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On edge neighborhood graphs-II

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Abstract. Let G be an undirected, simple, connected graph e and $e = uv$ be an edge of G. Let $N_G(e)$ be the subgraph of G induced by the set of all vertices of G which are not incident to e but are adjacent to at least one end vertex of e. $N_e$is the class of all graphs H such that, for some graph G, $N_G(e) \cong H$ for every edge e of G. Zelinka [6] studied edge neighborhood graphs and obtained some special graphs in N_e. Ali and Alsardary [1] obtained some other graphs in N_e. In this paper we give some new graphs in N_e and investigate some properties of the city graphs.

Key Words and Phrases: edge neighborhood graph

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1. Introduction

Let G be an undirected, simple, connected graph and $e = uv$ be an edge of G. Let U be the set of all vertices of G that are adjacent to at least one of the vertices $\{u, v\}$ and let $U_e = U - \{u, v\}$. Then the induced subgraph $< U_e >$ is called the *edge neighborhood graph of* e *in* $G$and is denoted by $N_G(e)$. Let $N(e)$ be the class of all graphs H such that, for some graph G, $N_G(e) \cong H$ for every edge e of G.

See [4] and [7] for the background material. We follow the notation and terminology of Harary [3] and Tutte [5].

Zelinka [6] has proved that N_e includes the following graphs:
(i) K_n for every positive integer n.
(ii) for every pair of positive integers m, n.
(iii) CyclesC_4, C_6, C_8.
(iv) Q_1, Q_2, Q_3 where $Q_n$is the cube of dimension n.
(v) $K^*_{n,n}$ where $K^*_{n,n}$is obtained from $K_{n,n}$ by deleting edges of a maximum matching.

Balasubramanian and Alsardary [2] has proved that N_e includes the following graphs:
(vi) nK_2 for every positive integer n.
(vii) $2K_1 \bigcup 2K_2$.
(viii)$K_{m-1,m-1,m, \ldots ,m}$, the complete k-partite graph for every positive integer $m \geq 2$ and $k \geq 3$.

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Moreover, Ali and Alsardary [1] has proved that N_e includes the following graphs:
(ix) nK_1 for every positive integer n.
(x) $K_1 \bigcup 2K_2$.
(xi) The line graph $L(K^+_{3,m})$, where$K^+_{3,m}$ is the graph obtained from $K_{3,m}$ by joining two vertices of the independent subset V_1, with $[V_1] = 3$ and $[V_2] = m$. (xii) $K_n \bigcup (K_2 \times K_m)$ for any positive integers m, n where K_n is disjoint from$K_2 \times K_m$,
and $K_2 \times K_m$ is the Cartesian product of K_2 and K_m.

Definition 1. *Let H and G be graphs such that $H \cong N_G(e)$ for every edge e of G. We call G a city (or required [6]) graph containing the neighborhood H and represent it by C_H.*

In the present work, we obtain new edge neighborhood graphs and give some properties of the city graphs.

2. New Edge Neighborhood Graphs

Proposition 1. *Every cubic connected graph G of girth≥ 5 is a city graph containing$4K_1$*

Proof. Obvious.◄

Note. Proposition 1 makes it clear that for $H \in N_e$, the city graph containing H is not necessarily unique. As for $H = 4K_1$, any of the following graphs can be taken as a city containing $4K_1$:

Cube, Dodecahedron, Heawood, Petersen, McGee, Tutte-Coxeter, Grinberg and Tutte graphs.

Definition 2. *Let G be a labeled graph and let $\{v_1, v_2, ..., v_p\}$ be an enumeration of the vertices of G. Let $(n_1, n_2, ..., n_p)$ be a finite sequence of non-negative integers such that $\sum_i n_i > 0$. We define $G(n_1, n_2, ..., n_p)$ as follows:*
For $i = 1, 2, ..., p$, let $V_i = \left\{v_i^1, v_i^2, ..., v_i^{n_i}\right\}$ if $n_i > 0$, $V_i = \phi$ if $n_i = 0$. Then the vertex set of $G(n_1, n_2, ..., n_p)$ is $\bigcup_{i=1}^p V_i$. We join v_i^α and v_j^β if and only if, $i \neq j$ and v_i and v_j are adjacent in G

Many interesting class of graphs can be brought under the definition.
(i) Let $G = K_1$. Then $G(n_1) = n_1K_1$.
(ii)Let $G = P_1$ be a path of length 1. Then $G(m, n) = K_{m,n}$.
(iii)Let $G = C_3$ be a three-cycle. Then $G(m, n, p) = K_{m,n,p}$.

Proposition 2. *Let $G = C_4$. Then for every positive integer m,$G(m-1, m-1, m, m) \in N_e$.*

Proof. Clearly $G(m, m, m, m)$ is a city containing the neighbourhood $G(m-1, m-1, m, m)$.◄

Proposition 3. *Let $H = P_3 = (v_1, v_2, v_3, v_4)$. Then $H(m, m-1, m-1, m) \in N_e$ for every positive integer m.*

Proof. Let $G = C_n = (v_1, v_2, ..., v_n, v_1)$, $n \geq 5$. Then $G(m, m, ..., m\ (n\ times))$ is a city containing the neighbourhood $P_3(m, m-1, m-1, m)$.◀

In connection with Proposition 2, we have the following result:

Proposition 4. *Let $G = K_{s,t}$ be a complete bipartite graph of vertices $v_1, v_2, \ \dots\ , v_s; v_{s+1}, v_{s+2}, \ \dots\ , v_{s+t}$. Then G(m, m, . . . ,m) is the city graph containing the neighbourhood G(m-1, m, m, . . . ,m ; m-1, m, m, . . . ,m).*

Proposition 5. *Let G be a complete graph with $V(G) = \{v_1, v_2, \ \dots\ , v_n\}$. Then $G(m-1, m-1, m, \dots, m) \in N_e$ for every positive integer m.*

Proof. Clearly, $G(m, m, \dots, m)$ is a city graph containing the neighbourhood $G(m-1, m-1, m, \dots, m)$. ◀

Proposition 6. *Let $G = Q_3$ and $V(G) = \{v_1, v_2, \ \dots\ , v_8\}$. Then $G(m-1, m-1, m, m, m, m, 0, 0) \in N_e$, for every positive integer m.*

Proof. One may easily check that $G(m, m, m, \dots, m(8times))$ is a city graph containing $G(m-1, m-1, m, m, m, m, 0, 0)$. ◀

We think that for $n \geq 4$, $Q_n(m, m, \ \dots\ , (2^n\ times))$ is a city graph containing $Q_n(m-1, m-1, m, \ \dots\ , m\ , \ 0\ , \ \dots\ 0)$ in which m is repeated $2(n-1)$ times, and 0 is repeated $(2^n - 2n)$ times .

In view of Propositions 2-6, we may propose the following conjecture:

Conjecture. Let G be a city graph containing a neighbourhood F, and let $V(G) = \{v_1, v_2, \ \dots\ , v_p\}$. Then for every positive integer m, $G(m, m, m, \dots, m(ptimes))$ is a city graph containing some neighbourhood graph.

3. Some Properties of City Graphs

In this section we study some useful properties of the city graphs, especially those not containing triangles. Let G be a city graph containing H. First we shall present some simple propositions.

Proposition 7. *If G contains no triangles, then for each edge $e = uv$ of G,*

$$d_G(u) + d_G(v) = |H| + 2,$$

where $|H|$ denotes the order of H, and $d_G(\ )$ is the degree of vertex ( ) in the graph G.

Proof. From the definition of the edge neighbourhood graphs, each vertex of $N_G(e)$ is adjacent with u or v but not with both of them. Thus,

$$d_G(u) + d_G(v) - 2 = |N_G(e)| = |H|\,,$$

for each edge $e = uv$ of .◄

Definition 3. *A graph G is called edge-regular r of degree r if each edge of G is adjacent with exactly r edges, i.e., $L(G)$ is r-regular.*
From Proposition 7 it is clear that if G has no triangles, then it is edge-regular of degree $|H|$.

Proposition 8. *Let $P = x_1x_2...x_k$, $k \geq 3$, be a path of a city graph G containing H. If G has no triangles, then*

$$d_G(x_i) = \begin{cases} r & , \ for\ all\ odd\ \ i \leq k \\ |H| + 2 - r, & for\ all\ even\ i \leq k \end{cases}$$

where $r = d_G(x_1)$.

Proof. From Proposition 7, for each $i = 1,\ 2,\ ,...,k-2$,

$$d_G(x_i) + d_G(x_{i+1}) = d_G(x_{i+1}) + d_G(x_{i+2}) = |H| + 2.$$

Thus,

$$d_G(x_i) = d_G(x_{i+2})\,,\ i = 1,\ 2,\ ...,\ k-2$$

Therefore,

$$d_G(x_i) = r,\ for\ odd\ \ i \leq k$$

and

$$d_G(x_i) = |H| + 2 - r,\ for\ even\ \ i \leq k.\ ◄$$

If the degree of each vertex of a graph G is r or s then it is called (r, s)-regular.

Theorem 1. *If G is a city graph containing H and G is without triangles, then is (r, s)-regular with $r + s = |H| + 2$.*

Proof. Let W be the set of all vertices of G of degree r or s such that $r + s = |H| + 2$. From Proposition 9, $W \neq \phi$. Let $< W >$ be the subgraph of G induced by W. If $< W > \neq G$, then there is a vertex x of G not in W which is adjacent to a vertex $y \in W$. From Proposition 7,

$$d_G(x) + d_G(y) = |H| + 2.$$

Thus, x is either of degree r or s, and hence x must belong to W. Hence $< W >= G$, and so G is (r, s)-regular. ◄

Theorem 2. *If G is a city graph containing a graph H without triangles and contains an odd cycle C_k of length $k \geq 5$, then is regular of degree $\frac{1}{2}|H| + 1$.*

Proof. Let $C_k = x_1x_2...x_kx_1$. Then, by Proposition 8,

$$d_G(x_1) = d_G(x_3) = \ ... = d_G(x_k),$$

$$d_G(x_2) = d_G(x_4) = \ ... = d_G(x_{k-1}) = d_G(x_1).$$

By Proposition 7,

$$d_G(x_1) + d_G(x_2) = |H| + 2$$

Thus,

$$d_G(x_i) = \tfrac{1}{2}|H| + 1, \ \ i = 1, \ 2, \ ..., \ k.$$

Now, let y be any vertex in not on C_k. Since is connected, then there is a path between y and x_1. Thus, by Proposition 9, either

$d_G(y) = d_G(x_1)$ or $d_G(y) + d_G(x_1) = |H| + 2$,

and so

$$d_G(y) = \tfrac{1}{2}|H| + 1.$$

Hence G is regular of degree $\frac{1}{2}|H| + 1$.◄

Corollary 1. *If H has an odd order, then any city graph G containing H is either bipartite or contains a triangle.*

Proof. If G has no triangles, then it has no cycles C_k of odd length $k \geq 5$, since otherwise G will be regular of degree $\frac{1}{2}|H| + 1$, which means that H should be of even order. Thus, G is bipartite. ◄

We deduce from Corollary 1 that every city graph of an odd cycle must contain a triangle. This fact may help to prove that $C_k \notin N_e$, for odd $k \geq 7$. Zelinka [6] proved that $C_5 \notin N_e$.

Corollary 2. *If G is a city graph containing H and is without triangles, then either G is $\frac{1}{2}|H| + 1$-regular or (r, s)-regular with $r \neq s$ and $r + s = |H| + 2$. In the latter case G is bipartite $(V_1, V_2; E)$ with V_1 (respectively V_2) is the set of all vertices of degree r (respectively).*

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