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Discrete Singular Operators and Equations in a Half-Space

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Abstract. Discrete multidimensional singular integral equations with Calderon-Zygmund kernels are considered in a discrete half-space. The solvability of such equations is studied using the properties of discrete Fourier transform and corresponding properties of Calderon-Zygmund operators.

Key Words and Phrases: discrete convolution, Calderon-Zygmund operator, periodic Riemann problem, symbol

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1. Introduction

We consider discrete operator generated by Calderon-Zygmund kernel $K(x)$, which is defined for discrete argument function $u_h(\tilde{x})$, $\tilde{x} \in \mathbf{Z}_h^m$, where $\mathbf{Z}_h^m$ is an integer lattice (modulo h) in $\mathbf{R}^m$. We also consider the corresponding equation

$$au_h(\tilde{x}) + \sum_{\tilde{y}\in\mathbf{Z}_{h,+}^m} K(\tilde{x}-\tilde{y})u_h(\tilde{y})h^m = v_h(\tilde{x}), \quad \tilde{x} \in \mathbf{Z}_{h,+}^m, \tag{1}$$

in discrete half-space $\mathbf{Z}_{h,+}^m = \{\tilde{x} \in \mathbf{Z}_h^m : \tilde{x_m} > 0\}$, $u_h, v_h \in L_2(\mathbf{Z}_{h,+}^m) \equiv l_h^2$.

By definition, we let $K(0) = 0$, and define the symbol of operator

$$u_h(\tilde{x}) \mapsto au(\tilde{x}) + \sum_{\tilde{y}\in\mathbf{Z}_h^m} K(\tilde{x}-\tilde{y})u_h(\tilde{y})h^m, \quad \tilde{x} \in \mathbf{Z}_h^m,$$

as a periodic function

$$\sigma_h(\xi) = a + \sum_{\tilde{x}\in\mathbf{Z}_h^m} e^{-i\xi\tilde{x}}K(\tilde{x})h^m, \tag{2}$$

with period $[-\pi h^{-1};\ \pi h^{-1}]^m$.

The sum in (2) is defined as a limit of partial sums over cubes Q_N

$$\lim_{N\to\infty} \sum_{\tilde{x}\in Q_N} e^{-i\xi\tilde{x}}K(\tilde{x})h^m,$$

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$$Q_N = \left\{ \tilde{x} \in \mathbf{Z}_h^m : \ |\tilde{x}| \le N, \ |\tilde{x}| = \max_{1 \le k \le m} |\tilde{x}_k| \right\}.$$

This reminds us of the symbol of classical Calderon-Zygmund operator [7] defined as a Fourier transform of kernel $K(x)$ in the sense of principal value

$$\sigma(\xi) = \lim_{\substack{N \to \infty \\ \varepsilon \to 0}} \int\limits_{\varepsilon < |x| < N} K(x) e^{i\xi x} dx.$$

It has been shown by earlier studies [9] that the images of $\sigma(\xi)$ and $\sigma_h(\xi)$ coincide with each other, and this makes it possible to treat such equations in more detail in a half-space.

2. Background

The continual analog of equation (1) is the equation

$$au(x) + \int\limits_{\mathbf{R}_+^m} K(x-y)u(y)dy = v(x), \quad x \in \mathbf{R}_+^m, \tag{3}$$

in the space $L_2(\mathbf{R}_+^m)$.

This is a well-studied equation [5]. Using the Fourier transform, it can be reduced to the classical Riemann boundary value problem for upper and lower half-planes [1] with the coefficient $\sigma(\xi', \xi_m)$, where ξ is a dual (in the Fourier sense) variable and $\xi' = (\xi_1, ..., \xi_{m-1})$ is a parameter.

For discrete convolution in a half-axis, one of the authors of this paper showed [8] that such equation is equivalent to certain Riemann boundary value problem in a strip. It is easy to verify that for discrete half-space we have the similar problem in a strip for which the coefficients are defined by symbol $\sigma_h(\xi', \xi_m)$, and ξ' is a parameter. For continual equation we have the Riemann boundary value problem with parameter ξ' and a coefficient defined by $\sigma(\xi', \xi_m)$. The unique solution of this problem is determined by topological index with respect to the variable ξ_m.

The topological index of such problem is determined, roughly speaking, by the variation of the argument of function $\sigma(\cdot, \xi_m)$, as the argument ξ_m varies from $-\infty$ to $+\infty$ and does not depend on $\xi' (m \ge 3)$. The same is true for the discrete equation (1), and its solvability is determined by the variation of argument $\sigma(\cdot, \xi_m)$, as the variable ξ_m varies in the interval $[-\pi h^{-1}, \pi h^{-1}]$.

The key moment is to get the following relation:

$$\lim_{h \to 0} \int\limits_{-\pi h^{-1}}^{\pi h^{-1}} d \arg \sigma_h(\cdot, t) = \int\limits_{-\infty}^{+\infty} d \arg \sigma(\cdot, t). \tag{4}$$

The validity of (4) implies the solvability (or unsolvability) of equations (1) and (3). Based on the results of [2], we can assert that the relation (4) is satisfied at least for continuous symbol $\sigma(\xi)$ on sphere S^{m-1}, if $\sigma(0; +1) = \sigma(0; -1)$.

3. Discrete Convolutions on a Half-Axis

A convolution of two functions f and g on a straight line is defined by the integral

$$(f \star g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy,$$

which exists if $f, g \in L_2(\mathbf{R})$. This is a continual convolution. Discrete convolution is defined in the same way. If f and g are functions of discrete argument, i.e. if they are sequences, then

$$(f \star g)(n) \equiv \sum_{k \in \mathbf{Z}} f(n-k)g(k) \equiv \sum_{k \in \mathbf{Z}} f_{n-k} g_k, \tag{5}$$

$$f_k \equiv f_k, \; g(k) \equiv g_k, \; k \in \mathbf{Z},$$

which exists for $f, g \in l_2$.

The Fourier transform of discrete function is defined by the following formula:

$$(Ff)(\xi) \equiv \tilde{f}(\xi) = \sum_{k \in \mathbf{Z}} f_k e^{-ik\xi}, \; \xi \in [-\pi, \pi].$$

Applying the Fourier transform to (5), we come to the standard formula

$$F(f \star g) = \tilde{f} \cdot \tilde{g},$$

which immediately provides a solvability condition for discrete convolution equation

$$au(n) + \sum_{k \in \mathbf{Z}} M(n-k)u(k) = v(n), \tag{6}$$

where a is a constant, M and v are the given discrete functions, and u is a sought function.

Function $a + \tilde{M}(\xi), \; \xi \in [-\pi, \pi]$, is called a symbol of the equation (6).

Thus, the equation (6) has a unique solution if its symbol never vanishes, $M, v \in l_2$.

The situation gets much more complicated if we suppose that the equation (6) is defined not on the whole space $\mathbf{Z}$, , but only on $\mathbf{Z}_+ = \{0, 1, 2, ...\}$, i.e.

$$au(n) + \sum_{k \in \mathbf{Z}_+} M(n-k)u(k) = v(n), \; n \in \mathbf{Z}_+, \tag{7}$$

where discrete function M is defined on the whole $\mathbf{Z}$, while the given v (and the sought u) are only defined on $\mathbf{Z}_+$.

Consider two projectors:

$$(P_+u)(n) = \begin{cases} u(n), & n \geq 0 \\ 0, & n < 0, \end{cases} \quad (P_-u)(n) = \begin{cases} 0, & n \geq 0 \\ u(n), & n < 0, \end{cases}$$

and discrete convolution operator $M: \ u(n) \longmapsto au(n) + \sum\limits_{k\in\mathbf{Z}_+} M(n-k)u(k)$. Then the equation (7) can be rewritten as follows:

$$P_+ M u_+ = f_+, \tag{8}$$

where functions u_+(the sought one) and f_+ (the given one) are defined on $\mathbf{Z}_+$. It is clear that the equation (8) is equivalent (from the viewpoint of solvability) to so called paired equation

$$(M_1 P_+ + M_2 P_-)U = F, \tag{9}$$

on the whole lattice $\mathbf{Z}$, $M_2 = I$.

The use of discrete Fourier transform leads us to the summation of divergent series

$$\sum_{k\in\mathbf{Z}_+} e^{-ik\xi}. \tag{10}$$

To get rid of divergence, we add a multiplier e^{is} and then pass to the limit as $s \to 0$. As a result, we obtain operator P_+ in terms of Fourier images.

So,

$$\sum_{k\in\mathbf{Z}_+} e^{-ik\xi} e^{iks} = \sum_{k\in\mathbf{Z}_+} e^{-ik(\xi+is)} = \sum_{k\in\mathbf{Z}_+} e^{-ik\zeta}, \ \zeta = \xi + is.$$

The obtained series is convergent and its sum is equal to

$$\sum_{k\in\mathbf{Z}_+} e^{-ik\zeta} = 1/2 - i/2\cot(\zeta/2).$$

Thus,

$$(FP_+u) = 1/2\tilde{u}(\xi) - i/2 \lim_{s\to 0+} \int\limits_{-\pi}^{\pi} \cot\frac{\zeta - \tau}{2}\tilde{u}(\tau)d\tau.$$

Note that we would come to the similar integral (in the sense of principal value) if we summed the series (10) in a usual way (using Dirichlet kernel and passage to the limit in partial sums [4]). That would lead us to the periodical version of Hilbert transform

$$(Hu)(x) = v.p. \int\limits_{-\pi}^{\pi} \cot\frac{x-t}{2} u(t)dt.$$

If projector P_-, is considered, then the sum (9) becomes

$$= i/2 + i/2\cot(\zeta/2),$$

and we get the following formula:

$$(FP_-u) = -1/2\tilde{u}(\xi) + i/2 \lim_{s\to 0+} \int\limits_{-\pi}^{\pi} \cot\frac{\zeta - \tau}{2}\tilde{u}(\tau)d\tau.$$

4. Periodic Riemann Boundary Value Problem

Consider the function

$$\Phi(\zeta) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \cot\frac{\zeta - t}{2}\phi(t)dt,$$

and suppose that $\phi(t)$ satisfies Hölder condition on $[-\pi,\pi]$:

$$|\phi(t_1) - \phi(t_2)| \leq c|t_1 - t_2|^{\alpha},$$

$\forall t_1, t_2 \in [-\pi,\pi],\ 0 < \alpha \leq 1,\ \phi(-\pi) = \phi(\pi)$.

Boundary values ($s \to \pm 0$) can be calculated by passing from $[-\pi,\pi]$ to the unit circumference and applying classical Sokhotskii-Plemelj formulas. As a result, we get

Theorem 1. *The formulas*

$$\Phi^{\pm}(\xi) = \pm\frac{\phi(t)}{2} + \frac{1}{2\pi i} v.p. \int_{-\pi}^{\pi} \cot\frac{\xi - t}{2}\phi(t)dt, \tag{11}$$

are true, where $\Phi^{\pm}(\xi)$ denote the boundary values $\Phi^{\pm}(\zeta)$ as $s \to \pm 0$.

These formulas lead us to the following formulation of periodic Riemann boundary value problem: find the pair of functions $\Phi^{\pm}(z)$, analytic in half-strips

$$\Pi_{\pm} = \{z \in \mathbf{C}:\ z = t + is,\ t \in [-\pi,\pi],\ \ \pm s > 0\},$$

for which their boundary values satisfy linear relation

$$\Phi^{+}(t) = G(t)\Phi^{-}(t) + g(t),\ t \in [-\pi,\pi],$$

as $s \to 0\pm$, where $G(t)$ and $g(t)$ are the given functions on $[-\pi,\pi]$.

If we suppose that $G(t) \in C[-\pi,\pi],\ G(-\pi) = G(\pi)$, then the index of function G on the interval $[-\pi,\pi]$ is defined as the variation of $\arg G(t)$ divided by 2π as t varies from $-\pi$ to π. This is an integer denoted by æ .

Theorem 2. *If $G(t)$ satisfies Hölder condition, æ=0, then the periodic Riemann boundary value problem has a unique solution $\Phi^{\pm}(t) \in L_2[-\pi,\pi]$, which is constructed using function $\Phi(\zeta)$.*

5. Equations in Continual Case and the Classical Riemann Boundary Value Problem

5.1. Half-Axis Case

Reduction of equation (9) to so called characteristic singular integral equation is realized with the help of special Hilbert transform [1], [2], [6]

$$(Hu)(x) \equiv \frac{1}{\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{u(s)}{s-x} ds$$

$$\equiv \frac{1}{\pi i} \lim_{\substack{N\to+\infty \\ \varepsilon\to 0+}} \left(\int\limits_{-N}^{x-\varepsilon} + \int\limits_{x+\varepsilon}^{N} \right) \frac{u(s)}{s-x} ds.$$

The properties of this operator are well-studied. In particular, operator $H : L_2(\mathbf{R}) \to \mathbf{L_2}(\mathbf{R})$ is a bounded linear operator with its spectrum consisting of two points ± 1, and $H^2 = I$.

Besides, the following two operators

$$P = 1/2(I + H),\ Q = 1/2(I - H),$$

are the projectors on the subspace $A(\mathbf{R}) \subset L_2(\mathbf{R})$ of functions admitting analytic extension to the upper complex half-plane $\mathbf{C}_+$ and on the subspace $B(\mathbf{R}) \subset L_2(\mathbf{R})$ of functions admitting analytic extension to the lower complex half-plane $\mathbf{C}_-$, respectively. So

$$A(\mathbf{R}) \oplus B(\mathbf{R}) = L_2(\mathbf{R}).$$

The following identities are true:

$$P^2 = P,\ Q = I - P,\ Q^2 = Q,\ PQ = QP = 0.$$

If we denote by P_+, and P_- the operators of restriction to the positive and negative half axes, respectively, then it is easy to verify [2] that

$$FP_+ = QF,\ FP_- = PF. \tag{12}$$

Next, by applying Fourier transform to one-dimensional equation (9) we get

$$\frac{1}{2}\sigma_{M_1}(\xi)(I - H)\tilde{U}(\xi) + \frac{1}{2}\sigma_{M_2}(\xi)(I + H)\tilde{U}(\xi) = \tilde{F}(\xi),$$

where $\sigma_{M_1}, \sigma_{M_2}$ are the symbols of operators M_1, M_2. By grouping terms, we can rewrite the last equation as follows:

$$\frac{\sigma_{M_1}(\xi) + \sigma_{M_2}(\xi)}{2}\tilde{U}(\xi) +$$

$$+ \frac{\sigma_{M_1}(\xi) + \sigma_{M_2}(\xi)}{2\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{\tilde{U}(\eta)}{\eta - \xi} d\eta = \tilde{F}(\xi). \tag{13}$$

Equation (13) is well-known in the theory of singular integral equations [1]. It is called a characteristic singular integral equation, and its solution is closely related to the classical Riemann boundary value problem for upper and lower half-planes $\mathbf{C}_\pm$. This problem is formulated as follows: finding two functions $\Phi^{\pm}(t)$, defined on $\mathbf{R}$, which admit analytic extension to $\mathbf{C}_{\pm}$ and satisfy the linear relation

$$\Phi^{+}(t) = G(t)\Phi_{-}(t) + g(t), \tag{14}$$

on straight line $\mathbf{R}$, where $G(t)$ and $g(t)$ are the given functions on $\mathbf{R}$. If we denote

$$a(t) = \frac{\sigma_{M_1}(t) + \sigma_{M_2}(t)}{2}, \; b(t) = \frac{\sigma_{M_1}(t) - \sigma_{M_2}(t)}{2},$$

then we will see that the equation (13) in space $L_2(\mathbf{R})$ and the problem (14) for $\Phi^{\pm} \in L_2(\mathbf{R})$ are equivalent [1], i.e. coefficient $G(t)$ and the right hand side $g(t)$ are easily calculated by a and b:

$$G(t) = \frac{a(t) + b(t)}{a(t) - b(t)}, \; g(t) = \frac{\tilde{F}(t)}{a(t) - b(t)},$$

and, vice versa, problem (14) corresponds to the characteristic singular integral equation (13). It is also known [1] that the solvability conditions of equation (13) are determined by certain topological invariants called indices. Note that in our case

$$G(t) = \sigma_{M_1}(t)\sigma_{M_2}^{-1}(t). \tag{15}$$

We assume that the following condition is satisfied for (15). Denote by $\overline{\mathbf{R}}$ the one-point compactification of $\mathbf{R}$ and suppose that $G(t)$ is continuous on $\overline{\mathbf{R}}$ and vanishes nowhere. The variation of argument of $G(t)$ divided by 2π, as t varies from $-\infty$ to $+\infty$, is called the index æ of this function. If æ $= 0$, then the solution of equation (13) is unique and can be written out explicitly using Hilbert transform [1].

5.2. Half-Space Case

Back to equation (9), where M_1, and M_2 are Calderon-Zygmund operators (as in equation (3)) and by P_+, P_- we mean the operators of restriction to the half-space $\mathbf{R}^m_{\pm} = \{x = (x_1, ..., x_m), \;\; \pm x_m > 0\}$.

It is evident that, slightly complemented, the previous reasoning stays true. If we denote by F the Fourier transform (as we did before), then we have the following relations:

$$FP_+ = Q_{\xi'}F, \; FP_- = P_{\xi'}F,$$

$$P = 1/2(I + H_{\xi'}), \; Q = 1/2(I - H_{\xi'}).$$

Here $H_{\xi'}$ is a Hilbert transform in variables $\xi_m, \; \xi' = (\xi_1, ..., \xi_{m-1})$:

$$(H_{\xi'}u)(\xi', \xi_m) \equiv \frac{1}{\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{u(\xi', \tau)}{\tau - \xi_m} d\tau.$$

In such case, the equation (13) turns to the following one with the parameter ξ':

$$\frac{\sigma_{M_1}(\xi',\xi_m)+\sigma_{M_2}(\xi',\xi_m)}{2}\tilde{U}(\xi)+$$

$$+\frac{\sigma_{M_1}(\xi',\xi_m)+\sigma_{M_2}(\xi',\xi_m)}{2\pi i}v.p.\int_{-\infty}^{+\infty}\frac{\tilde{U}(\xi',\eta)}{\eta-\xi_m}d\eta=\tilde{F}(\xi). \tag{16}$$

This equation corresponds to the Riemann boundary value problem (with parameter ξ') with coefficient

$$G(\xi',\xi_m)=\sigma_{M_1}(\xi',\xi_m)\sigma^{-1}_{M_2}(\xi',\xi_m). \tag{17}$$

To ensure the unique solvability of equation (16), the index of $G(\xi',\xi_m)$ with respect to variable ξ_m needs to be equal to 0.

The symbol of Calderon-Zygmund operator has a very specific nature. It is a homogeneous function of degree 0, i.e. it is in fact defined on the unit sphere S^{m-1}. Let $m\geq 3$. Take $\xi'\in S^{m-2}$ and suppose that $G(0,-1)=G(0,+1)$. As ξ_m varies between $-\infty$ to $+\infty$, the function $G(\xi)$ will take values on the arc of large semi-circle joining the points $(0,-1)$ and $(0,+1)$. At the same time, the symbol will take values alongside the closed curve in the complex plane. These curves will be homotopic for different values of ξ' , i.e. they will have the same index æ with respect to 0. The condition æ $=0$ provides the uniqueness of the solution of equation (16).

6. Back To Discrete Case

We are back to discrete equations, assuming that $P_\pm$ in (9) are the operators of restriction to $\mathbf{Z}^m_{h,\pm}$, and M_1, M_2 are discrete Calderon-Zygmund operators generated by kernels $M_1(x)$, and $M_2(x)$, which are bounded in space $L_2(\mathbf{Z}^m_h)$.

Discrete Fourier transform for discrete argument functions defined on lattice $\mathbf{Z}^m_h$ is given by the formula

$$u(\tilde{x})\longmapsto\frac{1}{(2\pi)^m}\sum_{\tilde{x}\in\mathbf{Z}^m_h}u(\tilde{x})e^{-i\tilde{x}\cdot\xi}h^m\equiv\tilde{u}(\xi),\ \xi\in[-h^{-1}\pi,h^{-1}\pi]^m.$$

Such Fourier transform has the same properties as the classical one [3].

In accordance with Theorem 1 and Section 5, we define the periodical analog of Hilbert transform with respect to the variable ξ_m ($\xi\in[-\pi,\pi]^m,\ \xi'$ is fixed) by the formula

$$(H^{per}_{\xi'}u)(\xi_m)=\frac{1}{2\pi i}\int_{-\pi h^{-1}}^{\pi h^{-1}}u(t)\cot\frac{h(t-\xi_m)}{2}dt. \tag{18}$$

Periodical analogs of projectors (12) look as follows:

$$P^{per}_{\xi'}=1/2(I+H^{per}_{\xi'}),\ Q^{per}_{\xi'}=1/2(I-H^{per}_{\xi'}).$$

And the periodical analog of equation (16) will be

$$\frac{\sigma_{1,h}(\xi',\xi_m)+\sigma_{2,h}(\xi',\xi_m)}{2}\tilde{U}(\xi)+$$
$$+\frac{\sigma_{1,h}(\xi',\xi_m)+\sigma_{2,h}(\xi',\xi_m)}{4\pi i}\times$$
$$\times v.p.\int\limits_{-\pi h^{-1}}^{\pi h^{-1}}\tilde{U}(\xi',\eta)\cot\frac{h(\eta-\xi_m)}{2}d\eta=\tilde{F}(\xi), \tag{19}$$

where $\sigma_{1,h}, \sigma_{2,h}$ are the symbols (2) of discrete operators M_1, M_2. Of course, equation (19) is related to the corresponding Riemann boundary value problem, and the unique solvability condition for this problem is given in Theorem 2. In our case, this condition is

$$Ind\ \sigma_{1,h}(\cdot,\xi_m)\sigma_{2,h}^{-1}(\cdot,\xi_m)=0.$$

7. Passage From Discrete Case To Continual One

First we recall that the images of symbols σ and σ_h coincide with each other [9]. Moreover, index is an integer-valued characteristic in both continual (if the transmission condition $\sigma(0,-1)=\sigma(0,+1)$ is satisfied) and discrete (periodical) cases. Analyzing variations $\arg\sigma_h(\cdot,\xi_m)$ alongside the arcs of large semi-circumferences on S^{m-1} and taking into account that

$$\lim_{h\to 0}\sigma_h(\xi)=\sigma(\xi),\ \forall\xi\in S^{m-1},$$

we arrive at the conclusion that the following theorem is true:

Theorem 3. *The equations (1) and (3) are either both solvable or both unsolvable.*

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