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Coincidence and Fixed Point Theorems In Fuzzy Ultrametric Spaces

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Abstract. We prove common fixed point theorems for self maps satisfying contractive conditions on spherically complete fuzzy ultrametric spaces.

Key Words and Phrases: fuzzy ultrametric spaces, spherically complete, common fixed point

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1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [9] in 1965. To use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [5] and defined the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics, particularly in connection with both string and $E-$infinity theory which were introduced and studied by El Naschie [12,13]. They showed also that every metric induces a fuzzy metric. In [3,4,11,14,15,16], fixed point theorems in fuzzy (probabilistic) metric spaces have been proved. Recently, Gajic [10] proved a fixed point theorem in ultrametric spaces introduced by Van Rooij [2]. Rao et.al.[7,8] extended this result for two and more mappings. In this paper, we obtain some coincidence and fixed point theorems in fuzzy ultrametric spaces.First, we give some definitions.

Definition 1. *([3]). A binary operation* $* : [0,1]^2 \longrightarrow [0,1]$ *is called a continuous t-norm if it satisfies the following conditions:*
(1) $*$ *is associative and commutative,*
(2) $*$ *is continuous,*
(3) $a * 1 = a$ *for all* $a \in [0,1]$,
(4) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

Two typical examples of a continuous $t-$norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

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Definition 2. *([1]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,*

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ *if and only if* $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ *is continuous.*

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 1.1. ([1]) Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|},$$

for all $x, y \in X$. Then $(X, M, *)$ is a fuzzy metric space.

Definition 3. *Let $(X, M, *)$ be a fuzzy metric space.*

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 1. *([11]). For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

Definition 4. *Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 2. *M is a continuous function on $X^2 \times (0, \infty)$.*

Proof. See Proposition 1 of [6].◄

A class of implicit relation

Let Ψ be the set of all continuous and decreasing functions $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. Two typical examples of these functions are $\psi(x) = \frac{1}{x+1}$, $\psi(x) = \frac{1}{x^\alpha}$ for every $\alpha > 0$.

Lemma 3. *Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define*

$$M(x, y, t) = (\frac{1}{1 + d(x, y)})^{\psi(t)},$$

for all $x, y \in X$, where $d(x, y)$ is an ordinary metric and $\psi \in \Psi$. Then $(X, M, *)$ is a fuzzy metric space.

Proof. First of all, it is easy to see that $M(x, y, t) > 0$, $M(x, y, t) = 1 \Longleftrightarrow x = y$ and $M(x, y, t) = M(y, x, t)$. For each $x, y, z \in X$ and $t, s > 0$, we have

$$\begin{aligned} M(x, z, t + s) &= (\frac{1}{1 + d(x, y)})^{\psi(t+s)} \\ &\geq (\frac{1}{1 + d(x, z)} . \frac{1}{1 + d(z, y)})^{\psi(t+s)} \\ &\geq (\frac{1}{1 + d(x, z)})^{\psi(t+s)} . (\frac{1}{1 + d(z, y)})^{\psi(t+s)} \\ &\geq (\frac{1}{1 + d(x, z)})^{\psi(t)} . (\frac{1}{1 + d(z, y)})^{\psi(s)} \\ &\geq M(x, z, t) * M(z, y, s). \end{aligned}$$

◄

2. Main Results

Generally, to prove fixed or common fixed point theorems for maps satisfying strictly contractive conditions, the continuity of maps and complete metric spaces are necessary. In spherically complete ultrametric spaces, the continuity of maps are not necessary to obtain fixed points. First we state some known definitions.

Definition 5. *([2]). Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality:*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \forall x, y, z \in X,$$

then d is called an ultrametric on X and the pair (X, d) is called an ultrametric space.

Definition 6. *([2]). An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a non empty intersection.*

Rao et.al.[7] proved the following

Theorem 1. *Let (X, d) be an ultrametric space, $f, S, T : X \to X$ satisfying the following conditions:*
(1) $f(X)$ is spherically complete,
(2) $d(Sx, Ty) < max\{d(fx, fy), d(fx, Sx), d(fy, Ty)\}$ for $x, y \in X$, $x \neq y$,
(3) $fS = Sf, fT = Tf, ST = TS$,
(4) $S(X) \subseteq f(X)$, $T(X) \subseteq f(X)$.
Then either $fw = Sw$ or $fw = Tw$ for some $w \in X$.

Now we extend this Theorem for maps in fuzzy ultrametric spaces.

Definition 7. *Let $(X, M, *)$ be a fuzzy metric space. If the fuzzy metric M satisfies strong triangle inequality:*

$$M(x, y, t) \geq \min\{M(x, z, t), M(z, y, t)\} \forall x, y, z \in X, t > 0,$$

then M is called a fuzzy ultrametric on X and the $(X, M, *)$ is called a fuzzy ultrametric space.

Definition 8. *A fuzzy ultrametric space $(X, M, *)$ is said to be spherically complete if every shrinking collection of balls in X has a non empty intersection.*

Remark 1. *(i) Let d be an ultrametric on X and $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define*

$$M(x, y, t) = (\frac{1}{1 + d(x, y)})^{\psi(t)},$$

for all $x, y \in X$, where $\psi \in \Psi$.Then fuzzy metric M is also a fuzzy ultrametric .
(ii)Let an ultrametric space (X, d) be spherically complete and $a*b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = (\frac{1}{1 + d(x, y)})^{\psi(t)},$$

for all $x, y \in X$, where $\psi \in \Psi$.Then fuzzy ultrametric space $(X, M, *)$ is also a spherically complete.

Theorem 2. *Let $(X, M, *)$ be a fuzzy ultrametric space, $f, S, T : X \to X$ satisfying the following conditions:*
(1) $f(X)$ is spherically complete,
(2) $M(Sx, Ty, t) > min\{M(fx, fy, t), M(fx, Sx, t), M(fy, Ty, t)\}$
for all $x, y \in X$ such that $x \neq y$ and $t > 0$
(3) $fS = Sf, fT = Tf, ST = TS$,
(4) $S(X) \subseteq f(X)$, $T(X) \subseteq f(X)$.
Then either $fw = Sw$ or $fw = Tw$ for some $w \in X$.

Proof. For $a \in X$, let $B_a = (fa; 1 - min\{M(fa, Sa, t), M(fa, Ta, t)\})$ denote the closed sphere centered at fa with the radius $1 - min\{M(fa, Sa, t), M(fa, Ta, t)\}$.

Let A be the collection of all the spheres for all $a \in f(X)$.
Then the relation $B_a \leq B_b$ holds if $B_b \subseteq B_a$ is a partial order on A.
Let A_1 be a totally ordered subcollection of A.
Since $(f(X), M, *)$ is spherically complete, we have $\bigcap_{B_a \in A_1} B_a = B \neq \phi$.
Let $fb \in B$ where $b \in f(X)$ and $B_a \in A_1$. Then $fb \in B_a$. Hence,

$$\begin{aligned} M(fb, fa, t) &\geq 1 - (1 - \min\{M(fa, Sa, t), M(fa, Ta, t)\}) \\ &= \min\{M(fa, Sa, t), M(fa, Ta, t)\} \cdots\cdots (i) \end{aligned}$$

If $a = b$, then $B_a = B_b$. Assume that $a \neq b$.
Let $x \in B_b$. Then

$$\begin{aligned} M(x, fb, t) &\geq 1 - (1 - min\{M(fb, Sb, t), M(fb, Tb, t)\}) \\ &= \min\{M(fb, Sb, t), M(fb, Tb, t)\} \\ &\geq \min\left\{ \begin{array}{c} min\{M(fb, fa, t), M(fa, Ta, t), M(Ta, Sb, t)\}, \\ min\{M(fb, fa, t), Mfa, Sa, t), M(Sa, Tb, t)\} \end{array} \right\} \\ &> \min\left\{ \begin{array}{c} M(fb, fa, t), M(fa, Ta, t), M(fa, Sa, t), \\ min\{M(fb, fa, t), M(fb, Sb, t), M(fa, Ta, t)\} \\ min\{M(fa, fb, t), M(fa, Sa, t), M(fb, Tb, t)\} \end{array} \right\} \\ &= \min\{M(fa, Sa, t), M(fa, Ta, t)\} \ \ from(i) \end{aligned}$$

Thus, $M(x, fb, t) > min\{(fa, Sa), d(fa, Ta)\} \cdots\cdots$ (ii)
Now,

$$\begin{aligned} M(x, fa, t) &\geq \min\{M(x, fb, t), M(fb, fa, t)\} \\ &\geq \min\{M(fa, Sa, t), M(fa, Ta, t)\} \, from(i), (ii) \\ &= 1 - (1 - min\{M(fa, Sa, t), M(fa, Ta, t)\}) \end{aligned}$$

Thus, $x \in B_a$. Hence, $B_b \subseteq B_a$ for any $B_a \in A_1$.
Thus, B_b is an upper bound in A for the family A_1 and hence, by Zorn's Lemma, A has a maximal element , say B_z, $z \in f(X)$.There exists $w \in X$ such that $z = fw$.
Suppose $fw \neq Sw$ and $fw \neq Tw$. Now from(2) we have

$$\begin{aligned} M(Sfw, TSw, t) &> min\{M(f^2w, fSw, t), M(f^2w, Sfw, t), M(fSw, TSw, t)\} \\ &= M(f^2w, fSw, t) \cdots\cdots \text{(iii) since } fS = Sf. \end{aligned}$$

$$\begin{aligned} M(STw, Tfw, t) &> min\{M(fTw, f^2w, t), M(fTw, STw, t), M(f^2w, Tfw, t)\} \\ &= M(f^2w, fTw, t)........(iv) \text{ since } fT = Tf. \end{aligned}$$

$$\begin{aligned} M(Sfw, S^2w, t) &\geq \min\left\{M(Sfw, TSw, t), M(TSw, Tfw, t), M(Tfw, S^2w, t)\right\} \\ &\geq \min\left\{ \begin{array}{c} M(f^2w, fSw, t), M(f^2w, fTw, t), \\ min\{M(fSw, f^2w, t), M(fSw, S^2w, t), M(f^2w, Tfw, t)\} \end{array} \right\} \\ &= min\{M(f^2w, fSw, t), M(f^2w, fTw, t)\}........(v) \end{aligned}$$

From (iii), (v) we have

$$\min\{M(Sfw, TSw, t), M(Sfw, S^2w, t)\}$$
$$> \min\{M(f^2w, fSw, t), M(f^2w, fTw, t)\}.......(vi)$$

$$M(Tfw, T^2w, t) \geq min\{M(Tfw, TSw, t), M(TSw, Sfw, t), M(Sfw, T^2w, t)\}$$
$$> min\{M(f^2w, fTw, t), M(f^2w, fSw, t), min\{M(f^2w, fTw, t),$$
$$M(f^2w, Sfw, t), M(fTw, T^2w, t)\}\} \text{ from (iii), } (iv)$$
$$= min\{M(f^2w, fTw, t), M(f^2w, fSw, t)\}.......(vii)$$

From $(iv), (vii)$ we have
$$min\{M(STw, Tfw, t), M(Tfw, T^2w, t)\}$$
$$> min\{M(f^2w, fTw, t), M(f^2w, fSw, t)\}......(viii)$$
If $min\{M(f^2w, fTw, t), M(f^2w, fSw, t)\} = M(f^2w, fSw, t)$ then
from (vi), $min\{M(Sfw, TSw, t), M(Sfw, S^2w, t)\} > M(f^2w, fSw, t)$
which gives $f^2w \notin B_{Sw}$. Hence, $fz \notin B_{Sw}$. But $fz \in B_z$. Hence, $B_z \nsubseteq B_{Sw}$.
It is a contradiction to the maximality of B_z in A, since $Sw \in S(X) \subseteq f(X)$.
If $min\{M(f^2w, fTw, t), M(f^2w, fSw, t)\} = M(f^2w, fTw, t)$
then from $(viii)$, $min\{M(STw, Tfw, t), M(Tfw, T^2w, t)\} > M(f^2w, fTw, t)$
which gives $f^2w \notin B_{Tw}$. Hence, $fz \notin B_{Tw}$. But $fz \in B_z$. Hence, $B_z \nsubseteq B_{Tw}$.
It is a contradiction to the maximality of B_z in A, since $Tw \in T(X) \subseteq f(X)$.
Hence, either $fw = Sw$ or $fw = Tw$.◄

Corollary 1. *Let $(X, M, *)$ be a spherically complete fuzzy ultrametric space and $S, T : X \to X$ be commuting maps such that*
$M(Sx, Ty, t) > min\{M(x, y, t), M(x, Sx, t), M(y, Ty, t)\}$
for all $x, y \in X$ with $x \neq y$ and $t > 0$.
Then either S or T has a fixed point in X.

Corollary 2. *Let $(X, M, *)$ be a spherically complete fuzzy ultrametric space and $f, T : X \to X$ be such that $T(X) \subseteq f(X)$,*
(2.9.1) $M(Tx, Ty, t) > min\{M(fx, fy, t), M(fx, Tx, t), M(fy, Ty, t)\}$
for all $x, y \in X$ with $x \neq y$ and $t > 0$.
Then there exists $z \in X$ such that $fz = Tz$.
Further, if f and T are coincidentally commuting at z then z is a unique common fixed point of f and T .

Proof. Let $B_a = (fa; 1 - M(fa, Ta, t))$ denote the closed sphere centered at fa with radius $1 - M(fa, Ta, t)$ and let A be a collection of these spheres for all $a \in X$. Proceeding as in Theorem 2.7, we can conclude that A has a maximal element, say B_z, $z \in X$.
Suppose $fz \neq Tz$.
Since $Tz \in T(X) \subseteq f(X)$, there exists $w \in X$ such that $Tz = fw$.
Clearly, $w \neq z$. From (2.9.1) we have
$$M(fw, Tw, t) = M(Tz, Tw, t) > min\{M(fz, fw, t), M(fz, Tz, t), M(fw, Tw, t)\}$$
$$= M(fz, fw, t).$$
Thus, $fz \notin B_w$. Hence, $B_z \nsubseteq B_w$. It is a contradiction to the maximality of B_z. Hence, $fz = Tz$.

Further , assume that f and T are coincidentally commuting at z.
Then $f^2z = f(fz) = fTz = Tfz = T(Tz) = T^2z$.
Suppose $fz \neq z$. From (2.9.1) we have
$M(Tfz, Tz, t) > min\{M(f^2z, fz, t), M(f^2z, Tfz, t), M(fz, Tz, t)\}$
$= M(Tfz, Tz, t)$. It is a contradiction.
Hence, $fz = z$. Thus, $z = fz = Tz$.
Uniqueness of z follows easily from (2.9.1).◄

Corollary 3. *Let $(X, M, *)$ be a spherically complete fuzzy ultrametric space and T be self-mappings of X satisfying the following condition:*
(i)

$$M(Tx, Ty, t) > \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\},$$

for all $x, y \in X$ such that $x \neq y$ and $t > 0$. Then there exists a unique $z \in X$ such that $Tz = z$.

Corollary 4. *Let $(X, M, *)$ be a spherically complete fuzzy ultrametric space and T^n be self-mappings of X satisfying the following condition:*
(i)

$$M(T^nx, T^ny, t) > \min\{M(x, y, t), M(x, T^nx, t), M(y, T^ny, t)\},$$

for all $x, y \in X$, $\forall n \in \mathbb{N}$ such that $x \neq y$ and $t > 0$. Then there exists a unique $z \in X$ such that $Tz = z$.

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