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A Type of Limit Theorems for Random Truncated Functions Sequence of mth-Order Nonhomogeneous Markov Chains on Generalized Random Selection Systems

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Abstract. In this paper, a class of strong limit theorems for the sequence of multivariate random truncated functions of an mth-order nonhomogeneous Markov chain on the generalized random selection system is established by constructing the consistent distribution functions and a nonnegative super-martingale. As corollaries, some strong laws of large numbers represented by inequalities for the mth-order Markov chain on the generalized random selection system are obtained.

Key Words and Phrases: mth-order nonhomogeneous Markov chain; truncated function sequence; strong laws of large numbers; generalized random selection system; nonnegative super-martingale.

2000 Mathematics Subject Classifications: 60F15

1. Introduction

Suppose that $\{X_n, n \geq 0\}$ is an arbitrary stochastic sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ which takes values in a countable alphabet set $S = \{s_0, s_1, s_2, \cdots\}$, with the joint distribution:

$$P(X_0 = x_0, \cdots, X_n = x_n) = p(x_0, \cdots, x_n) > 0, \quad x_i \in S, \quad 0 \leq i \leq n. \tag{1}$$

We have by the definition of the conditional probability

$$p(x_0, \cdots, x_n) = p(x_0) \prod_{k=1}^{n} p(x_k | x_0, \cdots, x_{k-1}). \tag{2}$$

Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain on the measure P, with the m-dimensional initial distribution and mth-order transition probabilities defined as follows:

$$q_o(i_0, \cdots, i_{m-1}) = P(X_0 = i_0, \cdots, X_{m-1} = i_{m-1}), \quad i_0, \cdots, i_{m-1} \in S. \tag{3}$$

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$$p_n(j|i_1,\cdots,i_m) = P_n(X_n = j|X_{n-m} = i_1,\cdots,X_{n-1} = i_m), \quad i_1,\cdots,i_m, j \in S. \tag{4}$$

The joint distribution of $\{X_n, n \geq 0\}$ with respect to the measure P is

$$p(x_0,\cdots,x_n) = q_o(x_0,\cdots,x_{m-1}) \prod_{k=m}^{n} p_k(x_k|x_{k-m},\cdots,x_{k-1}),$$

$$x_i \in S, \quad 0 \leq i \leq n. \tag{5}$$

Lemma 1. *Suppose that p and q are two arbitrary probability measures, $\{\sigma_n, n \geq 0\}$ is a nonnegative stochastic sequence and $\sigma_n \uparrow \infty$. Then*

$$\limsup_{n\to\infty} \frac{1}{\sigma_n} \log \frac{q(X_0,\cdots,X_n)}{p(X_0,\cdots,X_n)} \leq 0. \qquad P-a.s. \tag{6}$$

The conception of random selection derives from gambling. We consider a sequence of Bernoulli trials, and suppose that at each trial the bettor has the free choice of whether or not to bet. A theorem on gambling systems asserts that under any non-anticipative system the successive bets form a sequence of Bernoulli trials with unchanged probability for success. The importance of this statement was recognized by von Mises, who introduced the impossibility of a successful gambling system as a fundamental axiom (see [11], [12]). This topic was discussed further by Kolmogorov (see[13]) and Liu and Wang (see [14] and [15]).

In order to explain the conception of random selection, which is the crucial part of the gambling system, we first give the notion of the generalized random selection system as following:

Definition 1. *Let $f_n(x_1,\cdots,x_n)$ be a set of real-valued functions defined on $S^n (n = 1,2,\cdots)$, which will be called the generalized random selection functions if they take values in the set $[0,b]$. Denote*

$$Y_1 = y \ (y \textit{ is an arbitrary real number})$$

$$Y_{n+1} = f_n(X_1,\cdots,X_n), \quad n \geq 1, \tag{7}$$

where $\{Y_n, n \geq 1\}$ is called the generalized gambling system (the generalized random selection system). The traditional random selection system $\{Y_n,\ n \geq 0\}$[4] takes values in the set of $\{0,1\}$. Let $\delta_i(j)$ be the Kronecker delta function on S, that is for $i,j \in S$

$$\delta_i(j) = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

In order to explain the real meaning of the notion of the random selection, we consider the traditional gambling model. Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain, and $\{g_n(x_0,\cdots,x_m), n \geq m\}$ be a real-valued function sequence defined on

S^{m+1}. Interpret X_n as the result of the nth trial, the type of which may change at each step. Let $\mu_n = Y_n g_n(X_{n-m}, \cdots, X_n)$ denote the gain of the bettor at the nth trial, where Y_n represents the bet size, $g_n(X_{n-m}, \cdots, X_n)$ is determined by the gambling rules, and $\{Y_n, n \geq 0\}$ is called a gambling system or a random selection system. The bettor's strategy is to determine $\{Y_n, n \geq 0\}$ by the results of the last m trials. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n depends on $X_{n-1}, \cdots, X_{n-m}$ as $n \geq m$, and $b_0, \cdots, b_{m-1}$ are constants. Thus, $\sum_{k=m}^{n} Y_k g_k(X_{k-m}, \cdots, X_k)$ represents the total gain, $\sum_{k=m}^{n} b_k$ the accumulated entrance fees, and $\sum_{k=m}^{n} [Y_k g_k(X_{k-m}, \cdots, X_k) - b_k]$ the accumulated net gain. Motivated by the classical definition of "fairness" of game of chance (see Kolmogorov[13]), we introduce the following definition:

Definition 2. . *The game is said to be fair, if for almost all $\omega \in \{\omega : \sum_{k=m}^{\infty} Y_k = \infty\}$, the accumulated net gain in the first n trials is to be of smaller order of magnitude than the accumulated stake $\sum_{k=m}^{n} Y_k$ as n tends to infinity, that is*

$$\lim_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} [Y_k g_k(X_{k-m}, \cdots, X_k) - b_k] = 0 \qquad a.s. \quad on \quad \{\omega : \sum\nolimits_{k=m}^{\infty} Y_k = \infty\}.$$

Study for strong limit properties of nonhomogeneous Markov chain is always one of central parts of the limit theory of probability theory. Many scholars have studied the subject until now. Liu and and Yang (see[1]) have studied the asymptotic equipartition properties (AEP) and limit properties of function sequence of nonhomogeneous Markov chain. Liu (see[2]) has discussed the strong limit theorems relative to the geometric average of random transition properties of finite nonhomogeneous Markov chain. Liu and Yang (see[3]) have investigated the strong deviation theorems of nonhomogeneous Markov chain relative to arbitrary stochastic sequence and AEP approximation of nonhomogeneous Markov information source. Liu (see[5]) has discussed the strong limit theorems for the harmonic mean of random transition probabilities of nonhomogeneous Markov chain. Liu and Wang (see[6]) have proved the strong limit properties for the state couples of nonhomogeneous Markov chain on the random selection system. Wang (see[7]) has studied the AEP and limit theorems for nonhomogeneous Markov chain on the generalized gambling system. Zhao and Wei (see[16]) have discussed some small deviation theorems for the truncated function sequence of the nonhomogeneous Markov chain. Afterward, many scholars (see[17-35]) have studied all kinds of stochastic processes and some limit properties with their applications for nonhomogeneous Markov chains on the generalized gambling system.

Many of practical information sources, such as language and image information, are often mth-order Markov chain, and always nonhomogeneous. mth-order nonhomogeneous Markov chain is a natural generalization of the general nonhomogeneous Markov chain. Hence, it is of importance to study the limit properties for the mth-order nonhomogeneous Markov chain in the information theory and the probability theory. Yang and Liu (see[9]) have proved the limit theorem for averages of the functions of $m+1$ variables of mth-order nonhomogeneous Markov chain and the AEP for mth-order nonhomogeneous Markov information source. Wang (see[10]) has discussed the Shannon-McMillan theorems for mth-order nonhomogeneous Markov information source.

The purpose of this paper is to establish a class of strong limit theorems for the sequence of multivariate truncated functions of mth-order nonhomogeneous Markov chains on the generalized random selection system by constructing the consistent distribution functions and a nonnegative super-martingale. As corollaries, we obtain a class of strong laws of large numbers represented by inequalities for mth-order nonhomogeneous Markov chains on the generalized random selection system. In the proof, we apply a new type of analytical techniques to the study of strong limit theorems.

We denote $X_m^n = \{X_m, \cdots, X_n\}$, and denote by x_m^n the realization of X_m^n.

2. Main Result and Its Proof

Theorem 1. *Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4), $\{f_n(x_0, \cdots, x_m),\ n \geq m\}$ be a real-valued function sequence defined on S^{m+1}, $\{Y_n, n \geq 0\}$ be the generalized random selection system defined by (7). Let $\{a_n(\omega), n \geq 0\}$ be a positive-valued increasing stochastic sequence, $\{g_n(x), n \geq 0\}$ be a series of continuous positive-valued even functions defined on $(-\infty, +\infty)$ which satisfy the conditions*

$$g_n(x) \uparrow, \qquad g_n(x)/x^2 \downarrow, \tag{8}$$

as $|x|$ increases.

Denote

$$\tilde{f}_k(X_{k-m}, \cdots X_k) = f_k(X_{k-m}, \cdots X_k) I_{\{|f_k(X_{k-m}, \cdots X_k)| \leq a_k\}}, \qquad k \geq m, \tag{9}$$

$$D(\omega) = \{\omega : \lim_{n\to\infty} \sum_{k=m}^{n} Y_k = \infty,$$

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{E_P[Y_k g_k(f_k(X_{k-m}^k)) | X_{k-m}^{k-1}]}{g_k(a_k)} = \sigma(\omega) < \infty\}. \tag{10}$$

Then

$$\lim_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k) | X_{k-m}^{k-1})]}{a_k} = 0,$$

$$P - a.s. \qquad \omega \in D(\omega), \tag{11}$$

where E_P represents the expectation relative to the measure P.

Proof. We consider the probability measure space $(\Omega, \mathcal{F}, \mathcal{P})$. Let λ be a constant. Denote

$$Q_k(\lambda; x_{k-m}^{k-1}) = E_P\left\{ \exp\left[\frac{\lambda Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k} \right] \middle| X_{k-m}^{k-1} = x_{k-m}^{k-1} \right\}$$

$$= \sum_{x_k \in S} p_k(x_k|x_{k-m}^{k-1}) \exp\left\{ \frac{\lambda Y_k[\tilde{f}_k(x_{k-m}^k) - E_P(\tilde{f}_k(x_{k-m}^{k-1}, X_k)|x_{k-m}^{k-1})]}{a_k} \right\}, \tag{12}$$

$$m_k(\lambda, x_{k-m}^k) = \frac{p_k(x_k|x_{k-m}^{k-1})}{Q_k(\lambda; x_{k-m}^{k-1})} \exp\left\{ \frac{\lambda Y_k[\tilde{f}_k(x_{k-m}^k) - E_P(\tilde{f}_k(x_{k-m}^{k-1}, X_k)|x_{k-m}^{k-1})]}{a_k} \right\}, \tag{13}$$

$$\mu(\lambda; x_0, \cdots, x_n) = q_o(x_0, \cdots, x_{m-1}) \prod\nolimits_{k=m}^n m_k(\lambda, x_{k-m}^k). \tag{14}$$

It follows from (12)-(14) that

$$\sum_{x_n \in S} \mu(\lambda; x_0, \cdots, x_n) =$$

$$= \sum_{x_n \in S} q_o(x_o^{m-1}) \prod_{k=m}^n \frac{p_k(x_k|x_{k-m}^{k-1})}{Q_k(\lambda; x_{k-m}^{k-1})} \exp\left\{ \frac{\lambda Y_k[\tilde{f}_k(x_{k-m}^k) - E_P(\tilde{f}_k(x_{k-m}^{k-1}, X_k)|x_{k-m}^{k-1})]}{a_k} \right\}$$

$$= \mu(\lambda; x_0, \cdots, x_{n-1}) \sum_{x_n \in S} \frac{p_n(x_n|x_{n-m}^{n-1})}{Q_n(\lambda; x_{n-m}^{n-1})} \exp\left\{ \frac{\lambda Y_n[\tilde{f}_n(x_{n-m}^n) - E_P(\tilde{f}_n(x_{n-m}^{n-1}, X_n)|x_{n-m}^{n-1})]}{a_n} \right\}$$

$$= \mu(\lambda; x_0, \cdots, x_{n-1}) \frac{Q_n(\lambda; x_{n-m}^{n-1})}{Q_n(\lambda; x_{n-m}^{n-1})} = \mu(\lambda; x_0, \cdots, x_{n-1}). \tag{15}$$

Therefore, $\mu(\lambda; x_0, \cdots, x_n)$, $n = 1, 2, \cdots$ are a family of consistent distribution functions on S^{n+1}. Denote

$$T_n(\lambda, \omega) = \frac{\mu(\lambda; X_0, \cdots, X_n)}{p(X_0, \cdots, X_n)}. \tag{16}$$

It is easy to see that $\{T_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$ (where $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$) is a nonnegative sup-martingale from Doob's martingale convergence theorem (see[8]). Moreover,

$$\lim_{n\to\infty} T_n(\lambda, \omega) = T_\infty(\lambda, \omega) < \infty, \qquad P-a.s. \tag{17}$$

By the first equation of (10), (17) and Lemma 1, we get

$$\limsup_{n\to\infty} (1 \Big/ \sum\nolimits_{k=m}^n Y_k) \ln T_n(\lambda, \omega) \leq 0, \qquad P-a.s. \quad \omega \in D(\omega). \tag{18}$$

By (5), (14) and (16), we can rewrite (18) as

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_k} \left\{ \sum_{k=m}^n \frac{\lambda Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k} - \sum_{k=m}^n \ln Q_k(\lambda; X_{k-m}^{k-1}) \right\} \leq 0,$$

$$P-a.s. \quad \omega\in D(\omega). \tag{19}$$

By (19), we easily obtain

$$\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{\lambda Y_k[\tilde f_k(X_{k-m}^{k})-E_P(\tilde f_k(X_{k-m}^{k})|X_{k-m}^{k-1})]}{a_k}$$

$$\leq \limsup_{n\to\infty}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\ln Q_k(\lambda;X_{k-m}^{k-1}), \qquad P-a.s. \qquad \omega\in D(\omega). \tag{20}$$

By (9) and the property of the conditional expectation, noticing that $Y_n\in[0,b]$, we obtain

$$E_P[\frac{\lambda Y_n[\tilde f_n(X_{n-m}^{n})-E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})]}{a_n}|X_{n-m}^{n-1}]$$

$$= \frac{\lambda Y_n[E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})-E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})]}{a_n}=0, \tag{21}$$

$$\left|\frac{Y_n[\tilde f_n(X_{n-m}^{n})-E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})]}{a_n}\right|\leq$$

$$\frac{Y_n\left|\tilde f_n(X_{n-m}^{n})\right|}{a_n}+\frac{Y_nE_P(\left|\tilde f_n(X_{n-m}^{n})\right||X_{n-m}^{n-1})}{a_n}\leq 2Y_n. \tag{22}$$

Take into account the inequality $e^x-1-x\leq(1/2)x^2e^{|x|}$, (12), (21) and (22), we arrive at

$$\begin{aligned}
&0\leq Q_n(\lambda;X_{n-m}^{n-1})-1\\
=\;& E_P\{\exp[\frac{\lambda Y_n[\tilde f_n(X_{n-m}^{n})-E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})]}{a_n}]-1\\
&\qquad -\frac{\lambda Y_n[\tilde f_n(X_{n-m}^{n})-E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})]}{a_n}|X_{n-m}^{n-1}\}\\
\leq\;& \frac{1}{2}\lambda^2E_P\left\{\exp\{|\lambda|\cdot\left|\frac{Y_n[\tilde f_n(X_{n-m}^{n})-E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})]}{a_n}\right|\}\right.\\
&\qquad\left.\cdot\frac{Y_n^2(\tilde f_n(X_{n-m}^{n})-E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1}))^2}{a_n^2}\right|X_{n-m}^{n-1}\Bigg\}\\
\leq\;& \frac{1}{2}\lambda^2e^{2|\lambda|\cdot Y_n}\frac{Y_n^2[E_P\{\tilde f_n(X_{n-m}^{n})^2|X_{n-m}^{n-1}\}-\{E_P(\tilde f_n(X_{n-m}^{n})|X_{n-m}^{n-1})\}^2]}{a_n^2}\\
\leq\;& \frac{1}{2}\lambda^2e^{2|\lambda|b}\frac{E_P\{(Y_n\tilde f_n(X_{n-m}^{n}))^2|X_{n-m}^{n-1}\}}{a_n^2}.
\end{aligned} \tag{23}$$

According to the assumption, $g_n(x)/x^2 \downarrow$ as $|x|$ increases, so we obtain

$$(x/a_n)^2 \le g_n(x)/g_n(a_n), \qquad |x| \le a_n. \tag{24}$$

On the other hand, $g_n(x) \uparrow$ as $|x|$ increases, therefore it follows from (23) that

$$\left(\frac{Y_n\tilde{f}_n(X_{n-m}^n)}{a_n}\right)^2 \le \frac{Y_n^2 g_n(\tilde{f}_n(X_{n-m}^n))}{g_n(a_n)} \le \frac{Y_n^2 g_n(f_n(X_{n-m}^n))}{g_n(a_n)}. \tag{25}$$

By (23) and (25), we get

$$0 \le Q_n(\lambda; X_{n-m}^{n-1}) - 1 \le \frac{1}{2}\lambda^2 e^{2|\lambda|b} E_P\left\{ \frac{Y_n^2 g_n(f_n(X_{n-m}^n))}{g_n(a_n)} \middle| X_{n-m}^{n-1} \right\}. \tag{26}$$

It follows from (10) and (26) that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_k} \sum_{k=m}^n [Q_k(\lambda; X_{k-m}^{k-1}) - 1] \\ &\le \frac{1}{2}\lambda^2 e^{2|\lambda|b} \limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_{kn}} \sum_{k=m}^n E_P\left\{ \frac{Y_k^2 g_k(f_k(X_{k-m}^k))}{g_k(a_k)} | X_{k-m}^{k-1} \right\} \\ &\le \frac{1}{2}\lambda^2 b e^{2|\lambda|b} \limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_{kn}} \sum_{k=m}^n E_P\left\{ \frac{Y_k g_k(f_k(X_{k-m}^k))}{g_k(a_k)} | X_{k-m}^{k-1} \right\} \\ &\le \frac{1}{2}\lambda^2 b e^{2|\lambda|b} \sigma(\omega), \qquad P-a.s. \quad \omega \in D(\omega). \end{aligned} \tag{27}$$

By use of the inequality $0 \le \ln x \le x - 1$ $(x > 1)$, we have from (27) that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_k} \sum_{k=m}^n \ln Q_k(\lambda; X_{k-m}^{k-1}) \\ &\le \limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_k} \sum_{k=m}^n [Q_k(\lambda; X_{k-m}^{k-1}) - 1] \\ &\le \frac{1}{2}\lambda^2 b e^{2|\lambda|b} \sigma(\omega), \qquad P-a.s. \quad \omega \in D(\omega). \end{aligned} \tag{28}$$

By applying (20) and (28), we obtain

$$\limsup_{n\to\infty} \frac{\lambda}{\sum_{k=m}^n Y_k} \sum_{k=m}^n \frac{Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k} \le \frac{1}{2}\lambda^2 b e^{2|\lambda|b} \sigma(\omega),$$

$$P-a.s. \quad \omega \in D(\omega). \tag{29}$$

In the case $\lambda > 0$, dividing two sides of (29) by λ, we have

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k} \leq \frac{1}{2}\lambda b e^{2\lambda b}\sigma(\omega),$$

$$P-a.s. \quad \omega \in D(\omega). \tag{30}$$

Taking the limit in (30) as $\lambda \to 0^+$, we obtain

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k} \leq 0,$$

$$P-a.s. \quad \omega \in D(\omega). \tag{31}$$

Letting $\lambda < 0$, dividing two sides of (29) by λ, we have

$$\liminf_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k} \geq \frac{1}{2}\lambda b e^{-2\lambda b}\sigma(\omega),$$

$$P-a.s. \quad \omega \in D(\omega). \tag{32}$$

Analogously, taking the limit in (32) as $\lambda \to 0^-$, we obtain

$$\liminf_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{Y_k[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k} \geq 0,$$

$$P-a.s. \quad \omega \in D(\omega). \tag{33}$$

Therefore, (11) follows from (31) and (33) immediately. ◀

Lemma 2. *Let $\{a_n, n \geq 0\}$ $\{\sigma_n, n \geq 0\}$ be increasing positive-valued sequences, $\{x_n, n \geq 0\}$ be a real number sequence. If either of $\{a_n, n \geq 0\}$ and $\{\sigma_n, n \geq 0\}$ is unbounded, and*

$$\limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \frac{x_k}{a_k} \leq b, \qquad \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \frac{x_k}{a_k} \geq a, \tag{34}$$

where $a \leq 0, \quad b \geq 0$, then

$$\limsup_{n\to\infty} \frac{1}{a_n \sigma_n} \sum_{k=1}^{n} x_k \leq b - a, \qquad \liminf_{n\to\infty} \frac{1}{a_n \sigma_n} \sum_{k=1}^{n} x_k \geq a - b. \tag{35}$$

3. Strong Laws of Large Numbers for mth-order Nonhomogeneous Markov Chains on the Generalized Gambling System

Theorem 2. *Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4), $\{Y_n, n \geq 0\}$, $\{a_n(\omega), n \geq 0\}$ and $\{g_n(x), n \geq 0\}$ be all defined as in Theorem 1.*

We replace (8) by the following conditions:

$$\frac{g_n(x)}{|x|} \uparrow, \qquad \frac{g_n(x)}{x^2} \downarrow, \tag{36}$$

as $|x|$ increases.

If for all $n \geq m$, $E_P(X_n | X_{n-m}, \cdots, X_{n-1}) = 0$ holds, and

$$\sum_{k=m}^{\infty} \sum_{|x_k| > a_k} p_k(x_k | x_{k-m}, \cdots, x_{k-1}) < \infty, \tag{37}$$

then

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_k} \sum_{k=m}^n \frac{Y_k X_k}{a_k} \leq \sigma(\omega), \qquad P-a.s. \quad \omega \in D_o(\omega), \tag{38}$$

$$\liminf_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_k} \sum_{k=m}^n \frac{Y_k X_k}{a_k} \geq -\sigma(\omega), \qquad P-a.s. \quad \omega \in D_o(\omega), \tag{39}$$

$$\limsup_{n\to\infty} \frac{1}{a_n \sum_{k=m}^n Y_k} \sum_{k=m}^n Y_k X_k \leq 2\sigma(\omega), \qquad P-a.s. \quad \omega \in D_o(\omega), \tag{40}$$

$$\liminf_{n\to\infty} \frac{1}{a_n \sum_{k=m}^n Y_k} \sum_{k=m}^n Y_k X_k \geq -2\sigma(\omega), \qquad P-a.s. \quad \omega \in D_o(\omega), \tag{41}$$

where

$$D_o(\omega) = \{\omega : \lim_{n\to\infty} \sum_{k=m}^n Y_k = +\infty, \ \limsup_{n\to\infty} \frac{1}{\sum_{k=m}^n Y_k} \sum_{k=m}^n \frac{E_P[Y_k g_k(X_k) | X_{k-m}^{k-1}]}{g_k(a_k)} = \sigma(\omega)\}. \tag{42}$$

Proof. Letting $f_k(X_{k-m}, \cdots X_k) = X_k$ in Theorem 1, we obtain $D_o(\omega) = D(\omega)$. Denote $\tilde{X}_k = X_k I_{|X_k| \leq a_k}$. Noticing that $g_n(x)/|x| \uparrow$ implies $g_n(x) \uparrow$, we see that under the assumption of Theorem 2, (11) still holds. By (37) we easily obtain that

$$\sum_{n=m}^{\infty} P(X_n \neq \tilde{X}_n | X_{n-m}^{n-1} = x_{n-m}^{n-1})$$

$$= \sum_{n=m}^{\infty} P(X_n I_{\{|X_n|>a_n\}} | X_{n-m}^{n-1} = x_{n-m}^{n-1})$$
$$= \sum_{n=m}^{\infty} \sum_{|x_n|>a_n} p_n(x_n | x_{n-m}^{n-1}) < \infty.$$

It means that $\{X_n, n \geq 0\}$ is equivalent to $\{\tilde{X}_n, n \geq 0\}$. Hence, by virtue of $\sum_{k=m}^{n} Y_k \to \infty, \quad a.s.$, $Y_k \in [0, b]$ and $a_n \uparrow$, we obtain

$$\lim_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{Y_k(\tilde{X}_k - X_k)}{a_k} = 0, \qquad P - a.s. \tag{43}$$

Since $g_n(x)/|x| \uparrow$ as $|x|$ increases, we have

$$\frac{|x|}{a_n} \leq \frac{g_n(x)}{g_n(a_n)}, \qquad |x| > a_n. \tag{44}$$

Denote by $F_{(X_n|X_{n-m},\cdots,X_{n-1})}(x|x_{n-m}, \cdots, x_{n-1}) = P(X_n \leq x | X_{n-m} = x_{n-m}, \cdots, X_{n-1} = x_{n-1})$ the conditional distribution function of X_n relative to $X_{n-m}, \cdots, X_{n-1}$. Noticing $E_P(X_n | X_{n-m}, \cdots, X_{n-1}) = 0$, we can write $\forall x_{n-m}, \cdots, x_{n-1} \in S$,

$$\frac{|Y_n E_P(\tilde{X}_n | X_{n-m}^{n-1})|}{a_n} = \frac{|E_P(Y_n X_n I_{|X_n| \leq a_n} | X_{n-m}^{n-1})|}{a_n}$$
$$= \frac{|\int_{|x|\leq a_n} Y_n x dF_{(X_n|X_{n-m},\cdots,X_{n-1})}(x|x_{n-m}, \cdots, x_{n-1})|}{a_n}$$
$$= \frac{|\int_{|x|> a_n} Y_n x dF_{(X_n|X_{n-m},\cdots,X_{n-1})}(x|x_{n-m}, \cdots, x_{n-1})|}{a_n}$$
$$\leq \frac{\int_{|x|> a_n} Y_n |x| dF_{(X_n|X_{n-m},\cdots,X_{n-1})}(x|x_{n-m}, \cdots, x_{n-1})}{a_n}$$
$$\leq \int_{|x|>a_n} \frac{Y_n g_n(x)}{g_n(a_n)} dF_{(X_n|X_{n-m},\cdots,X_{n-1})}(x|x_{n-m}, \cdots, x_{n-1})$$
$$\leq E_P\left(\frac{Y_n g_n(X_n)}{g_n(a_n)} | X_{n-m}, \cdots, X_{n-1}\right). \tag{45}$$

Owing to (42) and (45),

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{|Y_k E_P(\tilde{X}_k | X_{k-m}^{k-1})|}{a_k} \leq \sigma(\omega), \quad P - a.s. \quad \omega \in D_o(\omega), \tag{46}$$

which implies that

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} \frac{Y_k E_P(\tilde{X}_k | X_{k-m}^{k-1})}{a_k} \leq \sigma(\omega), \quad P - a.s. \quad \omega \in D_o(\omega), \tag{47}$$

$$\liminf_{n\to\infty}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{Y_kE_P(\tilde{X}_k|X_{k-m}^{k-1})}{a_k}\geq -\sigma(\omega),\quad P-a.s.\quad \omega\in D_o(\omega). \tag{48}$$

Noticing

$$\frac{Y_kX_k}{a_k}=\frac{Y_k[(\tilde{X}_k-E_P(\tilde{X}_k|X_{k-m}^{k-1}))+(X_k-\tilde{X}_k)+E_P(\tilde{X}_k|X_{k-m}^{k-1})]}{a_k}, \tag{49}$$

by use of (11), (43) and (47), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{Y_kX_k}{a_k}\\
\leq\ &\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{Y_k[\tilde{X}_k-E_P(\tilde{X}_k|X_{k-m}^{k-1})]}{a_k}\\
&\quad+\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{Y_k(X_k-\tilde{X}_k)}{a_k}+\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{Y_kE_P(\tilde{X}_k|X_{k-m}^{k-1})}{a_k}\\
\leq\ &\sigma(\omega),\qquad\qquad P-a.s.\quad \omega\in D_o(\omega).
\end{aligned}$$

Hence, (38) is valid. Analogously, by applying (11), (43) and (48), we acquire (39). By utilizing Lemma 2, (40) and (41) follow from (38) and (39), respectively.◀

Corollary 1. *Let $\{X_n,n\geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4). Let $\{a_n,n\geq 0\}$ be an increasing positive-valued stochastic sequence, $\{g_n(x),n\geq 0\}$ be defined as in Theorem 2. Denote*

$$J(\omega)=\{\omega:\lim_{n\to\infty}\sum_{k=m}^{n}Y_k=+\infty,\quad \sum_{k=m}^{\infty}\frac{E_P[Y_kg_k(X_k)|X_{k-m}^{k-1}]}{\sum\limits_{i=m}^{k}Y_ig_k(a_k)}<\infty\}. \tag{50}$$

Then

$$\lim_{n\to\infty}\frac{1}{\sum\limits_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{Y_kX_k}{a_k}=0,\qquad P-a.s.\quad \omega\in J(\omega), \tag{51}$$

$$\lim_{n\to\infty}\frac{1}{a_n\sum\limits_{k=m}^{n}Y_k}\sum_{k=m}^{n}Y_kX_k=0,\qquad P-a.s.\quad \omega\in J(\omega). \tag{52}$$

Proof. By Kronecker's lemma, we have

$$\sum_{k=m}^{\infty}\frac{E_P[Y_kg_k(X_k)|X_{k-m}^{k-1}]}{\sum_{i=m}^{k}Y_ig_k(a_k)}<\infty\Rightarrow\lim_{n}\frac{1}{\sum_{k=m}^{n}Y_k}\sum_{k=m}^{n}\frac{E_P[Y_kg_k(X_k)|X_{k-m}^{k-1}]}{g_k(a_k)}=0. \tag{53}$$

Hence, $J(\omega) \subseteq D_o(\omega)$. It is obvious $\sigma(\omega) = 0$ at the moment. Therefore, (51) follows from (38), (39). And (52) follows from (41), (40).◄

Corollary 2. *Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4), $\{f_n(x_0, \cdots, x_m),\ n \geq m\}$, $\{a_n(\omega), n \geq 0\}$ be defined as before. Denote $0 \leq p_n \leq 2,\ n \geq 0$,*

$$L(\omega) = \{\omega : \limsup_{n\to\infty} \frac{1}{n} \sum_{k=m}^{n} \frac{E_P[|f_k(X_{k-m}^k)|^{p_k} | X_{k-m}^{k-1}]}{a_k^{p_k}} = \sigma(\omega) < \infty\}. \tag{54}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m}^{n} \frac{[\tilde{f}_k(X_{k-m}^k) - E_P(\tilde{f}_k(X_{k-m}^k) | X_{k-m}^{k-1})]}{a_k} = 0,$$

$$P - a.s. \qquad \omega \in L(\omega). \tag{55}$$

Proof. Let $Y_n \equiv 1,\ g_n(x) = |x|^{p_n},\ n \geq 0$. Obviously, $\lim\limits_{n\to\infty} \sum\limits_{k=m}^{n} Y_k = \lim\limits_{n\to\infty} (n-m+1) = +\infty$, $\{\ |x|^{p_n},\ n \geq 0\}$ is a series of continuous positive-valued even functions defined on $(-\infty, +\infty)$ which satisfy (8). By (10) we obtain that $L(\omega) = D(\omega)$. Corollary 2 follows from Theorem 1.

◄

Corollary 3. *Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4). Denote by $\sigma_n(i_0, \cdots, i_m)$ the number of occurrence of the state group $(i_0, \cdots, i_m)$ in the random vectors $(X_0, \cdots X_m), (X_1, \cdots X_{m+1}), \cdots, (X_{n-m}, \cdots X_n)$ which are selected by the generalized random selection system $\{Y_n, n \geq m\}$. That is,*

$$\sigma_n(i_0, \cdots, i_m) = \sum_{k=m}^{n} Y_k \delta_{i_0 \cdots i_m}(X_{k-m}^k). \tag{56}$$

We put

$$B(\omega) = \{\omega : \lim_{n\to\infty} \sum_{k=m}^{n} Y_k = \infty\}. \tag{57}$$

Then

$$\lim_{n\to\infty} \frac{1}{\sum_{k=m}^{n} Y_k} [\sigma_n(i_0, \cdots, i_m) - \sum_{k=m}^{n} Y_k \delta_{i_0 \cdots i_{m-1}}(X_{k-m}^{k-1}) p_k(i_m | i_0^{m-1})] = 0,$$

$$P - a.s \quad \omega \in B(\omega). \tag{58}$$

Proof. Letting $f_k(X_{k-m},\cdots X_k)=\delta_{i_0\cdots i_m}(X_{k-m},\cdots X_k),\ a_k\equiv 1, k\geq m$ in Theorem 1, we can easily see that $|\delta_{i_0\cdots i_m}(X_{k-m},\cdots X_k)|\leq 1$, $\tilde{f}_k(X_{k-m},\cdots X_k)=f_k(X_{k-m},\cdots X_k)$. By (8) and (10) we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}\sum_{k=m}^n\frac{E_P[Y_k g_k(f_k(X_{k-m}^k))|X_{k-m}^{k-1}]}{g_k(a_k)}\\
=\ &\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}\sum_{k=m}^n\frac{E_P[Y_k g_k(\delta_{i_0\cdots i_m}(X_{k-m}^k))|X_{k-m}^{k-1}]}{g_k(1)}\\
\leq\ &\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}\sum_{k=m}^n\frac{E_P[Y_k g_k(1)|X_{k-m}^{k-1}]}{g_k(1)}\\
=\ &\limsup_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}\sum_{k=m}^n Y_k=1<\infty.
\end{aligned}
$$

Therefore, we have $D(\omega)=B(\omega)$. It follows from (11) that

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}\sum_{k=m}^n\frac{Y_k[\tilde{f}_k(X_{k-m}^k)-E_P(\tilde{f}_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{a_k}\\
=\ &\lim_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}\sum_{k=m}^n Y_k[\delta_{i_0\cdots i_m}(X_{k-m}^k)-E_P(\delta_{i_0\cdots i_m}(X_{k-m}^k)|X_{k-m}^{k-1})]\\
=\ &\lim_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}[\sigma_n(i_0,\cdots,i_m)-\sum_{k=m}^n\sum_{x_k\in S}Y_k\delta_{i_0\cdots i_m}(X_{k-m}^{k-1},x_k)p_k(x_k|X_{k-m}^{k-1})]\\
=\ &\lim_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}[\sigma_n(i_0,\cdots,i_m)-\sum_{k=m}^n Y_k\delta_{i_0\cdots i_{m-1}}(X_{k-m}^{k-1})p_k(i_m|X_{k-m}^{k-1})]\\
=\ &\lim_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}[\sigma_n(i_0,\cdots,i_m)-\sum_{k=m}^n Y_k\delta_{i_0\cdots i_{m-1}}(X_{k-m}^{k-1})p_k(i_m|i_0^{m-1})]=0.
\end{aligned}
$$

Hence, (58) follows from (11) immediately.◀

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