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On a Boundary Control Problem for Forced String Oscillations

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Abstract. Necessary and sufficient conditions for the existence of boundary controls at both ends of a string of length l are given for the critical case $T = l$. Being obtained in an explicit analytic form, these controls transform the process of forced string oscillations from an arbitrary initial state to any pre-assigned final state.

Key Words and Phrases: two-endpoint boundary control; inhomogeneous wave equation

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1. Introduction

In this paper, for generalized solutions of the inhomogeneous wave equation $u_{tt}(x,t) - u_{xx}(x,t) = f(x,t)$, $0 < x < l$, $0 < t < T$, with a finite energy, we study the problem of controlling vibrations on both endpoints of the string: $u(0,t) = \mu(t)$ and $u(l,t) = \nu(t)$.

The solution to this problem depends on the relation between the string's length l and the time T of control. In this paper we consider the case $T = l$ which is called critical.

In this case, for any five functions $\varphi(x)$, $\psi(x)$, $\varphi_1(x)$, $\psi_1(x)$ and $f(x,t)$ of the classes

$$\varphi(x) \in W_2^1[0,l], \quad \psi(x) \in L_2[0,l], \quad \varphi_1(x) \in W_2^1[0,l], \quad \psi_1(x) \in L_2[0,l]$$

$$f(x,t) \in L_2[(0 < x < l) \times (0 < t < T)] \tag{$*$}$$

we obtain necessary and sufficient conditions for the existence and uniqueness of boundary controls $\mu(t)$ and $\nu(t)$ which transform the oscillation process from the initial state $\{u(x,0) = \varphi(x), u_t(x,0) = \psi(x)\}$ to the final state $\{u(x,T) = \varphi_1(x), u_t(x,T) = \psi_1(x)\}$. These boundary controls are given in an explicit analytic form. We also show that the time interval $T = l$ is the smallest possible for the full controllability of the forced string vibrations under minimal restrictions.

To address various problems associated with the boundary control, V. A. Il'in and his disciples have published a series of papers (see, e.g., [1-6] and further references in [7]). Some earlier results related to this subject can be found in [8-12].

Note that all these papers study the process of free oscillations, i.e. the oscillations described by the homogeneous wave equation. The case of forced oscillations, i.e. the case when the oscillating system is affected by an external force, is studied in [13-15] for classical solutions.

1°. **Statement of the problem and basic definitions.** In an open rectangle $Q_T = (0 < x < l) \times (0 < t < T)$, let us consider the following three problems for the inhomogeneous wave equation.

Mixed problem I:

$$u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) \quad \text{in} \quad Q_T, \tag{1}$$

$$u(0,t) = \mu(t), \quad u(l,t) = \nu(t) \quad \text{for} \quad 0 \leqslant t \leqslant T, \tag{2}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \quad \text{for} \quad 0 \leqslant x \leqslant l, \tag{3}$$

in which $\mu(t),\ \nu(t) \in W_2^1[0,T]$, $\varphi(x)$, $\psi(x)$, $f(x,t)$ belong to the classes $(*)$ and the compatibility conditions

$$\mu(0) = \varphi(0), \quad \nu(0) = \varphi(l) \tag{4}$$

are satisfied.

Mixed problem II: Here the relations (1), (2) are supplied with

$$u(x,T) = \varphi_1(x), \quad u_t(x,T) = \psi_1(x) \quad \text{for} \quad 0 \leqslant x \leqslant l, \tag{5}$$

in which $\varphi_1(x)$, $\psi_1(x)$, $f(x,t)$ belong to the classes $(*)$, $\mu(t),\ \nu(t) \in W_2^1[0,T]$ and the compatibility conditions

$$\mu(T) = \varphi_1(0), \quad \nu(T) = \varphi_1(l) \tag{6}$$

are satisfied.

Boundary control problem III: Here we consider (1),(2),(3) and (5) all together in which $\varphi(x)$, $\varphi_1(x)$, $\psi(x)$, $\psi_1(x)$, $f(x,t)$ belong to the classes $(*)$, $\mu(t)$, $\nu(t) \in W_2^1[0,T]$ and the compatibility conditions (4) and (6) are satisfied.

The solution to these problems will be sought in the class $\widehat{W_2^1}(Q_T)$ introduced in [1].

Definition 1. *We say that a function of two variables $u(x,t)$ belongs to $\widehat{W_2^1}(Q_T)$ if it is continuous in the closed rectangle $\overline{Q}_T$ and has generalized first-order partial derivatives which belong to $L_2[0 \leqslant x \leqslant l]$ for any fixed $t \in [0,T]$ and belong to $L_2[0 \leqslant t \leqslant T]$ for any fixed $x \in [0,l]$.*

Definition 2. *We say that a function of one variable $\underline{\mu}(t)$ (respectively $\overline{\mu}(t)$) belongs to the class $\underline{W}_2^1[0,T]$ (respectively, to the class $\overline{W}_2^1[0,T]$) if it is defined for all $t \leqslant T$ (respectively, for all $t \geq 0$), belongs to $W_2^1[0,T]$ and satisfies $\underline{\mu}(t) \equiv 0$ for $t \leqslant 0$(respectively, satisfies $\overline{\mu}(t) \equiv 0$ for $t \geqslant T$).*

Definition 3. *A function $u(x,t)$ is called the solution from the class $\widehat{W_2^1}(Q_T)$ to the mixed problem I if $u(x,t) \in \widehat{W_2^1}(Q_T)$ and the identity*

$$\int_0^l \int_0^T u(x,t)[\Phi_{tt}(x,t) - \Phi_{xx}(x,t)]dxdt + \int_0^l [\varphi(x)\Phi_t(x,0) - \psi(x)\Phi(x,0)]dx -$$

$$-\int_0^T \mu(t)\Phi_x(0,t)dt + \int_0^T \nu(t)\Phi_x(l,t)dt - \int_0^l\int_0^T f(x,t)\Phi(x,t)dxdt = 0 \tag{7}$$

holds for any function $\Phi(x,t) \in C^2(Q_T)$ satisfying the conditions $\Phi(0,t) \equiv 0$, $\Phi(l,t) \equiv 0$ for $0 \le t \le T$ and $\Phi(x,T) \equiv 0$, $\Phi_t(x,T) \equiv 0$ for $0 \le x \le l$, boundary conditions (2), the first initial condition (3) in the classical sense and the second initial condition (3) almost everywhere (a.e.).

Definition 4. *A function $u(x,t)$ is called the solution from the class $\widehat{W}_2^1(Q_T)$ to the mixed problem II if $u(x,t) \in \widehat{W}_2^1(Q_T)$ and the identity*

$$\int_0^l\int_0^T u(x,t)[\Phi_{tt}(x,t) - \Phi_{xx}(x,t)]dxdt - \int_0^l [\varphi_1(x)\Phi_t(x,T) - \psi_1(x)\Phi(x,T)]dx-$$

$$-\int_0^T \mu(t)\Phi_x(0,t)dt + \int_0^T \nu(t)\Phi_x(l,t)dt - \int_0^l\int_0^T f(x,t)\Phi(x,t)dxdt = 0 \tag{8}$$

holds for any function $\Phi(x,t) \in C^2(Q_T)$ satisfying the conditions $\Phi(0,t) \equiv 0$, $\Phi(l,t) \equiv 0$ for $0 \le t \le T$ and $\Phi(x,0) \equiv 0$, $\Phi_t(x,0) \equiv 0$ for $0 \le x \le l$, boundary conditions (2), the first final condition (5) in the classical sense and the second final condition (5) a.e.

Definition 5. *A function $u(x,t)$ is called the solution from the class of $\widehat{W}_2^1(Q_T)$ to the boundary control problem III if $u(x,t)$ is a solution to the mixed problem I of this class and, moreover, it satisfies the first relation (5) in the classical sense and the second relation (5) a.e.*

2°. **Auxiliary statements.** *Let us start with two uniqueness results. The proof of these assertions are similar to those given in [3] for the homogeneous wave equation.*

Proposition 1. *For any $T > 0$, each of the mixed problems I and II has a unique solution of the class $\widehat{W}_2^1(Q_T)$.*

Proposition 2. *For any $0 < T \le l$, there is a unique solution of the class $\widehat{W}_2^1(Q_T)$ to the problem III.*

Now consider the mixed problem I in which $\varphi(x) \equiv 0$ on $[0,l]$, $\psi(x) = 0$ a.e. on $[0,l]$, and the boundary functions $\mu(t)$ and $\nu(t)$ in $W_2^1[0,T]$ are arbitrary. By virtue of the compatibility conditions (4) we have the relations

$$\mu(0) = 0, \quad \nu(0) = 0, \tag{9}$$

that allow to continue $\mu(t)$ and $\nu(t)$ as identical zeros for all $t < 0$ and turn them into functions $\underline{\mu}(t)$ and $\underline{\nu}(t)$ of the class $\underline{W}_2^1[0,T]$.

Proposition 3. *For $0 < T \leq l$, $\varphi(x) \equiv 0$ and $\psi(x) = 0$ a.e. on $[0, l]$, for any $f(x,t) \in L_2[Q_T]$ and arbitrary functions $\mu(t), \nu(t) \in W_2^1[0,T]$ satisfying (9), the unique solution $u(x,t)$ of the class $\widehat{W}_2^1(Q_T)$ to the mixed problem I is defined by the relation*

$$u(x,t) = \underline{\mu}(t-x) + \underline{\nu}(t+x-l) + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)\, d\xi\, d\tau. \tag{10}$$

Proof. Let us extend the function $f(x,t)$ to be odd in the first variable with respect to $x = 0$ and $x = l$; thus it will belong to the class $L_2[(-l \leqslant x \leqslant 2l) \times (0 \leqslant t \leqslant T)]$. Applying properties of $\mu(t)$ and $\nu(t)$ it is easy to verify that, for $0 < T \leq l$, function (10) satisfies the boundary conditions $u(0,t) = \mu(t)$, $u(l,t) = \nu(t)$, the first initial condition $u(x,0) \equiv 0$ $\forall x \in [0,l]$ in the classical sense, and the second initial condition $u_t(x,0) \equiv 0$ a.e. on $[0,l]$. Therefore, it suffices to show that this function satisfies (7) where $\varphi(x) \equiv 0$ $\forall x \in [0,l]$ and $\psi(x) = 0$ a.e. on $[0,l]$, i.e. to show that the relation

$$L_{u,f,\Phi} \equiv \int_0^l \int_0^T u(x,t)[\Phi_{tt}(x,t) - \Phi_{xx}(x,t)]dxdt - \int_0^T \mu(t)\Phi_x(0,t)dt+$$

$$+\int_0^T \nu(t)\Phi_x(l,t)dt - \int_0^l \int_0^T f(x,t)\Phi(x,t)dxdt = 0 \tag{11}$$

holds with any function $\Phi(x,t) \in C^2(Q_T)$ satisfying the conditions $\Phi(0,t) \equiv 0$, $\Phi(l,t) \equiv 0$ for $0 \leq t \leq T$ and $\Phi(x,T) \equiv 0$, $\Phi_t(x,T) \equiv 0$ for $0 \leq x \leq l$. Integrating by parts, we rewrite (11) as follows:

$$L_{u,f,\Phi} = \int_0^l \int_0^T u_x(x,t)\Phi_x(x,t)dxdt - \int_0^l \int_0^T u_t(x,t)\Phi_t(x,t)dxdt - \int_0^l \int_0^T f(x,t)\Phi(x,t)dxdt. \tag{12}$$

Thus, it suffices to prove that the right-hand side of (12) is zero. Denote by $\hat{f}(x,t)$ an arbitrary primitive of $f(x,t)$ with respect to x. Calculating $u_x(x,t)$ and $u_t(x,t)$ from (10) and substituting them in the right-hand side of (12) we get

$$\int_0^l \{\int_0^T [-\underline{\mu}'(t-x) + \underline{\nu}'(t+x-l)]\Phi_x(x,t)dt\}dx - \int_0^T \{\int_0^l [\underline{\mu}'(t-x) + \underline{\nu}'(t+x-l)]\Phi_t(x,t)dx\}dt+$$

$$+\frac{1}{2}\int_0^l \{\int_0^T (\int_0^t [f(x+t-\tau,\tau) - f(x-t+\tau,\tau)]d\tau)\Phi_x(x,t)dt\}dx - \frac{1}{2}\int_0^T \{\int_0^l (\int_0^t [f(x+t-\tau,\tau)+$$

$$+f(x-t+\tau,\tau)]d\tau)\Phi_t(x,t)dx\}dt - \int_0^l \int_0^T f(x,t)\Phi(x,t)dxdt =$$

$$= \int_0^l \{[-\underline{\mu}(T-x) + \underline{\nu}(T+x-l)]\Phi_x(x,T) - [-\underline{\mu}(-x) + \underline{\nu}(x-l)]\Phi_x(x,0)\}dx-$$

$$-\int_0^l \int_0^T [-\underline{\mu}(t-x) + \underline{\nu}(t+x-l)]\Phi_{xt}(x,t)dxdt - \int_0^T \{[-\underline{\mu}(t-l) + \underline{\nu}(t)]\Phi_t(l,t)-$$

$$-[-\underline{\mu}(t) + \underline{\nu}(t-l)]\Phi_t(0,t)\}dt+$$

$$+\int_0^l \int_0^T [-\underline{\mu}(t-x) + \underline{\nu}(t+x-l)]\Phi_{tx}(x,t)dxdt - \frac{1}{2}\int_0^l \int_0^T \{\int_0^t [\hat{f}(x+t-\tau,\tau)+$$

$$+\hat{f}(x-t+\tau,\tau)]d\tau\}\Phi_{xt}(x,t)dxdt + \int_0^l \int_0^T [\int_0^t \hat{f}(x,\tau)d\tau]\Phi_{xt}(x,t)dxdt+$$

$$+\frac{1}{2}\int_0^l \int_0^T \{\int_0^t [\hat{f}(x+t-\tau,\tau)+$$

$$+\hat{f}(x-t+\tau,\tau)]d\tau\}\Phi_{tx}(x,t)dxdt - \int_0^l \int_0^T f(x,t)\Phi(x,t)dxdt.$$

The right-hand side of this equation is zero as the double integrals cancel each other out and all other terms vanish as $\Phi_x(x,T) \equiv 0$, $-\underline{\mu}(-x) + \underline{\nu}(x-l) \equiv 0$ $\forall x \in [0,l]$ and $\Phi_t(0,t) \equiv 0$, $\Phi_t(l,t) \equiv 0$ $\forall t \in [0,T]$.

Assertion 3 is proved. Similarly we can prove the following ◀

Proposition 4. *For $0 < T \le l$, $\varphi(x) \equiv 0$ and $\psi(x) = 0$ a.e. on $[0,l]$, for any $f(x,t) \in L_2[Q_T]$ and arbitrary functions $\mu(t), \nu(t) \in W_2^1[0,T]$ satisfying $\mu(T) = 0$, $\nu(T) = 0$, the unique solution $u(x,t)$ of the class $\widehat{W}_2^1(Q_T)$ to the mixed problem II is defined by the relation*

$$u(x,t) = \overline{\mu}(t+x) + \overline{\nu}(t-x+l) - \frac{1}{2}\int_t^T \int_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)\,d\xi\,d\tau.$$

2. The Main Result

Our main result is the following

Theorem 1. *Let $T = l$. Then for five predetermined functions $\varphi(x)$, $\psi(x)$, $\varphi_1(x)$, $\psi_1(x)$ and $f(x,t)$ belonging to the classes (*), there exists a unique solution to the boundary control problem III of the class $\widehat{W}_2^1(Q_T)$ if and only if the following relations*

$$\hat{\psi}_1(0) + \varphi_1(0) - \hat{\psi}(l) - \varphi(l) - \int_0^l \hat{f}(l-\tau,\tau)d\tau = 0, \tag{13}$$

$$\hat{\psi}_1(l) - \varphi_1(l) - \hat{\psi}(0) + \varphi(0) - \int_0^l \hat{f}(\tau,\tau)d\tau = 0, \tag{14}$$

hold (here $\hat{\psi}(x)$, $\hat{\psi}_1(x)$ and $\hat{f}(x,t)$ denote the primitives of $\psi(x)$, $\psi_1(x)$ and $f(x,t)$ in x, respectively).

Under these conditions, the solution to this problem is as follows

$$u(x,t) = \begin{cases} \frac{1}{2}[\varphi(x+t) + \varphi(x-t) + \hat{\psi}(x+t) - \hat{\psi}(x-t) + \int\limits_0^t \int\limits_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)d\xi d\tau] & in \ \triangle_1, \\ \frac{1}{2}[\varphi(x+t) + \hat{\psi}(x+t) + \varphi_1(x-t+l) - \hat{\psi}_1(x-t+l) + N(x,t)] & in \ \triangle_2, \\ \frac{1}{2}[\varphi(x-t) - \hat{\psi}(x-t) + \varphi_1(x+t-l) + \hat{\psi}_1(x+t-l) + M(x,t)] & in \ \triangle_3, \\ \frac{1}{2}[\varphi_1(x+t-l) + \varphi_1(x-t+l) + \int\limits_{x-t+l}^{x+t-l} \psi_1(\xi)d\xi - \int\limits_t^l \int\limits_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)d\xi d\tau] & in \ \triangle_4 \end{cases} \tag{15}$$

where $\triangle_1$ denotes the triangle bounded by the lines $t-x=0$, $t+x-l=0$, $t=0$; $\triangle_2$ is the triangle bounded by the lines $t-x=0$, $t+x-l=0$, $x=0$; $\triangle_3$ is the triangle bounded by the lines $t-x=0$, $t+x-l=0$, $x=l$; $\triangle_4$ in the triangle bounded by the lines $t-x=0$, $t+x-l=0$, $t=l$, and $N(x,t)$ and $M(x,t)$ stand for $\int\limits_0^t \int\limits_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)d\xi d\tau + \int\limits_0^l \hat{f}(x-t+\tau,\tau)d\tau$, $\int\limits_0^t \int\limits_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)d\xi d\tau - \int\limits_0^l \hat{f}(x+t-\tau,\tau)d\tau$, respectively.

The desired boundary controls $u(0,t) = \mu(t)$ and $u(l,t) = \nu(t)$ which transform the oscillatory process are given explicitly:

$$\mu(t) = \frac{1}{2}[\varphi(t) + \hat{\psi}(t) + \varphi_1(l-t) - \hat{\psi}_1(l-t) + \int_0^l \hat{f}(t-\tau,\tau)d\tau], \tag{16}$$

$$\nu(t) = \frac{1}{2}[\varphi_1(t) + \hat{\psi}_1(t) + \varphi(l-t) - \hat{\psi}(l-t) - \int_0^l \hat{f}(t+l-\tau,\tau)d\tau]. \tag{17}$$

Proof of necessity. First we consider the special case when $\varphi(x) \equiv 0$ on $[0,l]$, and $\psi(x) = 0$ a.e. on $[0,l]$. The solution $u(x,t)$ of $\widehat{W}_2^1(Q_T)$ to the problem III (if it exists) is simultaneously a solution of the same class to the mixed problem I with $\varphi(x) \equiv 0$ on

$[0, l]$, and $\psi(x) = 0$ a.e. on $[0, l]$. But this solution, by virtue of Proposition 3, can be represented as in (10) whence we obtain the relations

$$u_t(x, l) = \psi_1(x) = \underline{\mu}'(l - x) + \underline{\nu}'(x) + \frac{1}{2}\int_0^l [f(x + l - \tau, \tau) + f(x - l + \tau, \tau)]\, d\tau, \qquad (18)$$

$$u_x(x, l) = \varphi_1'(x) = -\underline{\mu}'(l - x) + \underline{\nu}'(x) + \frac{1}{2}\int_0^l [f(x + l - \tau, \tau) - f(x - l + \tau, \tau)]\, d\tau \qquad (19)$$

which are valid in $L_2[0, l]$ sense. Adding (18) and (19), we arrive at the equality

$$\psi_1(x) + \varphi_1'(x) = 2\underline{\nu}'(x) + \int_0^l f(x + l - \tau, \tau) d\tau. \qquad (20)$$

Integrating (20) over $[0, l]$ and using the relations $\underline{\nu}(0) = 0$, $\underline{\nu}(l) = \varphi_1(l)$ we get

$$\hat{\psi}_1(l) - \varphi_1(l) - \int_0^l \hat{f}(\tau, \tau) d\tau = \hat{\psi}_1(0) + \varphi_1(0) - \int_0^l \hat{f}(l - \tau, \tau) d\tau. \qquad (21)$$

From (21) it follows that if we denote by $\hat{\psi}_1(x)$ and $\hat{f}(x, t)$ the primitive functions of $\psi_1(x)$ and $f(x, t)$ in the variable x which satisfy

$$\hat{\psi}_1(0) + \varphi_1(0) - \int_0^l \hat{f}(l - \tau, \tau) d\tau = 0, \qquad (22)$$

then we get

$$\hat{\psi}_1(l) - \varphi_1(l) - \int_0^l \hat{f}(\tau, \tau) d\tau = 0. \qquad (23)$$

Thus, for the special case when $\varphi(x) \equiv 0$ on $[0, l]$, and $\psi(x) = 0$ a.e. on $[0, l]$, the necessity of (13) and (14) is established.

Now let us consider the general case when $\varphi(x)$ is an arbitrary function of $W_2^1[0, l]$, and $\psi(x)$ is an arbitrary element of $L_2[0, l]$. To this end, we extend the functions $\varphi(x)$ and $\psi(x)$ on the segment $-l \leqslant x \leqslant 2l$ so that $\varphi(x)$ becomes odd with respect to $x = 0$ and $x = l$ and $\psi(x)$ keeps on to be a function in L_2. Also we extend the function $f(x, t)$ so that it becomes odd with respect to $x = 0$ and $x = l$. These extended functions $\varphi(x)$, $\psi(x)$ and $f(x, t)$ belong to $W_2^1[-l, 2l]$, $L_2[-l, 2l]$ and $L_2[(-l \leq x \leq 2l) \times (0 \leq t \leq l)]$, respectively.

Now using these extended functions $\varphi(x)$, $\psi(x)$ and $f(x, t)$, let us consider the function

$$\vartheta(x, t) = \frac{1}{2}[\varphi(x+t)+\varphi(x-t)]+\frac{1}{2}[\hat{\psi}(x+t)-\hat{\psi}(x-t)]+\frac{1}{2}\int_0^t [\hat{f}(x+t-\tau, \tau)-\hat{f}(x-t+\tau, \tau)]d\tau \qquad (24)$$

which satisfies $\vartheta(x,0)=\varphi(x)$ $\forall x\in[0,l]$ and $\vartheta_t(x,0)=\psi(x)$ a.e. on $[0,l]$. Let us show that (24) gives a solution of $\widehat{W}_2^1(Q_T)$ to the mixed problem I in which $u(x,t)$ is replaced by $\vartheta(x,t)$, $\mu(t)$ – by $\vartheta(0,t)$, and $\nu(t)$ – by $\vartheta(l,t)$. It suffices to show that it satisfies (7) where $u(x,t)$, $\mu(t)$, $\nu(t)$ are replaced by $\vartheta(x,t)$, $\vartheta(0,t)$, $\vartheta(l,t)$, respectively, for any function $\Phi(x,t)$ (see definition 3).

Integrating by parts we rewrite (7) as follows:

$$\int_0^l\int_0^l \vartheta_x(x,t)\Phi_x(x,t)dxdt-\int_0^l\int_0^l \vartheta_t(x,t)\Phi_t(x,t)dxdt-\int_0^l\int_0^l f(x,t)\Phi(x,t)dxdt=$$

$$=\int_0^l \vartheta_t(x,0)\Phi(x,0)dx. \tag{25}$$

By virtue of (24), the left-hand side of (25) equals

$$\frac{1}{2}\int_0^l\{[\varphi(x+l)-\varphi(x-l)+\hat{\psi}(x+l)+\hat{\psi}(x-l)]\Phi_x(x,l)-2\hat{\psi}(x)\Phi_x(x,0)\}dx-$$

$$-\frac{1}{2}\int_0^l\int_0^l[\varphi(x+t)-\varphi(x-t)+\hat{\psi}(x+t)+\hat{\psi}(x-t)]\Phi_{xt}(x,t)dxdt-$$

$$-\frac{1}{2}\int_0^l\{[\varphi(l+t)-\varphi(l-t)+\hat{\psi}(l+t)+\hat{\psi}(l-t)]\Phi_t(l,t)\}dt-\frac{1}{2}\int_0^l\{[\varphi(t)-$$

$$-\varphi(-t)+\hat{\psi}(t)+\hat{\psi}(-t)]\Phi_t(0,t)\}dt+$$

$$+\frac{1}{2}\int_0^l\int_0^l[\varphi(x+t)-\varphi(x-t)+\hat{\psi}(x+t)+\hat{\psi}(x-t)]\Phi_{tx}(x,t)dxdt-$$

$$-\frac{1}{2}\int_0^l\int_0^l\{\int_0^t[\hat{f}(x+t-\tau,\tau)+\hat{f}(x-t+\tau,\tau)]d\tau\}\Phi_{xt}(x,t)dxdt+$$

$$+\int_0^l\int_0^l[\int_0^t\hat{f}(x,\tau)d\tau]\Phi_{xt}(x,t)dxdt+$$

$$+\frac{1}{2}\int_0^l\int_0^l\{\int_0^t[\hat{f}(x+t-\tau,\tau)+\hat{f}(x-t+\tau,\tau)]d\tau\}\Phi_{tx}(x,t)dxdt-\int_0^l\int_0^l f(x,t)\Phi(x,t)dxdt=$$

$$= -\int\limits_0^l \hat{\psi}(x)\Phi_x(x,0)dx = \int\limits_0^l \psi(x)\Phi(x,0)dx = \int\limits_0^l \vartheta_t(x,0)\Phi(x,0)dx,$$

which proves (25). Thus, we showed that the function (24) is a solution of $\widehat{W}_2^1(Q_l)$ to the mixed problem I. Therefore, the difference $[u(x,t) - \vartheta(x,t)]$ is a solution of the same class to the homogeneous mixed problem I with zero initial conditions at $t = 0$. Following the above consideration of the special case one can easily show that this difference satisfies the relations similar to (22) and (23):

$$\hat{\psi}_1(0) + \varphi_1(0) - \hat{\vartheta}_t(0,l) - \vartheta(0,l) = 0, \tag{26}$$

$$\hat{\psi}_1(l) - \varphi_1(l) - \hat{\vartheta}_t(l,l) + \vartheta(l,l) = 0. \tag{27}$$

From (24) it follows

$$\vartheta(0,l) = 0,\ \hat{\vartheta}_t(0,l) = \varphi(l) + \hat{\psi}(l) + \int\limits_0^l \hat{f}(l-\tau,\tau)d\tau, \tag{28}$$

$$\vartheta(l,l) = 0,\ \hat{\vartheta}_t(l,l) = \hat{\psi}(0) - \varphi(0) + \int\limits_0^l \hat{f}(\tau,\tau)d\tau. \tag{29}$$

It is easy to show that due to (28) and (29) relations (26) and (27) transform into (13) and (14). The necessity of conditions (13) and (14) for the general case is proved.

Proof of sufficiency. Function (24) belongs to $\widehat{W}_2^1(Q_l)$ as in each of the domains $\triangle_i$, $i = \overline{1,4}$, it is an algebraic sum of functions depending on $x+t$ or $x-t$ with a square-integrable generalized derivative and, by (22) and (23), it retains its continuity on common borders of any two of these areas.

It is easy to verify the validity of the relations $u(x,0) = \varphi(x)$, $u(x,l) = \varphi_1(x)$ for all $x \in [0,l]$ and the equalities $u_t(x,0) = \psi(x)$, $u_t(x,l) = \psi_1(x)$ a.e. on $[0,l]$.

It suffices to prove the validity of (7) for $u(0,t) = \mu(t)$, $u(l,t) = \nu(t)$ and for any function $\Phi(x,t)$ in Definition 3. By (25) one has to prove the equality

$$\int\limits_0^l\int\limits_0^l u_x(x,t)\Phi_x(x,t)dxdt - \int\limits_0^l\int\limits_0^l u_t(x,t)\Phi_t(x,t)dxdt - \int\limits_0^l\int\limits_0^l f(x,t)\Phi(x,t)dxdt = \int\limits_0^l \psi(x)\Phi(x,0)dx. \tag{30}$$

Consider the function

$$U(x,t)=\begin{cases}\frac{1}{2}[\varphi(x+t)-\varphi(x-t)+\hat{\psi}(x+t)+\hat{\psi}(x-t)+I(x,t)]\text{in } \triangle_1,\\ \frac{1}{2}[\varphi(x+t)+\hat{\psi}(x+t)-\varphi_1(x-t+l)+\hat{\psi}_1(x-t+l)-\\ -\int\limits_0^l \hat{f}(x-t+\tau,\tau)d\tau+I(x,t)]\text{in } \triangle_2,\\ \frac{1}{2}[\hat{\psi}(x-t)-\varphi(x-t)+\varphi_1(x+t-l)+\hat{\psi}_1(x+t-l)-\\ -\int\limits_0^l \hat{f}(x+t-\tau,\tau)d\tau+I(x,t)]\text{in } \triangle_3,\\ \frac{1}{2}[\varphi_1(x+t-l)-\varphi_1(x-t+l)+\hat{\psi}_1(x+t-l)+\\ +\hat{\psi}_1(x-t+l)-K(x,t)+I(x,t)]\text{in } \triangle_4\end{cases}$$

where $I(x,t)=\int\limits_0^t \hat{f}(x+t-\tau,\tau)d\tau+\int\limits_0^t \hat{f}(x-t+\tau,\tau)d\tau,\ K(x,t)=\int\limits_0^l \hat{f}(x+t-\tau,\tau)d\tau+\int\limits_0^l \hat{f}(x-t+\tau,\tau)d\tau.$

Similarly, for the function $u(x,t)$ one can prove that $U(x,t)$ belongs to $\widehat{W}_2^1(Q_l)$ and easily verify that the relations $U_x(x,t)=u_t(x,t)$, $U_t(x,t)-\hat{f}(x,t)=u_x(x,t)$ hold a.e. in the rectangle Q_l.

Using these relations and the properties of $\Phi(x,t)$ from Definition 3 we obtain

$$\int\limits_0^l\int\limits_0^l u_x(x,t)\Phi_x(x,t)dxdt-\int\limits_0^l\int\limits_0^l u_t(x,t)\Phi_t(x,t)dxdt-\int\limits_0^l\int\limits_0^l f(x,t)\Phi(x,t)dxdt=$$

$$=\int\limits_0^l\{\int\limits_0^l U_t(x,t)\Phi_x(x,t)dt\}dx-\int\limits_0^l\{\int\limits_0^l \hat{f}(x,t)\Phi_x(x,t)dx\}dt-\int\limits_0^l\{\int\limits_0^l U_x(x,t)\Phi_t(x,t)dx\}dt-$$

$$-\int\limits_0^l\int\limits_0^l f(x,t)\Phi(x,t)dxdt=\int\limits_0^l U(x,l)\Phi_x(x,l)dx-\int\limits_0^l U(x,0)\Phi_x(x,0)dx-$$

$$-\int\limits_0^l\int\limits_0^l U(x,t)\Phi_{xt}(x,t)dxdt-$$

$$-\int\limits_0^l U(l,t)\Phi_t(l,t)dt+\int\limits_0^l U(0,t)\Phi_t(0,t)dt+\int\limits_0^l\int\limits_0^l U(x,t)\Phi_{tx}(x,t)dxdt-\int\limits_0^l \hat{f}(l,t)\Phi(l,t)dt+$$

$$+\int\limits_0^l \hat{f}(0,t)\Phi(0,t)dt+\int\limits_0^l\int\limits_0^l f(x,t)\Phi(x,t)dxdt-\int\limits_0^l\int\limits_0^l f(x,t)\Phi(x,t)dxdt=$$

$$= -\int_0^l U(x,0)\Phi_x(x,0)dx = \int_0^l u_t(x,0)\Phi(x,0)dx = \int_0^l \psi(x)\Phi(x,0)dx.$$

Equality (30) is established. Thus, the theorem is proved.

Remark 1. *In [12] it is shown that the interval $(0,l)$ is the minimum time interval over which for arbitrary five functions $\varphi(x), \varphi_1(x), \psi(x), \psi_1(x)$ and $f(x,t)$ which belong to the classes (*) and satisfy the conditions (13) and (14), one can transfer the oscillatory system from the initial state to the final one. In the case when $T < l$, in order to implement such a transition one needs to impose additional conditions on all of these functions.*

Remark 2. *Important special cases of the problem under consideration are as follows.*

1)The Problem of Damping the Oscillatory Process, i.e. the problem of finding the boundary controls $\mu(t)$ and $\nu(t)$ that for arbitrarily given initial shift $\varphi(x) \in W_2^1[0,l]$ and initial velocity $\psi(x) \in L_2[0,l]$, transit the process to the full rest at $t = l$.

2) The problem of finding boundary controls $\mu(t)$ and $\nu(t)$ which transfer the string from its initial rest (i.e. when the initial conditions equal zero) to the state with any given shift $\varphi_1(x) \in W_2^1[0,l]$ and any given velocity $\psi_1(x) \in L_2[0,l]$ (*the excitation of an oscillatory process*).

One can easily derive the relevant statements from the main theorem.

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