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Generalizations of Prime Ideals of Semirings

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Abstract. In this paper, we analyze some properties and possible structures of almost prime ideals of a commutative semiring with non-zero identity.

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1. Introduction

Almost prime ideals in commutative rings with non-zero identity arise from the study of factorization in Noetherian domains. They were introduced by S. M. Bhatwadekar and P. K. Sharma in [4]. Weakly prime ideals arise from the study of factorization in commutative rings with zero divisors. They were studied by D. D. Anderson and E. Smith in [1]. Also, almost prime ideals and weakly prime ideals have been studied by D. D. Anderson and M. Bataineh in [2]. A proper ideal P of a commutative ring is almost prime if $ab \in P - P^2$ implies $a \in P$ or $b \in P$. A proper ideal P of a commutative ring is weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. As every prime ideal is a weakly prime ideal, and a weakly prime ideal is an almost prime ideal, weakly prime ideals and almost prime ideals are both generalizations of prime ideals. However, since $\{0\}$ is always a weakly prime ideal (by definition) and hence almost prime ideal, as is every proper ideal of a local ring (R, P) (recall that a local ring R is a commutative ring with unique maximal ideal P) with $P^2 = 0$, weakly prime ideals and almost prime ideals need not be prime (see [1] and [2]).

The concept of semiring was first introduced by H. S. Vandiver in 1935 and has since then been studied by many authors. Semirings constitute a fairly natural generalization of rings, with broad applications in the mathematical foundations of computer science (see [9] and [10]). In this paper we explore various properties of almost prime ideals over a commutative semiring with identity (see Definition 1 and [4]). We give several equivalent conditions for an ideal of a semiring to be almost prime ideal which are reminiscent of conditions for an ideal to be prime, and explore the relationship between almost prime and weakly prime ideals. In fact, this paper is concerned with generalizing some results of almost prime ideals and weakly prime ideals listed in [2], from commutative rings theory to commutative semiring theory (see Section 2).

For the sake of completeness, we state some definitions and notations which are used throughout the paper. A commutative semiring R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b+c) = ab + ac$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings. A semiring R is said to be a semidomain if $ab = 0$ $(a, b \in R)$, then either $a = 0$ or $b = 0$. A semifield is a semiring in which non-zero elements form a group under multiplication.

A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal (= k-ideal) K is an ideal such that if $x, x + y \in K$ then $y \in K$(so $\{0\}$ is a k-ideal of R). The k-closure $\mathrm{cl}(I)$ of I is defined as $\mathrm{cl}(I) = \{a \in R : a+c = d \text{ for some } c, d \in I\}$ which is an ideal of R satisfying $I \subseteq \mathrm{cl}(I)$ and $\mathrm{cl}(\mathrm{cl}(I)) = \mathrm{cl}(I)$. So an ideal I of R is a k-ideal if and only if $I = \mathrm{cl}(I)$. A prime ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that $R = \cup\{q+I : q \in Q\}$ and if $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. Allen [3] has presented the notion of Q-ideal I in the semiring R and constructed the quotient semiring R/I (also see [5] and [7]).

2. Almost prime ideals

Our starting point is the following remark:

Remark 1. *Let R be a semiring.*

(i) If an ideal of R is the union of two k-ideals, then it is equal to one of them.

(ii) Let I and J be ideals of R. The product of I and J, denoted by IJ, is defined to be the ideal of R generated by the set $\{ab : a \in I, b \in J\}$. Then we have $IJ = \{\sum_{i=1}^{n} a_i b_i : n \text{ is a positive integer}, a_1, ..., a_n \in I, b_1, ..., b_n \in J\}$. In particular, I^n is an ideal of R for every positive integer n.

(iii) Assume that I, J are ideals of R with I being a k-ideal and let $x \in R$. It is easy to see that $(I : J) = \{r \in R : rJ \subseteq I\}$, $(0 : x)$ and $(I : x)$ are k-ideals of R.

Definition 1. *Let R be a commutative semiring.*

(i) An n-almost prime $(n \geq 2)$ ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P - P^n$. In particular, the almost prime ideals are just the 2-almost prime ideals.

(ii) A proper ideal P of R is weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$.

Lemma 1. *Let R be a semiring. Then every weakly prime k-ideal of R is an almost prime ideal.*

Proof. Assume that P is a weakly prime k-ideal of R and let $x, y \in R$ such that $xy \in P - P^2$ with $x \notin P$. Clearly, $P \cup (0 : x) \subseteq (P : x)$. Let $z \in (P : x)$. If $zx \neq 0$, then P weakly prime gives $z \in P$. If $zx = 0$, then we have $z \in (0 : x)$; hence, $P \cup (0 : x) = (P; x)$.

Therefore, $(P:x)=P$ or $(P:x)=(0:x)$ by Remark 1(i). Since $y\notin(0:x)$ and $y\in(P:x)$, we must have $y\in P$, as required. ◄

Let R be a semiring. A non-zero element a of R is said to be semi-unit in R if there exist $r,s\in R$ such that $1+ra=sa$. R is said to be a local semiring if and only if R has a unique maximal k-ideal. Moreover, a is a semi-unit of R if and only if a lies outside each maximal k-ideal of R (see [6, Lemma 4]). Recall that if I is a proper Q-ideal of R, then there exists a maximal k-ideal P of R with $I\subseteq P$ (see [6, Theorem 3]). Clearly, a prime ideal is a weakly prime ideal, so it is an almost prime ideal by Lemma 1. The ideal $\{0\}$ is always weakly prime and hence almost prime, but it is prime if and only if R is a semidomain. Thus, weakly prime ideals and almost prime ideals need not be prime ideals. Moreover, an idempotent ideal I $(I=I^2)$ is almost prime. We next give a non-trivial example of an almost prime ideal which is not a prime.

Example 1. *Let (R,P) be a local semiring with $P^2=0$. Let I be a proper k-ideal of R such that $0\neq ab\in I$. Since $P^2=0$ and $ab\neq 0$, either a or b does not lie in P. If $a\notin P$, then a is a semi-unit; hence, $1+ra=sa$ for some $r,s\in R$. Therefore, $b+rab=sab$, and so $b\in I$ since I is a k-ideal of R. Then every proper k-ideal of I is weakly prime and hence almost prime. However, if $I\subset P$, I is not prime since P is the unique prime k-ideal of R.*

Example 2. *(i) Let $R=\{0,1,...,n\}$ and define $x+y=\max\{x,y\}$ and $xy=\min\{x,y\}$ for each $x,y\in R$. R together with the two defined operations forms a semiring with an identity element n. If $m\in R$, then the set $P=\{r\in R:r\leq m\}$ is an ideal of R. It is clear from the definition of addition in R that $0+P=P$ and $s+P=\{s\}$ for each $s>m$. Thus, P is a Q-ideal when $Q=\{0\}\cup\{s\in R:s>m\}$. Moreover, it is easy to see that $P=P^t$ is a Q-ideal (so k-ideal) of R for every positive integer t. Moreover, P is an almost prime ideal of R by definition.*

(ii) Let R denote the semiring of the set of all non-negative integers with the usual addition and multiplication. Let P denote the ideal generated by 6. Since $1\in R$, it follows that $P=\{6k:k\in R\}$. As $2\notin P$, $3\notin P$ and $2.3=6\in P$, it is clear that P is not prime. Moreover, since $3\notin P$, $8\notin P$ and $3.8=24\in P-P^2$, we get P is not almost prime.

Theorem 1. *Assume that I is a Q-ideal of a semiring R and let P be an almost prime k-ideal of R such that $I\subseteq P$. If P^2 is a k-ideal of R, then P/I is an almost prime ideal of R/I.*

Proof. Recall that $P/I=\{q+I:q\in P\cap Q\}$ by [5, Proposition 2.2]. By [5, Lemma 2.12], P^2+I is a k-ideal of R. First we show that $(P^2+I)/I=(P/I)^2$. Let $X=\sum_{i=1}^n a_ib_i\in(P/I)^2$ for some $a_i,b_i\in P/I$. It suffices to show that for each i $(1\leq i\leq n)$, $a_ib_i\in(P^2+I)/I$. Since I is a Q-ideal of R, there are elements $q_i,q_i'\in Q\cap P$ such that $a_i=q_i+I$ and $b_i=q_i'+I$, so there is a unique element t of Q such that $a_ib_i=t+I$, where $q_iq_i'+I\subseteq t+I$; hence, $q_iq_i'+a=t+b$ for some $a,b\in I$. Therefore, $t\in(P^2+I)\cap Q$ and so $a_ib_i\in(P^2+I)/I$. Thus, $(P/I)^2\subseteq(P^2+I)/I$. For the reverse inclusion assume that

$q + I \in (P^2 + I)/I$, where $q \in (P^2 + I) \cap Q$. Then there exist $c_i, d_i \in P$ $(1 \leq i \leq m)$ and $c \in I$ such that $q = \sum_{i=1}^{m} c_i d_i + c$. Since I is a Q-ideal and P is a k-ideal of R, there are elements $u_i, w_i \in P \cap Q$ such that $q = \sum_{i=1}^{m} u_i w_i + d$ for some $d \in I$. An inspection will show that $q + I = \sum_{i=1}^{m} (u_i + I) \odot (w_i + I) \in (P/I)^2$, and so we have equality.

Let $q_1 + I, q_2 + I \in R/I$ such that $(q_1 + I) \odot (q_2 + I) \in P/I - (P/I)^2$, where $q_1, q_2 \in Q$. Then there exists the unique element $q_3 \in Q$ such that $q_1 q_2 + I \subseteq q_3 + I \in P/I - (P^2 + I)/I$, so $q_3 \in P \cap Q$; hence, $q_1 q_2 \in P$ and $q_1 q_2 \notin P^2 + I$ and then $q_1 q_2 \notin P^2$. Since P is almost prime, then $q_1 \in P$ or $q_2 \in P$; hence, $q_1 + I \in P/I$ or $q_2 + I \in P/I$, as required.◄

Let R be a given semiring, and let S be the set of all multiplicatively cancellable elements of R (so $1 \in S$). For the structure of the semiring of fractions R_S of R with respect to S we refer the readers to [9, 11] (also see [5]).

Theorem 2. *Let P be an almost prime ideal of a semiring R with $P \cap S = \emptyset$. Then PR_S is an almost prime ideal of R_S.*

Proof. Let $r/s, t/u \in R_S$ such that $(r/s)(t/u) \in PR_S - (PR_S)^2$. Then there exist $p \in P$ and $w \in S$ such that $(rt)/(su) = p/w$, so $rtw = psu \in P$. Moreover, $rtz \notin P^2$ for every $z \in S$ and so $rtw \in P - P^2$. Then P almost prime gives $wt \in P$ or $r \in P$; hence, either $r/s \in PR_S$ or $t/u \in PR_S$ (see [6, Lemma 6]), as required.◄

Let I be a Q-ideal of a semiring R. An element $c \in R$ is called a zero divisor in R/I if there exists $d \in R - I$ such that $cd \in I$. An ideal I of R is called invertible if there is an ideal J of R (denoted by J^{-1}) such that $IJ = R$.

Proposition 1. *Let R be a semiring . If I, J are k-ideals of R with $IJ = I \cap J$, then IJ is a k-ideal of R.*

Proof. (i) It suffices to show that $IJ = \mathrm{cl}(IJ)$. Since the inclusion $IJ \subseteq \mathrm{cl}(IJ)$ is clear, we will prove the reverse inclusion. Let $z \in \mathrm{cl}(IJ)$. Then $z + c = d$ for some $c, d \in IJ$; hence, $z \in I \cap J = IJ$, and we have equality. ◄

Proposition 2. *Let P be an n-almost prime Q-ideal of a semiring R such that P^n is a k-ideal. Then the following hold:*

(i) If $x \in R$ is a zero divisor in R/P, then either $x \in P$ or $xP \subseteq P^n$.

(ii) If for any ideal I of R, $I \subseteq P$ and I consists of zero divisors on R/P, then $IP^{n-1} = P^n$.

(iii) If P is invertible, then P is a prime k-ideal of R.

Proof. (i) By assumption, there exists $y \in R - P$ with $xy \in P$. We may assume that $x \notin P$. Then $xy \in P^n$ since P is n-almost prime. It suffices to show that $xp \in P^n$ for every $p \in P$. So suppose that $p \in P$. Then $p + y \notin P$ and $x(p + y) \in P$ since every Q-ideal is a k-ideal. Hence, as P is n-almost prime, $x(y + p) \in P^n$. Thus, as $xy \in P^n$, $xp \in P^n$ since P^n is a k-ideal. Consequently, $xP \subseteq P^n$.

(ii) Let $x \in I$ and $y \in P^{n-1}$. It suffices to show that $xy \in P^n$. Since x is a zero divisor of R/P, then by (i) either $x \in P$ or $xP \subseteq P^n$. If $x \in P$, then the result is clear. So suppose that $xP \subseteq P^n$. It follows that $xy \in xP^{n-1} \subseteq xP \subseteq P^n$.

(iii) Let $xy \in P$ and $y \notin P$. Then if $y \in P$, we are done. So suppose that $y \notin P$, then $x \notin P$ and $y \notin P$, but $xy \in P$. So y is a zero divisor in R/P. This implies $yP \subseteq P^n$ by (i). As P is invertible, $yPP^{-1} \subset P^{-1}P^n$, so $Ry \subseteq P^{n-1}$ implies $y \in P^{n-1} \subseteq P$, which is a contradiction. Thus, P is a prime k-ideal of R. ◄

Theorem 3. *Let R be a local semiring with unique maximal k-ideal P and let I be a Q-ideal of R such that $P^2 \subseteq I \subseteq P$ and I^2 is a k-ideal. Then I is almost prime if and only if $P^2 = I^2$.*

Proof. Let I be an almost prime ideal. As $P^2 \subseteq I$, for any $x, y \in P$, $xy \in P^2 \subseteq I$. We will show that $xy \in I^2$. If not, then I almost prime gives $x \in I$ or $y \in I$. Let $x \in I$. Then $y \notin I$, since otherwise $xy \in I^2$. Now as $y^2 \in P^2 \subseteq I$, y is a zero divisor in R/I. Hence, by Propositoin 2, $xy \in yI \subseteq I^2$ is a contradiction. Thus $P^2 - I^2$. Conversely, assume that $P^2 = I^2$. Let $x, y \in R$ with $xy \in I - I^2$. If $x \notin P$, then it is a semi-unit in R, so $1 + sx = tx$ for some $s, t \in R$; hence, $y + sxy = txy$. It follows that $y \in I$ since I is a k-ideal. Thus, assume $x, y \in P$. In this case $xy \in P^2 = I^2$, which is not true. Therefore it is clear that I is almost prime. ◄

An ideal I of a semiring R is said to be a strong ideal if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$. Let $R = \{0, 1, 2, ..., 20\}$, and define $a + b = \max\{a, b\}$, $a.b = \min\{a, b\}$ for each $a, b \in R$. Then $(R, +, .)$ is easily checked to be a commutative semiring with 20 as identity. Let J_4 denote the ring integer modulo 4. Let $J_4 \oplus R = \{(a, b) : a \in J_4, b \in R\}$ denote the direct sum of semirings J_4 and R. Then $J_4 \oplus R$ is a commutative semiring. An inspection will show that $I_0 = \{(a, 0) : a \in J_4\}$ is a proper strong ideal in $J_4 \oplus R$ and $I_{10} = \{(0, n) : n \leq 10\}$ is a proper ideal of $J_4 \oplus R$ which is not a strong ideal. Also, since $\{0\}$ is a proper strong k-ideal of R, the set Δ of all proper strong k-ideals of R is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ, and by applying Zorn's Lemma to this partially ordered set we obtain that a strong maximal k-ideal of R is just a maximal member of the partially ordered set $(\Delta, \subseteq)$. An ideal P of R is called strong maximal k-ideal if it is maximal in the lattice of strong k-ideals of R. A semiring R is said to be a strong local semiring if and only if R has a unique strong maximal k-ideal.

Theorem 4. *Let (R, P) be a strong local semiring with every proper principal ideal almost prime. Then $P^2 = 0$.*

Proof. Let $a, b \in P - \{0\}$. Consider the ideal $< ab >$. Suppose that $< ab > \neq 0$. Then $< ab >$ is almost prime, so $ab \in < ab >$. Then either $a \in < ab >$, $b \in < ab >$ or $ab \in < ab >^2$. Suppose first that $a \in < ab >$. Then $a = rab$ for some $r \in R$. By assumption, there is an element $b' \in P$ such that $b + b' = 0$, so $a(1 + rb') = 0$. By [8, Lemma 3.4], $1 + rb'$ is a semi-unit; hence, $1 + (1 + rb')s = (1 + rb')t$. Therefore, $< a > = 0$.

Similarly, if $b \in < ab >$, then $< b >= 0$ and if $ab \in < ab >^2$, then $< ab >= 0$. So in any case we get a contradiction to our assumption that $a \neq 0$, $b \neq 0$, and $ab \neq 0$.◄

A semiring R is called cancellative if whenever $ac = ab$ for some elements a, b, and c of R with $a \neq 0$, then $b = c$.

Theorem 5. *Let R be a strong local semiring with unique maximal k-ideal P. If every proper principal ideal of R is almost prime, then R is semifield.*

Proof. It suffices to show that $P = 0$. Suppose not. By Theorem 4, there is a non-zero element $a \in P$ such that $0 \neq < a >$ is an almost prime ideal of R with $a \in < a >$. So $a = ra$ for some $r \in R$. If $r \in P$, then $a = 0$, which is not true. If $r \notin P$, then r is a semi-unit by [6, Lemma 1], so $1 + rs = tr$ for some $t, s \in R$, which is a contradiction. So suppose that $a^2 \in < a^2 >^2$. By a similar argument, we have a^2 is a semi-unit in R, which is not true. Thus $P = 0$, and the proof is complete.◄

Lemma 2. *Let R be a semiring and $a \in R$. Then $\mathrm{cl}(Ra^2) = (\mathrm{cl}(Ra))^2$. In particular, $(\mathrm{cl}(Ra))^2$ is a k-ideal of R.*

Proof. Let $y \in \mathrm{cl}(Ra^2)$. Then $y + ra^2 = sa^2$ for some $r, s \in R$. Since $ra^2, sa^2 \in (Ra)^2 \subseteq (\mathrm{cl}(Ra))^2$, we have $\mathrm{cl}(Ra^2) \subseteq (\mathrm{cl}(Ra))^2$. For the reverse inclusion, assume that $x = st^2 \in (\mathrm{cl}(Ra))^2$, where $t \in \mathrm{cl}(Ra)$ and $s \in R$. Then there exist $u, v \in R$ such that $t + ua = va$, so $st^2 + 2su^2a^2 + 2stua = sv^2a^2 + su^2a^2$ and $2stua + 2su^2a^2 = 2suva^2$; hence, $st^2 + 2suva^2 = (sv^2 + su^2)a^2$. Therefore, $x \in \mathrm{cl}(Ra^2)$, and we have equality.◄

Theorem 6. *Let R be a cancellative semiring and $x \in R$. Then $\mathrm{cl}(Rx)$ is almost prime if and only if $\mathrm{cl}(Rx)$ is a prime ideal of R.*

Proof. Assume that $\mathrm{cl}(Rx)$ is almost prime and let $a, b \in R$ such that $ab \in \mathrm{cl}(Rx)$, but $a \notin \mathrm{cl}(Rx)$ and $b \notin \mathrm{cl}(Rx)$. Then $\mathrm{cl}(Rx)$ almost prime gives $ab \in (\mathrm{cl}(Rx))^2$, so $a(b + x) \in \mathrm{cl}(Rx)$ and $a, b + x \notin \mathrm{cl}(Rx)$ since it is a k-ideal of R. It follows from Lemma 2 that $a(b + x) \in (\mathrm{cl}(Rx))^2 = \mathrm{cl}(Rx^2)$; hence, $ax \in \mathrm{cl}(Rx^2)$, and this implies that $ax + rx^2 = sx^2$ for some $r, s \in R$, so $a + rx = sx$, which is a contradiction. Thus, $\mathrm{cl}(Rx)$ is a prime ideal of R. The converse is trivial for all semirings R.◄

If R_1 and R_2 are semirings, then $R = R_1 \times R_2 = \{(r_1, r_2) : r_1 \in R_1, r_2 \in R_2\}$ is a semiring under coordinate-wise multiplication. A semiring R is called decomposable if $R = R_1 \times R_2$ for some non-trivial semirings R_1, R_2. Otherwise, R is said to be indecomposable. Now we show that almost prime ideals are really only of interest in indecomposable semirings.

Theorem 7. *Let R_1 and R_2 be semirings. An ideal I of $R = R_1 \times R_2$ is almost prime if and only if I has one of the following forms:*

(i) $I = P_1 \times R_2$ for some almost prime ideal P_1 of R_1.

(ii) $I = R_1 \times P_2$ *for some almost prime ideal* P_2 *of* R_2.

(iii) $I = P_1 \times P_2$ *for some idempotent ideals* P_1 *and* P_2 *of* R_1 *and* R_2*, respectively.*

Proof. Let $I = P_1 \times P_2$ be almost prime ideal of R, where P_1 is an ideal of R_1 and P_2 is an ideal of R_2, so $I \neq R$. We split the proof into two cases.

Case 1: $P_2 = R_2$. It suffices to show that P_1 is an almost prime ideal of R_1. Let $x, y \in R_1$ such that $xy \in P_1 - P_1^2$. Then $(x,1)(y,1) \in (P_1 - P_1^2) \times R_2 = I - I^2$; hence, I almost prime gives either $x \in P_1$ or $y \in P_1$. Therefore, P_1 is an almost prime ideal of R_1. Similarly, if $P_1 = R_1$, then P_2 is an almost prime ideal of R_2.

Case 2: $P_1 \neq R_1$ and $P_2 \neq R_2$. If $P_1 \neq P_1^2$, then there is an element $x \in P_1$ such that $x \notin P_1^2$. Then $(x,1)(1,0) = (x,0) \in I - I^2 = ((P_1 - P_1^2) \times P_2) \cup (P_1 \times (P_2 - P_2^2))$, so either $1 \in P_1$ or $1 \in P_2$, which is a contradiction. The similar reasoning is true for $P_2 = P_2^2$.

Conversely, assume that $I = P_1 \times R_2$, where P_1 is an almost prime ideal of R_1; we show that I is an almost prime ideal of R. Let $(r_1, r_2), (s_1, s_2) \in R$ such that $(r_1 s_1, r_2 s_2) \in I - I^2 = (P_1 - P_1^2) \times R_2$, so $r_1 s_1 \in P_1 - P_1^2$; hence, P_1 almost prime gives either $r_1 \in P_1$ or $s_1 \in P_1$ and therefore either $(r_1, r_2) \in I$ or $(s_1, s_2) \in I$. Thus, I is almost prime. The similar reasoning is true for $I = R_1 \times P_2$, where P_2 is an almost prime ideal of R_2. Finally, suppose that $I = P_1 \times P_2$, where $P_1 = P_1^2$ and $P_2 = P_2^2$. Then $I = I^2$; hence, I is almost prime by definition, as needed. ◄

We next give three other characterizations of n-almost prime ideals.

Theorem 8. *For a proper* k*-ideal* P *of a semiring* R *the following statements are equivalent:*

(i) P *is* n*-almost prime.*

(ii) For $x \in R - P$, $(P : x) = P \cup (P^n : x)$.

(iii) For $x \in R - P$, $(P : x) = P$ *or* $(P : x) = (P^n : x)$.

(iv) For ideals I, J *of* R *with* $IJ \subseteq P$ *and* $IJ \nsubseteq P^n$, $I \subseteq P$ *or* $J \subseteq P$.

Proof. *(i)* $\to$ *(ii)*. Let $a \in (P : x)$ where $x \in R - P$. Then $ax \in P$. If $ax \in P^n$, then $a \in (P^n : x)$; while if $ax \notin P^n$, then P n-almost prime gives $a \in P$. So $(P : x) \subseteq P \cup (P^n : x)$. As the reverse containment holds for any ideal P, we have equality. *(ii)* $\to$ *(iii)* follows from Remark 1 (i).

(iii) $\to$ *(iv)* Suppose on contrary that $I \nsubseteq P$ and $J \nsubseteq P$. Then there exists $b \in I - P$ such that $bJ \subseteq P$ and hence $J \subseteq (P : b)$, but $J \nsubseteq P$, so by (iii), $(P : b) = (P^n : b)$ and therefore $J \subseteq (P^n : b)$ which implies that $bJ \subseteq P^n$. Similarly, there is an element $c \in J - P$ such that $cI \subseteq P^n$. Finally, for any $b \in I \cap P$ and $c \in J \cap P$ we must have $bc \in P^n$. Therefore, $IJ \subseteq P^n$, which is a contradiction. Thus, $I \subseteq P$ or $J \subseteq P$.

(iv) $\to$ *(i)* Assume that $ab \in P - P^n$ $(a, b \in R)$ and let $I = Ra$ and $J = Rb$. Then $IJ \subseteq P$, but $IJ \nsubseteq P^n$. By (iv), either $I \subseteq P$ or $J \subseteq P$ and this implies that P is n-almost prime ideal of R. ◄

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