\pi}{2p} \quad (0 \leq \beta < p;\ p \in \mathbb{N};\ z \in \mathbb{U}).$$

A function $f(z) \in A(p)$ is said to be in the class of p-valently strongly close-to-convex functions of order β, denoted by $\overline{CK_p}(\beta)$, if

$$\left|\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}\right| < \frac{\beta\pi}{2p} \quad (0 \leq \beta < p;\ p \in \mathbb{N};\ z \in \mathbb{U}).$$

The classes $\overline{S_p^*}(\beta)$, $\overline{K_p}(\beta)$ and $\overline{CK_p}(\beta)$ were introduced by Sharma and Srivastava [8].

For $f(z) \in A(p)$ of the form (1.1) and $g(z) \in A(p)$ of the form (1.3), we have (see Chen et al. [3])

$$(f*g)^{(m)}(z) = \delta(p,m)\, z^{p-m} + \sum_{k=p+1}^{\infty} \delta(k,m)\, a_k b_k z^{k-m} \quad \left(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right), \tag{1.6}$$

where

$$\delta(p,m) = .1 \qquad (m=0) p(p-1)\ldots(p-m-1) \qquad (m \neq 0). \tag{1.7}$$

Definition 1. *A function $f(z) \in A(p)$ is said to be in the class $K(g,h,p,m,\lambda,\beta)$ if and only if for any functions $g, h \in A(p)$ of the form* (1.3) *and* (1.4)*, respectively, we have*

$$\left|\frac{z(f*g)^{(m+1)}(z) + \lambda z^2 (f*g)^{(m+2)}(z)}{(1-\lambda)(f*h)^{(m)}(z) + \lambda z (f*h)^{(m+1)}(z)} - (p-m)\right| < \beta\}, \tag{1.8}$$

$$(0 \leq \lambda \leq 1;\ 0 < \beta \leq p - m;\ p \in \mathbb{N};\ p > m;\ m \in \mathbb{N}_0;\ z \in \mathbb{U}),$$

or, equivalently,

$$Re\left\{ \frac{z(f * g)^{(m+1)}(z) + \lambda z^2 (f * g)^{(m+2)}(z)}{(1-\lambda)(f * h)^{(m)}(z) + \lambda z (f * h)^{(m+1)}(z)} \right\} > p - m - \beta \tag{1.9}$$

$$(0 \leq \lambda \leq 1;\ 0 < \beta \leq p - m;\ p \in \mathbb{N};\ p > m;\ m \in \mathbb{N}_0;\ z \in \mathbb{U}).$$

Putting $h(z) = g(z)$ *in* (1.8), *we obtain:*

$$K(g, g, p, m, \lambda, \beta) = KL(g, p, m, \lambda, \beta)$$

$$= \left\{ f \in A(p) : \left| \frac{z(f * g)^{(m+1)}(z) + \lambda z^2 (f * g)^{(m+2)}(z)}{(1-\lambda)(f * g)^{(m)}(z) + \lambda z (f * g)^{(m+1)}(z)} - (p - m) \right| < \beta \right\}. \tag{1.10}$$

Remark 1. *Putting* $\lambda = 0$ *in* (1.8) *or* (1.9) *reduces the class* $K(g, h, p, m, \lambda, \beta)$ *to the class* $R_h^g(p, m, \beta)$ *introduced and studied by Sharma and Srivastava* [8].

We can obtain the following new classes for various choices of g, h, λ, p, m *and* β :
(i) $KL(\frac{z^p}{1-z}, p, m, \lambda, \beta) = KL(p, m, \lambda, \beta)$

$$= \left\{ f \in A(p) : \left| \frac{z f^{(m+1)}(z) + \lambda z^2 f^{(m+2)}(z)}{(1-\lambda) f^{(m)}(z) + \lambda z f^{(m+1)}(z)} - (p - m) \right| < \beta \right\};$$

(ii) $KL(g, p, m, 1, \beta) = KL(g, p, m, \beta)$

$$= \left\{ f \in A(p) : \left| 1 + \frac{z(f * g)^{(m+2)}(z)}{(f * g)^{(m+1)}(z)} - (p - m) \right| < \beta \right\};$$

(iii) $KL(z^p + \sum\limits_{k=p+1}^{\infty} \left[\frac{\ell + p + \mu(k-p)}{\ell + p}\right]^n z^k, p, m, \lambda, \beta) = KL(n, \mu, \ell; p, m, \lambda, \beta)$

$$= \left\{ f \in A(p) : \left| \frac{z(I_p^n(\mu, \ell) f(z))^{(m+1)} + \lambda z^2 (I_p^n(\mu, \ell) f(z))^{(m+2)}}{(1-\lambda)(I_p^n(\mu, \ell) f(z))^{(m)} + \lambda z (I_p^n(\mu, \ell) f(z))^{(m+1)}} - (p - m) \right| < \beta \right\},$$

where $n \in \mathbb{N}_0$, $\mu, \ell \geq 0$ *and the operator* $I_p^n(\mu, \ell)$ *was defined by Cătaş* [2], *which is a generalization of many other linear operators considered earlier.*

In order to prove our results, we shall make use of the following lemmas.

Lemma 1. [4] *(Jack's lemma). Let $w(z)$ be analytic in $\mathbb{U}$ and such that $w(0)=0$. Then if $|w(z)|$ attains its maximum value on a circle $|z|=r<1$ at a point $z_0\in\mathbb{U}$, we have*

$$z_0 w'(z_0)=\zeta w(z_0),$$

where $\zeta\geq 1$ is a real number.

Lemma 2. [5]. *Let $\varphi(u,v)$ be a complex valued function:*

$$\varphi : D \longrightarrow \mathbb{C},\ \ (D\subset \mathbb{C}\times\mathbb{C};\ \mathbb{C}\ \text{is a complex plane}),$$

and let $u=u_1+iu_2$ and $v=v_1+iv_2$. Suppose that the function $\varphi(u,v)$ satisfies the following conditions:
(i) $\varphi(u,v)$ is continuous in D;
(ii) $(1,0)\in D$ and $Re\ (\varphi(1,0))>0$;
(iii) $Re\ (\varphi(iu_2,v_1))\leq 0$ for all $(iu_2,v_1)\in D$ and such that $v_1\leq -\left(1+u_2^2\right)/2$.
Let $p(z)=1+p_1z+p_2z^2+...$ be regular in $\mathbb{U}$ such that $(p(z),zp'(z))\in D$ for all $z\in U$. If $Re\left(\varphi\left(p(z),zp'(z)\right)\right)>0\ (z\in\mathbb{U})$, then $Re\left(p(z)\right)>0\ (z\in\mathbb{U})$.

Lemma 3. [7]. *Let a function $p(z)$ be analytic in $\mathbb{U}$, $p(0)=1$ and $p(z)\neq 0\ (z\in\mathbb{U})$. If there exists a point $z_0\in\mathbb{U}$ such that*

$$|\arg p(z)|<\frac{\pi}{2}\delta\ \text{for}\ |z|<|z_0|,$$

and

$$|\arg p(z_0)|=\frac{\pi}{2}\delta,$$

with $0<\delta\leq 1$, then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=it\delta,$$

where

$$t\geq 1\ \text{when}\ \arg p(z_0)=\frac{\pi}{2}\delta,$$

and

$$t\leq -1\ \text{when}\ \arg p(z_0)=-\frac{\pi}{2}\delta.$$

2. Main results for the class $K(g,h,p,m,\lambda,\beta)$

Theorem 1. *Let the functions $f(z), g(z), h(z) \in A(p)$ be defined by (1.1), (1.3) and (1.4), respectively. If*

$$\left|1+\frac{z(1+\lambda)(f*g)^{(m+2)}(z)+\lambda z^2(f*g)^{(m+3)}(z)}{(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)}-\frac{z(f*h)^{(m+1)}(z)+\lambda z^2(f*h)^{(m+2)}(z)}{(1-\lambda)(f*h)^{(m)}(z)+\lambda z(f*h)^{(m+1)}(z)}\right|<\frac{\beta}{p-m+\beta}, \quad (2.1)$$

then $f(z) \in K(g,h,p,m,\lambda,\beta)$.

Proof. Let $w(z)$ be defined by

$$\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{(1-\lambda)(f*h)^{(m)}(z)+\lambda z(f*h)^{(m+1)}(z)}=(p-m)+\beta w(z), \quad (2.2)$$

where $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. Differentiating (2.2) logarithmically with respect to z, we obtain

$$1+\frac{z(1+\lambda)(f*g)^{(m+2)}(z)+\lambda z^2(f*g)^{(m+3)}(z)}{(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)}-\frac{z(f*h)^{(m+1)}(z)+\lambda z^2(f*h)^{(m+2)}(z)}{(1-\lambda)(f*h)^{(m)}(z)+\lambda z(f*h)^{(m+1)}(z)}=\frac{\beta z w'(z)}{(p-m)+\beta w(z)}.$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|}|w(z)|=|w(z_0)|=1 \ \ (w(z_0)\neq 1).$$

By using Lemma 1 and taking $w(z_0)=e^{i\theta}$ $(\theta\neq 0)$, we have

$$\begin{aligned}&\left|1+\frac{z_0(1+\lambda)(f*g)^{(m+2)}(z_0)+\lambda z_0^2(f*g)^{(m+3)}(z_0)}{(f*g)^{(m+1)}(z_0)+\lambda z_0(f*g)^{(m+2)}(z_0)}-\frac{z_0(f*h)^{(m+1)}(z_0)+\lambda z_0^2(f*h)^{(m+2)}(z_0)}{(1-\lambda)(f*h)^{(m)}(z_0)+\lambda z_0(f*h)^{(m+1)}(z_0)}\right|\\ = & \left|\frac{z_0\beta w'(z_0)}{[(p-m)+\beta w(z_0)]}\right|=\frac{\beta\zeta}{\left[(p-m)^2+\beta^2+2\beta(p-m)\cos\theta\right]^{\frac{1}{2}}}\\ \geq & \frac{\beta}{p-m+\beta} \ \ (\zeta\geq 1),\end{aligned}$$

which contradicts the condition (2.1) of Theorem 1. Then we have $|w(z)|<1$ for all $z_0\in\mathbb{U}$. Consequently, we conclude that $f(z)\in K(g,h,p,m,\lambda,\beta)$, which complete the proof of Theorem 1.

Theorem 2. *If $p>m$, then $KL(g,p,m+1,\lambda,\beta)\subset KL(g,p,m,\lambda,\alpha)$, where*

$$0<\alpha\leq\frac{-(p-m-\beta+1)+\sqrt{(p-m-\beta+1)^2+4\beta(p-m)}}{2}\leq p-m. \quad (2.3)$$

Proof. Let $f(z) \in KL(g,p,m+1,\lambda,\beta)$. Then we have

$$\left| \frac{z(f*g)^{(m+2)}(z)+\lambda z^2(f*g)^{(m+3)}(z)}{\lambda z(f*g)^{(m+2)}(z)+(1-\lambda)(f*g)^{(m+1)}(z)} - (p-m-1) \right| < \beta, \tag{2.4}$$

and let $w(z)$ be defined by

$$\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*g)^{(m+1)}(z)+(1-\lambda)(f*g)^{(m)}(z)} - (p-m) = \alpha w(z). \tag{2.5}$$

Clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. Differentiating (2.5) logarithmically with respect to z, we obtain

$$1+\frac{z(1+\lambda)(f*g)^{(m+2)}(z)+\lambda z^2(f*g)^{(m+3)}(z)}{(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)} - \frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{(1-\lambda)(f*g)^{(m)}(z)+\lambda z(f*g)^{(m+1)}(z)} = \frac{\alpha z w'(z)}{(p-m)+\alpha w(z)},$$

that is,

$$\frac{z(1+\lambda)(f*g)^{(m+2)}(z)+\lambda z^2(f*g)^{(m+3)}(z)}{(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)} - (p-m-1) = \alpha w(z)\left[1+\frac{zw'(z)}{w(z)[(p-m)+\alpha w(z)]}\right].$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1\,(w(z_0)\neq 1)\,.$$

By using Lemma 1 and taking $w(z_0)=e^{i\theta}$ $(\theta \neq 0)$, we have

$$\left|\frac{z_0(f*g)^{(m+2)}(z_0)+\lambda z_0^2(f*g)^{(m+3)}(z_0)}{(1-\lambda)(f*g)^{(m+1)}(z_0)+\lambda z_0(f*g)^{(m+2)}(z_0)} - (p-m-1)\right| = |\alpha w(z_0)|\left|1+\frac{z_0 w'(z_0)}{w(z_0)[(p-m)+\alpha w(z_0)]}\right|$$

$$= \alpha\left|1+\frac{\zeta}{[(p-m)+\alpha e^{i\theta}]}\right| \geq \alpha\left\{1+\frac{\zeta}{(p-m)}Re\left[\frac{1+\frac{\alpha}{(p-m)}\cos\theta-\frac{i\alpha}{(p-m)}\sin\theta}{1+\frac{2\alpha}{(p-m)}\cos\theta-\frac{2i\alpha}{(p-m)}\sin\theta+\frac{\alpha^2}{(p-m)^2}}\right]\right\}$$

$$\geq \alpha\left\{1+\frac{1}{(p-m)}\left[\frac{1+\frac{\alpha}{(p-m)}\cos\theta}{1+\frac{2\alpha}{(p-m)}\cos\theta+\frac{\alpha^2}{(p-m)^2}}\right]\right\} = \alpha\left\{1+\frac{1}{(p-m)}\left[\frac{1}{2+\frac{\frac{\alpha^2}{(p-m)^2}-1}{1+\frac{\alpha}{(p-m)}\cos\theta}}\right]\right\}$$

$$\geq \alpha\left\{1+\frac{1}{(p-m)}\left[\frac{(p-m+\alpha)(p-m)}{2(p-m+\alpha)(p-m)+\alpha^2-(p-m)^2}\right]\right\} = \alpha\left[\frac{p-m+\alpha+1}{p-m+\alpha}\right].$$

From (2.3) we have

$$\left| \frac{z(f*g)^{(m+2)}(z)+\lambda z^2(f*g)^{(m+3)}(z)}{\lambda z(f*g)^{(m+2)}(z)+(1-\lambda)(f*g)^{(m+1)}(z)} - (p-m-1) \right| \geq \beta,$$

which contradicts (2.4). Hence, $|w(z)|<1$ for all $z\in\mathbb{U}$. Consequently, we conclude that $f(z)\in KL(g,p,m,\lambda,\alpha)$. Thus, the proof of Theorem 2 is completed.

Theorem 3. *Let $f(z) \in A(p)$. If*

$$Re\left\{\psi\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}+(1-\psi)\,z\left[\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right]'\right\}>\gamma, \tag{2.6}$$

for some $\gamma < \psi(p-m)$, $0 \le \psi \le 1$, $z \in \mathbb{U}$, then $f(z) \in K(g,h,p,m,\lambda,\beta)$, where $\beta = \frac{2[\psi(p-m)-\gamma]}{1+\psi} \le p$.

Proof. If $\psi = 1$, the result holds true. Let $0 \le \psi < 1$ and define the function $p(z)$ by

$$\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}=(p-m-\beta)+\beta p(z). \tag{2.7}$$

Then $p(z) = 1 + p_1 z + p_2 z^2 + ...$ is regular in $\mathbb{U}$. Differentiating (2.7) logarithmically with respect to z, we have

$$1+\frac{z(1+\lambda)(f*g)^{(m+2)}(z)+\lambda z^2(f*g)^{(m+3)}(z)}{(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)}-\frac{z(f*h)^{(m+1)}(z)+\lambda z^2(f*h)^{(m+2)}(z)}{(1-\lambda)(f*h)^{(m)}(z)+\lambda z(f*h)^{(m+1)}(z)}=\frac{\beta zp'(z)}{[(p-m-\beta)+\beta p(z)]},$$

that is,

$$z\frac{\left[\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)\right]}{\left[(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)\right]}\left\{\frac{(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right\}'=\frac{\beta zp'(z)-[(p-m-\beta)+\beta p(z)]}{[(p-m-\beta)+\beta p(z)]},$$

or, equivalently,

$$z^2\left\{\frac{(f*g)^{(m+1)}(z)+\lambda z(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right\}'=\beta zp'(z)-[(p-m-\beta)+\beta p(z)].$$

Therefore, we have

$$Re\left\{\left\{\psi\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}+(1-\psi)\,z\left[\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right]'\right\}-\gamma\right\}$$

$$=Re\left\{\left\{\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}+(1-\psi)\,z^2\left[\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right]'\right\}-\gamma\right\}$$

$$=Re\left\{\psi(p-m-\beta)+\beta\psi p(z)+(1-\psi)\beta zp'(z)-\gamma\right\}>0.$$

If we define a function $\varphi(u,v)$ by

$$\varphi(u,v)=\psi(p-m-\beta)+\beta\psi u+(1-\psi)\beta v-\gamma \tag{2.8}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then
(*i*) $\varphi(u,v)$ is continuous in D;
(*ii*) $(1,0) \in D$ and $Re\,(\varphi(1,0)) = \psi(p-m) - \gamma > 0$;
(*iii*) For all $(iu_2, v_1) \in D$ and such that $v_1 \le -\left(1+u_2^2\right)/2$, we have

$$Re\ (\varphi(iu_2,v_1))=\psi(p-m-\beta)+(1-\psi)\beta v_1-\gamma$$

$$\leq \psi (p-m-\beta) - (1-\psi) \beta \left(1+u_2^2\right)/2 - \gamma \leq -(1-\psi) \beta u_2^2/2 \leq 0.$$

Therefore, $\varphi(u,v)$ satisfies the conditions of Lemma 2. This shows that $Re\,(p(z)) > 0$ $(z \in \mathbb{U})$, that is,

$$Re\left\{\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right\} > p-m-\beta \;\; (z \in \mathbb{U}),$$

which proves that $f(z) \in K(g,h,p,m,\lambda,\beta)$. This completes the proof of Theorem 3.

Theorem 4. *If*

$$\left|\arg\left\{\frac{1}{(p-m)}\left\{\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)} + z\left[\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right]'\right\}\right\}\right|$$

$$< \frac{\delta\pi}{2p} + \tan^{-1}\left(\frac{\delta}{p}\right) \quad (z \in \mathbb{U}), \tag{2.9}$$

then

$$\left|\arg\left\{\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right\}\right| < \frac{\delta\pi}{2p} \;\; (z \in \mathbb{U}).$$

In particular, if

$$\left|\arg\left\{\frac{zf'(z)}{pf(z)}\left[2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]\right\}\right| < \frac{\delta\pi}{2p} + \tan^{-1}\left(\frac{\delta}{p}\right) \quad (z \in \mathbb{U}),$$

then $f(z) \in \overline{S_p^*}(\beta)$.

Proof. Let

$$\varphi(z) = \frac{1}{(p-m)}\left\{\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right\}.$$

Then

$$z\varphi'(z) = \frac{z}{(p-m)}\left\{\frac{z(f*g)^{(m+1)}(z)+\lambda z^2(f*g)^{(m+2)}(z)}{\lambda z(f*h)^{(m+1)}(z)+(1-\lambda)(f*h)^{(m)}(z)}\right\}'.$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg\varphi(z)| < \frac{\delta\pi}{2p} \text{ for } |z| < |z_0|, \;\; |\arg\varphi(z_0)| = \frac{\delta\pi}{2p}.$$

By using Lemma 3, we have

$$\arg\left\{\frac{1}{(p-m)}\left\{\frac{z_0(f*g)^{(m+1)}(z_0)+\lambda z_0^2(f*g)^{(m+2)}(z_0)}{\lambda z_0(f*h)^{(m+1)}(z_0)+(1-\lambda)(f*h)^{(m)}(z_0)} + z_0\left[\frac{z_0(f*g)^{(m+1)}(z_0)+\lambda z_0^2(f*g)^{(m+2)}(z_0)}{\lambda z_0(f*h)^{(m+1)}(z_0)+(1-\lambda)(f*h)^{(m)}(z_0)}\right]'\right\}\right\}$$

$$= \arg\left\{\varphi(z_0) + z_0\varphi'(z_0)\right\} = \arg\left\{\varphi(z_0)\left[1 + \frac{z_0\varphi'(z_0)}{\varphi(z_0)}\right]\right\}$$

$$= \arg\varphi(z_0) + \arg\left(1 + it\frac{\delta}{p}\right) = \arg\left(\varphi(z_0)\right) + \tan^{-1}\left(t\frac{\delta}{p}\right). \qquad (2.10)$$

When $\arg\varphi(z_0) = \frac{\delta\pi}{2p}$, we have

$$\arg\left\{\frac{1}{(p-m)}\left\{\frac{z_0(f*g)^{(m+1)}(z_0)+\lambda z_0^2(f*g)^{(m+2)}(z_0)}{\lambda z_0(f*h)^{(m+1)}(z_0)+(1-\lambda)(f*h)^{(m)}(z_0)} + z_0\left[\frac{z_0(f*g)^{(m+1)}(z_0)+\lambda z_0^2(f*g)^{(m+2)}(z_0)}{\lambda z_0(f*h)^{m+1}(z_0)+(1-\lambda)(f*h)^{(m)}(z_0)}\right]'\right\}\right\}$$

$$\geq \frac{\delta\pi}{2p} + \tan^{-1}\left(\frac{\delta}{p}\right). \qquad (2.11)$$

Similarly, if $\arg\varphi(z_0) = -\frac{\delta\pi}{2p}$, then we have

$$\arg\left\{\frac{1}{(p-m)}\left\{\frac{z_0(f*g)^{(m+1)}(z_0)+\lambda z_0^2(f*g)^{(m+2)}(z_0)}{\lambda z_0(f*h)^{(m+1)}(z_0)+(1-\lambda)(f*h)^{(m)}(z_0)} + z_0\left[\frac{z_0(f*g)^{(m+1)}(z_0)+\lambda z_0^2(f*g)^{(m+2)}(z_0)}{\lambda z_0(f*h)^{(m+1)}(z_0)+(1-\lambda)(f*h)^{(m)}(z_0)}\right]'\right\}\right\}$$

$$\leq -\left(\frac{\delta\pi}{2p} + \tan^{-1}\left(\frac{\delta}{p}\right)\right). \qquad (2.12)$$

We find that (2.11) and (2.12) conradict the condition (2.9). Consequently, we conclude that

$$|\arg\varphi(z)| < \frac{\delta\pi}{2p} \ \ (z \in \mathbb{U}).$$

This completes the proof of Theorem 4.

Remark 2. (i) *Putting* $\lambda = 0$ *in the above results, we obtain the corresponding results of Sharma and Srivastava* [8, *Theorem 3, 4, 5 and 6*];
(ii) *For special choices of* g, h, λ, p, m *and* β, *we can obtain corresponding results for different classes defined in the introduction.*

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M. K. Aouf
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
*E-mail:*mkaouf127@yahoo.com

A. A. Shamandy
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
*E-mail:*aashamandy@hotmail.com

A. O. Mostafa
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
E-mail: adelaeg254@yahoo.com

A. K. Wagdy
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
*E-mail:*awagdyfos@yahoo.com

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