Azerbaijan Journal of Mathematics
V. 2, No 2, 2012, July
ISSN 2218-6816

$D-$Recurrent Hopf Hypersurfaces of Sasakian Space Form

E.Abedi*, M.Ilmakchi, Z.Nazari

Abstract. In this paper, we are studying recurrent Hopf hypersurfaces in the Sasakian space form and prove that such hypersurface is the product of the Sasakian space form and the geodesic curve.

Key Words and Phrases: recurrent hypersurfaces; Sasakian manifold

2010 Mathematics Subject Classifications: 53C25; 53C40

---

1. Introduction

A differentiable manifold $\widetilde{M}^{2m+1}$ is said to have an almost contact structure if it admits a (non-vanishing) vector field ξ , a one-form η and a $(1,1)-$tensor field ϕ satisfying

$$\eta(\xi) = 1 \quad , \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the field of identity transformations of the tangent spaces at all points. These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$, and that the endomorphism ϕ has rank $2m$ at every point in $\widetilde{M}^{2m+1}$. A manifold $\widetilde{M}^{2m+1}$, equipped with an almost contact structure (ϕ, ξ, η) is called an almost contact manifold and will be denoted by $(\widetilde{M}^{2m+1}, (\phi, \xi, \eta))$.

Suppose that $\widetilde{M}^{2m+1}$ is a manifold carrying an almost contact structure. A Riemannian metric g on $\widetilde{M}^{2m+1}$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y is called compatible with the almost contact structure, and $(\widetilde{M}^{2m+1}, (\phi, \xi, \eta, g))$ is said to be an almost contact metric structure on $\widetilde{M}^{2m+1}$. It is known that an almost contact manifold always admits at least one compatible metric. Note that putting $Y = \xi$ yields

$$\eta(X) = g(X, \xi)$$

for all vector fields X tangent to $\widetilde{M}^{2m+1}$, which means that η is the metric dual of the characteristic vector field ξ.

---

*Corresponding author.

A manifold $\widetilde{M}^{2m+1}$ is said to be a contact manifold if it carries a global one-form η such that

$$\eta \wedge (d\eta)^m \neq 0$$

everywhere on M. The one-form η is called the contact form.

A submanifold M of a contact manifold $\widetilde{M}^{2m+1}$ tangent to ξ is called an invariant (resp. anti-invariant) submanifold if $\phi(T_pM) \subset T_pM, \forall p \in M$ (resp. $\phi(T_pM) \subset T_p^{\perp}M, \forall p \in M$).

A submanifold M tangent to ξ of a Riemannian contact manifold $\widetilde{M}^{2m+1}$ is called a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions D and $D^{\perp}$ on M such that:

1. $TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi$, where $\mathbb{R}\xi$ is the $1-$dimensional distribution spanned by ξ;
2. D is invariant by ϕ, i.e., $\phi(D_p) \subset D_p, \forall p \in M$;
3. $D^{\perp}$ is anti-invariant by ϕ, i.e., $\phi(D_p^{\perp}) \subset T_p^{\perp}M, \forall p \in M$.

Let $(\widetilde{M}, \phi, \xi, \eta, \widetilde{g})$ be a $(2n+1)$-dimensional contact manifold such that

$$\overline{\nabla}_X \xi = -\phi X \qquad , \qquad (\overline{\nabla}_X \phi)Y = \widetilde{g}(X,Y)\xi - \eta(Y)X,$$

where $\overline{\nabla}$ is a Levi-Chivita connection of $\widetilde{M}$. Then $\widetilde{M}$ is called a Sasakian manifold.

A plane section π in T_pM is called a $\phi-$section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a $\phi-$section is called a $\phi-$sectional curvature. A Sasakian manifold with constant $\phi-$sectional curvature c is said to be a Sasakian space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor of a Sasakian space form $\widetilde{M}(c)$ is given by [2]:

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\} \\ & -\frac{c-1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [\widetilde{g}(Y,Z)\eta(X) - \widetilde{g}(X,Z)\eta(Y)]\xi \\ & -\widetilde{g}(\phi Y, Z)\phi X + \widetilde{g}(\phi X, Z)\phi Y + 2\widetilde{g}(\phi X, Y)\phi Z\}\end{aligned}$$

for any tangent vector fields X, Y, Z on $\widetilde{M}(c)$.

Definition 1. *Let A be the shape operator of hypersurface M in $\widetilde{M}$ and the plane spanned by $\{\xi, U\}$ be invariant subspace of A. Then the hypersurface M is called a Hopf hypersurface of $\widetilde{M}$.*

Definition 2. *Let T be a $(1,1)$ tensor field on the Riemannian manifold M. Then T is called recurrent tensor field if $(\nabla_X T)Y = \omega(X)TY$ where ω is a one-form and X, Y are vector fields on M.*

Definition 3. *Let (M, g) be a Riemannian manifold. Let S_pM be the set of unit vectors in T_pM, that is*

$$S_pM = \{z \in T_pM \mid g(z, z) = 1\}.$$

Then

$$SM = \bigcup_{p \in M} S_pM = \{z \in TM \mid g(z, z) = 1\}$$

is called the unit sphere bundle of (M, g).

Definition 4. *Let (M, g) be a Riemannian manifold and $z \in SM$. Then the restriction $R_z : z^{\perp} \to z^{\perp}$ of the linear map $R(., z)z$ to $z^{\perp}$ is called the Jacobi operator with respect to z, that is $R_z x = R(x, z)z$, where $x \in z^{\perp}$.*

Definition 5. *Let M be a hypersurface in $\widetilde{M}$ and the Jacobi operator with respect to ξ be recurrent all X in D. Then the hypersurface M is called $D-$Recurrent hypersurface of $\widetilde{M}$.*

2. $D-$Recurrent Hopf hypersurfaces of Sasakian space form

Let (M, g) be a real hypersurface tangent to ξ of Sasakian space form $\overline{M}(c)$ and N be a unit normal vector field on M. Then we have

$$TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi,$$

where D is ϕ-invariant subspace and $D^{\perp}$ is a one-dimensional subspace spanned by $U = \phi(N)$ which is orthogonal component of D. Let M be $D-$Recurrent Hopf hypersurface of $\overline{M}$:

$$(\nabla_X R_{\xi})Y = \omega(X) R_{\xi}(Y) \tag{1}$$

for all X in D and Y in span $\{\xi, \phi X\}^{\perp}$ where ω is a one-form on $\overline{M}$.

Lemma 1. *Suppose M is a hypersurface of Sasakian space form $\overline{M}(c)$ with the unit normal vector field N on M. Then $\nabla_X U = -\phi AX$ for all X in D.*

Proof. From Gauss formula and Sasakian equation we compute

$$\nabla_X U + g(AX, U)N = -\phi AX$$

for all X in D. Considering the tangential and normal part, we have $\nabla_X U = -\phi AX$. ◀

Lemma 2. *Suppose M is a hypersurface of Sasakian space form $\overline{M}(c)$ with the unit normal vector field N on M. Then $A\xi = U$.*

Proof. From Gauss formula and Sasakian equation we compute

$$\nabla_U \xi + g(AU, \xi)N = -\phi U = N.$$

Considering the tangential and normal part of the last relation we conclude

$$\nabla_U \xi = 0 \quad g(AU, \xi) = 1. \tag{2}$$

We compute again to obtain

$$\nabla_\xi \xi + g(A\xi, \xi)N = -\phi\xi = 0.$$

Considering the tangential and normal part of this relation we conclude

$$\nabla_\xi \xi = 0, \quad g(A\xi, \xi) = 0 \tag{3}$$

which implies that $A\xi = U$. ◀

From Gauss formula and Sasakian equation with the Weingarten formula and above lemma we compute

$$\nabla_\xi U + g(AU, \xi)N = N.$$

and let $AU = \alpha U + \beta\xi$ we have

$$\nabla_U U + g(AU, U)N = -\phi AU = -\alpha N.$$

Considering the tangential and normal part, we compute

$$\nabla_\xi U = 0 \quad , \quad \nabla_U U = 0 \tag{4}$$

and $AU = \xi + \alpha U$.

By analyzing with curvature tensor field we have

$$\overline{R}_\xi Y = Y - g(y, \xi)\xi$$

for all Y in TM where $\overline{R}$ is Jacobi operator with respect to ξ of $\overline{M}$. Now by the Codazzi equation we have

$$R_\xi Y = Y - g(Y, \xi)\xi - g(Y, U)U$$

for all Y in TM. So we have

$$(\nabla_X R_\xi)Y = g(y, \phi X)\xi + g(Y, \xi)\phi X - g(Y, U)\phi AX.$$

Hence from (1) we have

$$g(y, \phi X)\xi + g(Y, \xi)\phi X - g(Y, U)\phi AX = \omega(X)\{y - g(Y, \xi)\xi - g(Y, U)U\} \tag{5}$$

Lemma 3. *Let M be a hypersurface of Sasakian space form $\overline{M}(c)$ with the unit normal vector field N on M. Then $AX = 0$ and $\omega(X) = 0$ for all X in D.*

Proof. In the equation (5) we set $Y = U$. Then $\phi AX = 0$ for all X in D. Thus, $AX = 0$. If Y belongs to $(span\{\phi X\})^\perp$, then from 5 we have

$$g(Y, \phi X)\xi = \omega(X)Y$$

for all X in D. Thus, $\omega(X) = 0$. ◀

Lemma 4. *The distribution $D \oplus span\{\xi\}$ is involutive in M.*

Proof. Let us choose X, Y in D. Then, using Lemma 1, we have

$$\begin{aligned} g([X,Y],U) &= g(\nabla_X Y - \nabla_Y X, U) \\ &= -g(Y, \nabla_X U) + g(X, \nabla_Y U) \\ &= g(Y, \phi AX) - g(X, \phi AY) = 0. \end{aligned}$$

On the other hand, using lemma 1 and 2, we obtain

$$\begin{aligned} g([X,\xi],U) &= g(\nabla_X \xi - \nabla_\xi X, U) \\ &= -g(\phi X, U) - g(X, \phi A\xi) = 0. \end{aligned}$$

This shows that the distribution $D \oplus span\{\xi\}$ is involutive. ◄

Now we consider the integral submanifold M' for distribution $D \oplus span\{\xi\}$ in M. Then the following lemma holds:

Lemma 5. *The integral submanifold M' is totally geodesic in M.*

Proof. As M' is a hypersurface of M, we have

$$\nabla_X Y = \nabla'_X Y + g(A'X, Y)U \qquad , \qquad \nabla_X U = -A'X$$

for all X, Y in TM', where ∇' is a Levi-Chivita connection of M' and A' is a shape operator of M' in M. Using lemmas 1, 2, 3 and equation (4), we conclude $A'X = 0$ for all X in TM'. As a result, M' is a totally geodesic hypersurface of M. ◄

Lemma 6. *The submanifold M' in is totally geodesic $\overline{M}$.*

Proof. For all X, Y in TM', from lemma 3 and 5 we have

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + g(AX, Y)N \\ &= \nabla'_X Y + g(A'X, Y)U + g(AX, Y)N \\ &= \nabla'_X Y. \end{aligned}$$

Otherwise, if one of X and Y is ξ or both of them are ξ, then using lemma 2 and equation (3), we have

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + g(AX, Y)N \\ &= \nabla'_X Y + g(A'X, Y)U + g(AX, Y)N \\ &= \nabla'_X Y. \end{aligned}$$

These results show that M' is a totally geodesic in $\overline{M}$. submanifold. ◄

Lemma 7. *The manifold M' is a Sasakian space form.*

Proof. We first take $\phi' = \phi|_M$ and use the fact where D is invariant under ϕ. As M' is a totally geodesic submanifold in $\overline{M}$ from lemma 6 for all X, Y in M' we have

$$\nabla'_X Y = \overline{\nabla}_X Y$$

and

$$\begin{aligned}(\nabla_X \phi)Y &= \nabla_X(\phi Y) - \phi(\nabla_X Y)\\ &= \overline{\nabla}_X(\phi Y) - \phi(\overline{\nabla}_X Y)\\ &= (\overline{\nabla}_X \phi)Y\\ &= \widetilde{g}(X,Y)\xi - \eta(Y)X\\ &= g'(X,Y)\xi - \eta'(Y)X,\end{aligned}$$

where η' and g' are restrictions of η and g on M, respectively. So $(M', \phi', \xi, \eta', g')$ is a $(2n-1)$-dimensional contact manifold. Now, by the Gauss equation, the curvature tensor R' of M' satisfies

$$\begin{aligned}g'(R'(X,Y)Z,W) &= g(R(X,Y)Z,W)\\ &\quad +g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W)\\ &\quad +g(A'Y,Z)g(A'X,W) - g(A'X,Z)g(A'Y,W)\\ &= (\frac{c+3}{4})[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &\quad +(\frac{c-1}{4})[g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W)\\ &\qquad +2g(X,\phi Y)g(\phi Z,W)]\end{aligned}$$

for all X, Y, Z, W in TM, so

$$H(X') = R^{\lambda}(X', \phi X') = g^{\lambda}(R^{\lambda}(X', \phi X')\phi X', X') = c$$

This shows that the M' is a Sasakian space form. ◀

Now consider the integral curve of the vector field U and denote it as $\gamma(t)$. In other words, $\gamma'(t) = U$. Hence the following theorem holds:

Theorem 1. *Let $\overline{M}$ be a Sasakian space form with the condition (1), and let M be a Hopf hypersurface of $\overline{M}$. Then M is a locally product of $M' \times \gamma$, where M' is a totally geodesic Sasakian space form and γ is a geodesic curve of M.*

Proof. With above conditions it is sufficient to show that

$$\nabla_{TM'}TM' \subseteq TM' \quad , \quad \nabla_U U = 0 \quad , \quad \nabla_U TM' = 0 \quad , \quad \nabla_{TM'}U = 0.$$

For this purpose first note that

$$g(\nabla_X Y, U) = -g(Y, \nabla_X U)$$

for all X, Y in TM'. Equations (2), (3) and (4) and lemmas 1, 2 and 3 imply the above relation. Hence, by de Rham decomposition theorem [7], M is locally isometric to the Riemannian product of the maximal integral manifolds M' and γ. By lemma 7, M is locally isometric to the Riemannian product of Sasakian space form $M'(c)$ and curve γ. ◀

References

[1] A.Bejancu and N.Papaghiuc, Semi-invariant submanifolds of a Sasakian manifold, *an. St. Univ. "Al.I.Cuja" Iasi, Matem.*, 1:163-170, 1981.

[2] D.E.Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. Vol 509, Springer-verlag. Berlin - Heidelberg - New York, 1979.

[3] F.Bouten and L.Verstraelen, Recurrent Hypersurfaces of Symmetric Spaces, *J. Korean Math. Soc.*18(1):7-13, 1981.

[4] T.Hamada, On Real Hypersurfacesof a Complex Projective Space with Recurrent Ricci Tensor, *Glasgow Math. J.* 41:297-302, 1999.

[5] T.Hamada, Note on Real Hypersurfaces of Complex Space Forms with Recurrent Ricci Tensor, *Differential Geometry - Dynamical Systems*5(1):27-30, 2003.

[6] Y.Mashiko, S.Kurosu and Y.Matsuyama, On a Kaehler Hypersurface of a Complex Space Form with the Recurrent Second Fundamental Form, *Differential Geometry - Dynamical Systems*10:221-225, 2008.

[7] G.de Rham, Sur la réductibilité d'un espace de Riemann, *Comment. Math. Helv.* 268:328344, 1952.

Esmaiel Abedi
Department of Mathematics Azarbaijan University of Tarbiat Moallem, Tabriz 53751 71379, Iran
E-mail: esabedi@azaruniv.edu

Mohammad Ilmakchi
Department of Mathematics Azarbaijan University of Tarbiat Moallem, Tabriz 53751 71379, Iran
E-mail: mohammad_ilmakchi@yahoo.com

Zahra Nazari
Department of Mathematics Azarbaijan University of Tarbiat Moallem, Tabriz 53751 71379, Iran
E-mail: z_nazari48@yahoo.com

Received 29 August 2011
Accepted 15 March 2012