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Optimization of Initial Data for Linear Neutral Functional-Differential Equations with the Discontinuous Initial Condition

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Abstract. The necessary optimality conditions are obtained for initial data. Here initial data implies the collection of initial moment, delay parameter in phase coordinates, initial vector and functions. The discontinuous initial condition means that the values of the initial function and trajectory, generally, do not coincide at the initial moment. In this paper, the essential novelty are the optimality conditions of the initial moment and delay and the effect of discontinuity of the initial condition. An example is considered.

Key Words and Phrases: optimization; necessary optimality conditions; neutral functional-differential equation; discontinuous initial condition.

2010 Mathematics Subject Classifications: 34K35; 49K25

1. Statement of the problem

Let $t_{01} < t_{02} < t_1, 0 < \tau_1 < \tau_2, \eta > 0$ be given numbers with $t_1 - t_{02} > \hat{\eta} = \max\{\tau_2, \eta\}$ and R^n be the n-dimensional vector space of points

$$x = (x^1, ..., x^n)^T, \| x \|^2 = \sum_{i=1}^{n} (x^i)^2,$$

where T means transpose. Suppose that $K_1 \subset R^n, K_2 \subset R^n$ and $X \subset R^n$ are compact and convex sets. By Φ and G we denote, respectively, the sets of measurable initial functions $\varphi : [\hat{\tau}, t_{02}] \to K_1$ and $g : [\hat{\tau}, t_{02}] \to K_2$, where $\hat{\tau} = t_{01} - \hat{\eta}$.

The collection of initial moment $t_0 \in [t_{01}, t_{02}]$, delay parameter $\tau \in [\tau_1, \tau_2]$, initial vector $x_0 \in X$, initial functions $\varphi \in \Phi$ and $g \in G$ is said to be initial data and we denote it by $w = (t_0, \tau, x_0, \varphi, g)$.

To each initial data

$$w = (t_0, \tau, x_0, \varphi, g) \in W = [t_{01}, t_{02}] \times [\tau_1, \tau_2] \times X \times \Phi \times G$$

we assign the linear neutral functional-differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + C(t)\dot{x}(t-\eta) + f(t), t \in [t_0, t_1] \tag{1.1}$$

with the initial conditions

$$\begin{cases} x(t) = \varphi(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0, \\ \dot{x}(t) = g(t), t \in [\hat{\tau}, t_0), \end{cases} \tag{1.2}$$

where $A(t), B(t), C(t), t \in [t_{01}, t_1]$ are given continuous $n \times n$ -dimensional matrix-functions; $f(t), t \in [t_{01}, t_1]$ is a given continuous n -dimensional vector-function.
The condition (1.2) is said to be discontinuous initial condition since, generally, $x(t_0) \neq \varphi(t_0)$.

Remark 1.1. Here and in the sequel the symbol $\dot{x}(t)$ on the interval $[\hat{\tau}, t_0)$ is not related to the derivative of the function $\varphi(t)$ (see (1.2)). Thus, if $t \in [t_0, t_0 + \eta]$ we have

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + C(t)g(t-\eta) + f(t).$$

Definition 1.1. Let $w = (t_0, \tau, x_0, \varphi, g) \in W$. A function $x(t) = x(t; w) \in R^n, t \in [\hat{\tau}, t_1]$ is called a solution corresponding to the initial data w, if it satisfies condition (1.2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1.1) almost everywhere (a.e.) on $[t_0, t_1]$.

For each initial data $w \in W$ there exists the unique solution $x(t; w), t \in [\hat{\tau}, t_1]$, [1].
Let $q^i(t_0, \tau, x_0, x_1), i = \overline{0, l}$ be continuously differentiable scalar functions on the set $[t_{01}, t_{02}] \times [\tau_1, \tau_2] \times X \times R^n$.

Definition 1.2. An initial data $w = (t_0, \tau, x_0, \varphi, g) \in W$ is said to be admissible if the conditions

$$q^i(t_0, \tau, x_0, x(t_1)) = 0, i = \overline{1, l} \tag{1.3}$$

hold, where $x(t) = x(t; w)$.

The set of admissible initial data is denoted by W_0.

Definition 1.3. An initial data $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, g_0) \in W_0$ is said to be optimal if for arbitrary $w \in W_0$ the inequality

$$q^0(t_{00}, \tau_0, x_{00}, x_0(t_1)) \leq q^0(t_0, \tau, x_0, x(t_1)) \tag{1.4}$$

is fulfilled, where $x_0(t) = x(t; w_0)$.

The problem (1.1)-(1.4) is called initial data optimization problem. It consists in finding an optimal initial data w_0.

Theorem 1.1([2]). *There exists an optimal initial data* w_0 if $W_0 \neq \emptyset$.

Let $l = 0$ and

$$q^0(t_0, \tau, x_0, x_1) = \frac{1}{2} \parallel x_1 - y \parallel^2,$$

where $y \in R^n$ is a given point. In this case, every $w \in W$ is an admissible initial data and $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, g_0) \in W$ is said to be optimal if it minimizes the functional

$$\frac{1}{2} \parallel x(t_1; w) - y \parallel^2 . \tag{1.5}$$

Such type of functionals plays an important role in the investigation of inverse problems.

2. Formulation of the main results

Let us introduce the following notation:

$$Q(t_0, \tau, x_0, x_1) = (q^0(t_0, \tau, x_0, x_1), ..., q^l(t_0, \tau, x_0, x_1))^T, Q_{0x} = Q_x(t_{00}, \tau_0, x_{00}, x_0(t_1)),$$

$$\chi(t) = (\chi_1(t), ..., \chi_n(t)), \psi(t) = (\psi_1(t), ..., \psi_n(t)).$$

Theorem 2.1. *Let $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, g_0) \in W_0$ be an optimal initial data with $t_{00} \in (t_{01}, t_{02}), \tau_0 \in (\tau_1, \tau_2)$ and the following conditions hold:*

2.1. $t_{00}, t_{00} + \eta, t_{00} + \tau_0 \notin \{t_1 - \eta, t_1 - 2\eta, ...\}$;

2.2. the function $\varphi_0(t), t \in [\hat{\tau}, t_{02}]$ is absolutely continuous and the function $\dot{\varphi}_0(t)$ is bounded;

2.3. there exist the finite limits

$$\lim_{t \to t_{00} - \eta +} g_0(t) = g_1, \lim_{t \to t_{00} -} g_0(t) = g_2.$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0, \pi_0 \leq 0$ and the solution $(\chi(t), \psi(t))$ of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t)A(t) - \psi(t + \tau_0)B(t + \tau_0), \\ \psi(t) = \chi(t) + \psi(t + \eta)C(t + \eta), \\ \chi(t_1) = \psi(t_1) = \pi Q_{0x_1}, \\ \chi(t) = \psi(t) = 0, t > t_1 \end{cases} \tag{2.1}$$

such that the following conditions are fulfilled:

2.4. the integral maximum principle for initial functions $\varphi_0(t)$ and $g_0(t)$

$$\int_{t_{00}-\tau_0}^{t_{00}} \psi(t + \tau_0)B(t + \tau_0)\varphi_0(t)dt = \max_{\varphi \in \Phi} \int_{t_{00}-\tau_0}^{t_{00}} \psi(t + \tau_0)B(t + \tau_0)\varphi(t)dt,$$

$$\int_{t_{00}-\eta}^{t_{00}} \psi(t + \eta)C(t + \eta)g_0(t)dt = \max_{g \in G} \int_{t_{00}-\eta}^{t_{00}} \psi(t + \eta)C(t + \eta)g(t)dt;$$

2.5. the condition for the initial vector x_{00} :

$$[\pi Q_{0x_0} + \chi(t_{00})]x_{00} = \max_{x_0 \in X} [\pi Q_{0x_0} + \chi(t_{00})]x_0;$$

2.6. the condition for the initial moment t_{00} :

$$\pi Q_{0t_0} = -\psi(t_{00}+\eta)C(t_{00}+\eta)g_2 + \psi(t_{00})[A(t_{00})x_{00} + B(t_{00})\varphi_0(t_{00}-\tau_0)$$

$$+C(t_{00})g_1 + f(t_{00})] + \psi(t_{00}+\tau_0)B(t_{00}+\tau_0)[x_{00} - \varphi_0(t_{00})];$$

2.7. the condition for the delay parameter τ_0 :

$$\pi Q_{0\tau} = \psi(t_{00}+\tau_0)B(t_{00}+\tau_0)[x_{00} - \varphi_0(t_{00})] + \int_{t_{00}}^{t_{00}+\tau_0} \psi(t)B(t)\dot{\varphi}_0(t-\tau_0)dt$$

$$+ \int_{t_{00}+\tau_0}^{t_1} \psi(t)B(t)\dot{x}_0(t-\tau_0)dt.$$

Theorem 2.2. *Let* $w_0 \in W_0$ *be an optimal initial data with* $t_{00} \in [t_{01}, t_{02}), \tau_0 \in [\tau_1, \tau_2)$ *and the conditions 2.2 and 2.3 of Theorem 2.1 hold. Then there exist a vector* $\pi = (\pi_0, ..., \pi_l) \neq 0, \pi_0 \leq 0$ *and the solution* $(\chi(t), \psi(t))$ *of the system (2.1) such that the conditions 2.4 and 2.5 are fulfilled. Moreover,*

$$\pi Q_{0t_0} \leq -\psi(t_{00}+\eta+)C(t_{00}+\eta)g_1 + \psi(t_{00}+)[A(t_{00})x_{00} + B(t_{00})\varphi_0(t_{00}-\tau_0)$$

$$+C(t_{00})g_2 + f(t_{00})] + \psi(t_{00}+\tau_0+)B(t_{00}+\tau_0)[x_{00} - \varphi_0(t_{00})];$$

$$\pi Q_{0\tau} \leq \psi(t_{00}+\tau_0+)B(t_{00}+\tau_0)[x_{00} - \varphi_0(t_{00})] + \int_{t_{00}}^{t_{00}+\tau_0} \psi(t)B(t)\dot{\varphi}_0(t-\tau_0)dt$$

$$+ \int_{t_{00}+\tau_0}^{t_1} \psi(t)B(t)\dot{x}_0(t-\tau_0)dt.$$

Theorem 2.3. *Let* $w_0 \in W_0$ *be an optimal initial data with* $t_{00} \in (t_{01}, t_{02}], \tau_0 \in (\tau_1, \tau_2]$ *and the conditions 2.2 and 2.3 hold. Then there exist a vector* $\pi = (\pi_0, ..., \pi_l) \neq 0, \pi_0 \leq 0$ *and a solution* $(\chi(t), \psi(t))$ *of the system (2.1) such that the conditions 2.4 and 2.5 are fulfilled. Moreover,*

$$\pi Q_{0t_0} \geq -\psi(t_{00}+\eta-)C(t_{00}+\eta)g_1 + \psi(t_{00}-)[A(t_{00})x_{00} + B(t_{00})\varphi_0(t_{00}-\tau_0)$$

$$+C(t_{00})g_2 + f(t_{00})] + \psi(t_{00}+\tau_0-)B(t_{00}+\tau_0)[x_{00} - \varphi_0(t_{00})];$$

$$\pi Q_{0\tau} \geq \psi(t_{00}+\tau_0-)B(t_{00}+\tau_0)[x_{00} - \varphi_0(t_{00})] + \int_{t_{00}}^{t_{00}+\tau_0} \psi(t)B(t)\dot{\varphi}_0(t-\tau_0)dt$$

$$+ \int_{t_{00}+\tau_0}^{t_1} \psi(t)B(t)\dot{x}_0(t-\tau_0)dt.$$

Some comments.
c1. Theorems 2.1-2.3 are proved by a method given in [3].

c2. Theorem 2.1 corresponds to the case when two-sided variations at the points t_{00} and τ_0 are performed simultaneously. Theorem 2.2 corresponds to the case when the variations at the points t_{00} and τ_0 are performed simultaneously on the right. Theorem 2.3 corresponds to the case when the variations at the points t_{00} and τ_0 are performed simultaneously on the left.
c3. The function $\psi(t)$, generally, is discontinuous at the points $t_1 - k\eta > t_{01}, k = 0, 1, ...$ (see (2.1)).
c4. The expression

$$\psi(t_{00} + \tau_0)B(t_{00} + \tau_0)[x_{00} - \varphi_0(t_{00})]$$

in the conditions 2.6 and 2.7 **is the effect of discontinuity**.
c5. From the condition 2.4 it follows the pointwise maximum principle

$$\psi(t + \tau_0)B(t + \tau_0)\varphi_0(t) = \max_{p_1 \in K_1} \psi(t + \tau_0)B(t + \tau_0)p_1, \quad \text{a.e. on } [t_{00} - \tau_0, t_{00}],$$

$$\psi(t + \eta)C(t + \eta)g_0(t) = \max_{p_2 \in K_2} \psi(t + \eta)C(t + \eta)p_2, \quad \text{a.e. on } [t_{00} - \eta, t_{00}].$$

c6. Let

$$rank(Q_{0t_0}, Q_{0\tau}, Q_{0x_1}) = 1 + l.$$

then the solution $(\chi(t), \psi(t))$ in Theorem 2.1 is nontrivial.
c7. Let $x_{00} \in intX$. Then

$$\pi Q_{0x_0} = -\chi(t_{00}).$$

In this case, the solution $(\chi(t), \psi(t))$ is nontrivial if

$$rank(Q_{0t_0}, Q_{0\tau}, Q_{0x_0}, Q_{0x_1}) = 1 + l.$$

c8. Initial data optimization problems and related inverse problems for various classes of functional-differential equations are investigated in [4-10].

Now we formulate the necessary optimality conditions for the problem (1.1),(1.2),(1.5).

Theorem 2.4. *Let $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, g_0) \in W$ be an optimal initial data with $t_{00} \in (t_{01}, t_{02}), \tau_0 \in (\tau_1, \tau_2)$ and the conditions 2.1-2.3 hold. Then there exists the solution $(\chi(t), \psi(t))$ of the system*

$$\begin{cases} \dot{\chi}(t) = -\psi(t)A(t) - \psi(t + \tau_0)B(t + \tau_0), \\ \psi(t) = \chi(t) + \psi(t + \eta)C(t + \eta), \\ \chi(t_1) = \psi(t_1) = (y - x_0(t_1))^T, \\ \chi(t) = \psi(t) = 0, t > t_1 \end{cases} \tag{2.2}$$

such that the condition 2.4 is fulfilled. Moreover,

$$\chi(t_{00})x_{00} = \max_{x_0 \in X} \chi(t_{00})x_0; \tag{2.3}$$

$$-\psi(t_{00}+\eta)C(t_{00}+\eta)g_2+\psi(t_{00})[A(t_{00})x_{00}+B(t_{00})\varphi_0(t_{00}-\tau_0)$$
$$+C(t_{00})g_1+f(t_{00})]+\psi(t_{00}+\tau_0)B(t_{00}+\tau_0)[x_{00}-\varphi_0(t_{00})]=0;$$
$$\psi(t_{00}+\tau_0)B(t_{00}+\tau_0)[x_{00}-\varphi_0(t_{00})]+\int_{t_{00}}^{t_{00}+\tau_0}\psi(t)B(t)\dot{\varphi}_0(t-\tau_0)dt$$
$$+\int_{t_{00}+\tau_0}^{t_1}\psi(t)B(t)\dot{x}_0(t-\tau_0)dt=0.$$

Theorem 2.5. *Let $w_0\in W$ be an optimal initial data with $t_{00}\in[t_{01},t_{02}),\tau_0\in[\tau_1,\tau_2)$ and the conditions 2.2 and 2.3 hold. Then there exists the solution $(\chi(t),\psi(t))$ of the system (2.2) such that the conditions 2.4 and (2.3) are fulfilled. Moreover,*

$$-\psi(t_{00}+\eta+)C(t_{00}+\eta)g_1+\psi(t_{00}+)[A(t_{00})x_{00}+B(t_{00})\varphi_0(t_{00}-\tau_0)$$
$$+C(t_{00})g_2+f(t_{00})]+\psi(t_{00}+\tau_0+)B(t_{00}+\tau_0)[x_{00}-\varphi_0(t_{00})]\geq 0;$$
$$\psi(t_{00}+\tau_0+)B(t_{00}+\tau_0)[x_{00}-\varphi_0(t_{00})]+\int_{t_{00}}^{t_{00}+\tau_0}\psi(t)B(t)\dot{\varphi}_0(t-\tau_0)dt$$
$$\int_{t_{00}+\tau_0}^{t_1}\psi(t)B(t)\dot{x}_0(t-\tau_0)dt\geq 0.$$

Theorem 2.6. *Let $w_0\in W$ be an optimal initial data with $t_{00}\in(t_{01},t_{02}],\tau_0\in(\tau_1,\tau_2]$ and the conditions 2.2 and 2.3 hold. Then there exists the solution $(\chi(t),\psi(t))$ of the system (2.2) such that the conditions 2.4 and (2.3) are fulfilled. Moreover,*

$$-\psi(t_{00}+\eta-)C(t_{00}+\eta)g_1+\psi(t_{00}-)[A(t_{00})x_{00}+B(t_{00})\varphi_0(t_{00}-\tau_0)$$
$$+C(t_{00})g_2+f(t_{00})]+\psi(t_{00}+\tau_0-)B(t_{00}+\tau_0)[x_{00}-\varphi_0(t_{00})]\leq 0;$$
$$\psi(t_{00}+\tau_0-)B(t_{00}+\tau_0)[x_{00}-\varphi_0(t_{00})]+\int_{t_{00}}^{t_{00}+\tau_0}\psi(t)B(t)\dot{\varphi}_0(t-\tau_0)dt$$
$$+\int_{t_{00}+\tau_0}^{t_1}\psi(t)B(t)\dot{x}_0(t-\tau_0)dt\leq 0.$$

It is easy to see that for the problem 1.1, 1.2, 1.5 we have

$$Q(x_1)=\frac{1}{2}\parallel x_1-y\parallel^2,Q_{0x_1}=(x_0(t_1)-y)^T,Q_{0t_0}=0,Q_{0\tau}=0$$

and

$$\pi=\pi_0\neq 0,\pi_0\leq 0.$$

Thus, as π_0 we can take -1. After that, Theorems 2.4-2.6 follow from Theorems 2.1-2.3, respectively.

3. Example

Let

$$x=(x^1,x^2)^T\in R^2, t_{01}=\frac{1}{4}, t_{02}=\frac{3}{4}, t_1=2, \eta=1, \tau_1=\tau_2=1,$$

$$X=\{(0,0)^T\}, \Phi=\{(0,-\frac{1}{2})^T\}, G=\{(0,t)^T\}.$$

In this case, delay parameter τ, initial vector x_0, initial functions $\varphi(t)$ and $g(t)$ are fixed. Namely,

$$\tau=1, x_0=(0,0)^T, \varphi(t)=(0,-\frac{1}{2})^T, g(t)=(0,t)^T.$$

To each initial moment $t_0\in[\frac{1}{4},\frac{3}{4}]$ we assign the equation

$$\dot{x}(t)=Ax(t)+Bx(t-1)+C\dot{x}(t-1)+f(t), t\in[t_0,2]$$

with the initial condition

$$\begin{cases} x(t)=(x^1(t),x^2(t))^T=\varphi(t), t\in[t_0-1,t_0), x(t_0)=x_0,\\ \dot{x}(t)=g(t), t\in[t_0-1,t_0), \end{cases}$$

where

$$A=\begin{pmatrix} 0\,1\\ 0\,0 \end{pmatrix}, B=\begin{pmatrix} 0\,0\\ 0\,1 \end{pmatrix}, C=\begin{pmatrix} 0\,0\\ 0\,1 \end{pmatrix}, f(t)=\begin{pmatrix} 0\\ 2-2t \end{pmatrix}. \tag{3.1}$$

An initial moment $t_{00}\in[\frac{1}{4},\frac{3}{4}]$ is said to be optimal if for arbitrary $t_0\in[\frac{1}{4},\frac{3}{4}]$ the inequality

$$x_0^1(2)\le x^1(2)$$

is fulfilled, where $x_0^1(t)=x^1(t;t_{00}), x^1(t)=x^1(t;t_0)$. For the above considered example we have

$$l=0, q^0(x_1)=x_1^1, Q(x_1)=q^0(x_1), Q_0=q^0(x_0(2))=x_0^1(2),$$

$$Q_{0t_0}=0, Q_{0x_1}=(q^0_{x_1^1},q^0_{x_1^2})=(1,0).$$

From Theorem 1.1 it follows the existence of an optimal initial moment $t_{00}\in[\frac{1}{4},\frac{3}{4}]$. It is clear that

$$t_0, t_0+1\notin\{2,1,0,...\}, \forall t_0\in\left[\frac{1}{4},\frac{3}{4}\right], g_1=(0,t_{00}-1)^T, g_2=(0,t_{00})^T.$$

Thus, all the assumptions of Theorems 2.1-2.3 hold.

a) Let $t_{00}\in(\frac{1}{4},\frac{3}{4})$ be an optimal initial moment. According to Theorem 2.1 we have:

a1) $\pi=\pi_0\ne0$ and $\pi_0\le0$, therefore we can take $\pi_0=-1$.

a2) The function $(\chi(t),\psi(t))=(\chi_1(t),\chi_2(t),\psi_1(t),\psi_2(t))$ satisfies the system

$$\begin{cases} \dot{\chi}(t)=-\psi(t)A-\psi(t+1)B,\\ \psi(t)=\chi(t)+\psi(t+1)C,\\ \chi(2)=\psi(2)=\pi Q_{0x_1}=(-1,0),\\ \chi(t)=\psi(t)=0, t>2. \end{cases} \tag{3.2}$$

From (3.2), according to (3.1), we get

$$\begin{cases} \dot{\chi}_1(t) = 0, \\ \dot{\chi}_2(t) = -\psi_1(t) - \psi_2(t+1), \\ \psi_1(t) = \chi_1(t), \\ \psi_2(t) = \chi_2(t) + \psi_2(t+1), \\ \chi_1(2) = \psi_1(2) = -1, \chi_2(2) = \psi_2(2) = 0, \\ \chi_1(t) = \chi_2(t) = \psi_1(t) = \psi_2(t) = 0, t > 2. \end{cases}$$

It is clear that

$$\chi_1(t) = \psi_1(t) = -1.$$

Since

$$\psi_2(t) = 0, t \geq 2$$

and

$$\psi_2(t) = \chi_2(t) + \psi_2(t+1),$$

$\psi_2(t)$ is continuously differentiable, i.e.

$$\dot{\chi}_2(t) = \dot{\psi}_2(t) - \dot{\psi}_2(t+1).$$

Taking into account last relations, we obtain

$$\begin{cases} \dot{\psi}_2(t) = 1 - \psi_2(t+1) + \dot{\psi}_2(t+1), \\ \psi_2(t) = 0, t \geq 2. \end{cases}$$

Elementary computation shows that the solution $\psi_2(t)$ is

$$\psi_2(t) = \begin{cases} -\frac{1}{2}(3-t)^2 + 1, t \in [0,1), \\ t-2, t \in [1,2]. \end{cases}$$

a3) The necessary condition for t_{00} is:

$$0 = -\psi(t_{00}+1)Cg_2 + \psi(t_{00})[Ax_0 + B\varphi(t_{00}-1) + Cg_1 + f(t_{00})]$$

$$+\psi(t_{00}+1)B[x_{00} - \varphi(t_{00})] = -t_{00}\psi_2(t_{00}+1) + \Big[-\frac{1}{2} + t_{00} - 1 + 2 - 2t_{00}\Big]\psi_2(t_{00})$$

$$+\frac{1}{2}\psi_2(t_{00}+1) = \Big(\frac{1}{2} - t_{00}\Big)\Big[\psi_2(t_{00}) + \psi_2(t_{00}+1)\Big].$$

If $t_0 \in [\frac{1}{4}, \frac{3}{4}]$, then $t_0 + 1 \in [1,2]$. Therefore,

$$\psi_2(t_0) + \psi_2(t_0+1) = -\frac{1}{2}(3-t_0)^2 + 1 + t_0 + 1 - 2$$

$$= -\frac{1}{2}[t_0^2 - 8t_0 + 9] < 0, \forall t_0 \in \Big[\frac{1}{4}, \frac{3}{4}\Big].$$

Thus, $t_{00} = \frac{1}{2}$.

b) If $t_{00} = \frac{1}{4}$, then, according to Theorem 2.2, we have

$$\alpha(t_{00}) =: \Big(\frac{1}{2} - t_{00}\Big)\Big[\psi_2(t_{00}) + \psi_2(t_{00}+1)\Big] \geq 0.$$

But for $t_{00} = \frac{1}{4}$ we get

$$\alpha\Big(\frac{1}{4}\Big) = \frac{1}{4}\Big[\psi_2\Big(\frac{1}{4}\Big) + \psi_2\Big(\frac{5}{4}\Big)\Big] < 0,$$

i.e. $t_{00} \neq \frac{1}{4}$.

c) If $t_{00} = \frac{3}{4}$, then, according to Theorem 2.3, we have

$$\alpha(t_{00}) \leq 0.$$

But for $t_{00} = \frac{3}{4}$ we get

$$\alpha\Big(\frac{3}{4}\Big) = -\frac{1}{4}\Big[\psi_2\Big(\frac{3}{4}\Big) + \psi_2\Big(\frac{7}{4}\Big)\Big] > 0,$$

i.e. $t_{00} \neq \frac{3}{4}$.

Consequently, the optimal initial moment is $t_{00} = \frac{1}{2}$.

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