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Cyclic Contraction of Kannan Type Mappings in Generalized Menger Space Using a Control Function

B. S. Choudhury *, K. Das, S. K. Bhandari

Abstract. Generalized Menger space introduced by the first two authors of the present paper [5] is a generalization of Menger space as well as probabilistic generalization of generalized metric space introduced by Branciari [Publ. Math. Debrecen 57 (2000, no. 1-2, 31-37)]. Cyclic contractions are a class of recently introduced nonself contractive mappings. Kannan type mappings are contractive mappings different in their characteristics from Banach contractions which are well known in fixed point theory. In this paper we establish a cyclic Kannan type fixed point theorem in generalized Menger spaces. The fixed point theorem established here utilizes a control function. The result obtained is illustrated with an example.

Key Words and Phrases: generalized Menger space; Cauchy sequence; fixed point; Φ-function; Ψ-function; cyclic contraction

2010 Mathematics Subject Classifications: 47H10; 54H25

1. Introduction

Branciari[1] introduced the idea of generalized metric spaces. He replaced the triangular inequality by a quadrangular inequality and generalized the metric space as follows:

Definition 1. *[1] Let X be a nonempty set, R^+ be the set of all nonnegative real numbers and $d : X \times X \to R^+$ be a mapping such that for all $x, y \in X$ and for all points $\xi, \eta \in X$, each of them different from x and y, the following holds:*

1. $d(x, y) = 0 \Longleftrightarrow x = y$,
2. $d(x, y) = d(y, x)$ *and*
3. $d(x, y) \leq d(x, \xi) + d(\xi, \eta) + d(\eta, y)$.

*Corresponding author.

Banach contraction mapping theorem in generalized metric space was also established in the same work. By an example Branciari also showed that there exist generalized metric spaces which are not metric spaces. The references [9, 10] and [20] established some other fixed point results in generalized metric spaces.

Probabilistic metric spaces are probabilistic generalizations of metric spaces in which every pair of elements is assigned to a distribution function. The theory of these spaces is an important part of stochastic analysis. Schweizer and Sklar have given a comprehensive account of several aspects of such spaces in their book [26].

Definition 2. ***Probabilistic metric space [11, 26]***
A probabilistic metric space (briefly, a PM-space) is an ordered pair (X, F), where X is a non empty set and F is a mapping from $X \times X$ into the set of all distribution functions. We denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(t)$ represents the value of $F_{x,y}$ at $t \in R$. The function $F_{x,y}$ is assumed to satisfy the following conditions for all $x, y, z \in X$:

(i) $F_{x,y}(0) = 0$,
(ii) $F_{x,y}(t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,
(iii) $F_{x,y}(t) = F_{y,x}(t)$ *for all* $t > 0$,
(iv) if $F_{x,y}(t_1) = 1$ *and* $F_{y,z}(t_2) = 1$ *then* $F_{x,z}(t_1 + t_2) = 1$ *for* $t_1, t_2 > 0$.

A particular type of probabilistic metric space is Menger space in which the triangular inequality is postulated with the help of a t-norm.

Definition 3. ***n-th order t-norm [29]***
A mapping $T : \Pi_{i=1}^{n}[0, 1] \to [0, 1]$ is called a n-th order t-norm if the following conditions are satisfied:

(i) $T(0, 0, ..., 0) = 0$, $T(a, 1, 1, ..., 1) = a$ *for all* $a \in [0, 1]$,
(ii) $T(a_1, a_{2,}, a_3, ..., a_n) = T(a_2, a_1, a_3, ..., a_n) = T(a_2, a_3, a_1, ..., a_n)$
$= .... = T(a_2, a_3, a_4, ..., a_n, a_1)$,
(iii) $a_i \geq b_i$, *i=1,2,3,....,n implies* $T(a_1, a_2, a_3, ..., a_n) \geq T(b_1, b_2, b_3, ..., b_n)$,
(iv) $T(T(a_1, a_2, a_3, ..., a_n), b_2, b_3, ..., b_n)$
$= T(a_1, T(a_2, a_3, ..., a_n, b_2), b_3, ..., b_n)$
$= T(a_1, a_2, T(a_3, a_4 ...., a_n, b_2, b_3), b_4, ..., b_n)$
=..............................
$= T(a_1, a_2, ..., a_{n-1}, T(a_n, b_2, b_3, ..., b_n))$.

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

Definition 4. ***Menger space [11, 26]***
A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a (binary) t-norm, such that the following are satisfied:

1. $F_{x,y}(0) = 0$ *for all* $x, y \in X$,

2. $F_{x,y}(s) = 1$ *for all* $s > 0$ *and* $x, y \in X$ *if and only if* $x = y$,

3. $F_{x,y}(s) = F_{y,x}(s)$ *for all* $x, y \in X$, $s > 0$ *and*

4. $F_{x,y}(u+v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ *for all* $u, v \geq 0$ *and* $x, y, z \in X$.

The first fixed point theorem in probabilistic metric spaces was proved by Sehgal and Bharucha-Reid [27] in 1972. They proved their result for mappings satisfying some contractive conditions. Subsequently, fixed point theory in probabilistic metric spaces has developed in a large way. A comprehensive survey of this line of research up to 2001 is described by Hadzic and Pap in [11]. Some of the more recent references dealing with probabilistic contraction may be noted in [2, 3, 7, 8, 21] and [23]. Incorporating the approach of Branciari, the first two of the present authors had introduced the concept of generalized Menger spaces in their paper [5]. The definition is as follows:

Definition 5. ***Generalized Menger space [5]***
Let X be a non-empty set and F is a function from $X \times X$ to the set of all distribution functions. Then (X, F, Δ) is said to be a generalized Menger space if for all $x, y \in X$ and all distinct points $z, w \in X$ each of them different from x and y, the following conditions are satisfied:

(i) $F_{x,y}(0) = 0$,
(ii) $F_{x,y}(t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,
(iii) $F_{x,y}(t) = F_{y,x}(t)$ *for all* $t > 0$,
(iv) $F_{x,y}(t) \geq \Delta(F_{x,z}(t_1), F_{z,w}(t_2), F_{w,y}(t_3))$, *where* $t_1 + t_2 + t_3 = t$ *and* Δ *is a 3-rd order t-norm (Definition 1.3).*

Generalized Menger space is a generalization of Menger space. It is also a probabilistic generalization of generalized metric spaces introduced by Branciari.

Definition 6. *Let (X, F, Δ) be a generalized Menger space. A sequence $\{x_n\} \subset X$ is said to converge to some point $x \in X$ if given $\epsilon > 0$, $\lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$*

$$F_{x_n,x}(\epsilon) > 1 - \lambda.$$

Definition 7. *A sequence $\{x_n\}$ in a generalized Menger space (X, F, Δ) is said to be a Cauchy sequence in X if given $\epsilon > 0, \lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that*

$$F_{x_n,x_m}(\epsilon) > 1 - \lambda \text{ for all } m, n > N_{\epsilon,\lambda}. \qquad (1.1)$$

Definition 8. *A generalized Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X.*

In [12] Khan, Swaleh and Sessa introduced a new category of contractive fixed point problems in metric space. They introduced the concept of "altering distance function", which is a control function that alters the distance between two points in a metric space. This concept was further generalized in a number of works. There are several works in fixed point theory involving altering distance function, some of these are noted in [22, 24] and [25].

Recently the first two of the present authors had extended the concept of altering distance function to the context of Menger spaces [2]. The definition is as follows:

Definition 9. ***Φ-function [2]***
A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

1. *$\phi(t) = 0$ if and only if $t = 0$,*
2. *$\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,*
3. *ϕ is left continuous in $(0, \infty)$,*
4. *ϕ is continuous at 0.*

With the help of this function an extension of Sehgal's contraction was established in [2]. Other fixed point results using the Φ-function were obtained in probabilistic metric spaces, which may be noted in [3, 4, 7] and [21].

The following is the definition of Kannan type mapping:

Definition 10. *[13, 14] Let (X, d) be a metric space and f be a mapping on X. The mapping f is called a Kannan type mapping if there exists $0 \le \alpha < \frac{1}{2}$ such that*
$d(fx, fy) \le \alpha[d(x, fx) + d(y, fy)]$ for all $x, y \in X$.

This type of mappings have an important role in fixed point theory. They are different from Banach contractions. It is known that every Banach contraction and every Kannan type mappings in a complete metric space have unique fixed points. But they are separate classes of mappings. While the Banach contraction is always continuous, there are examples of discontinuous mappings which are Kannan type mappings. Again Banach contraction does not characterize metric completeness. In [28] it has been proved that every metric space X is complete if and only if every Kannan type mapping has a fixed point. But this is not the case with the Banach contraction. In fact, Connell in [6] has given an example of a metric space which is not complete but every Banach contraction defined on it has a fixed point. There are a large number of works dealing with Kannan type mappings. Some of these works are noted in [5, 17, 18, 19].

In recent years cyclic contraction and cyclic contractive type mapping have appeared in several works. This line of research was initiated by Kirk, et al [16], where they established the following generalization of the contraction mapping principle:

Theorem 1. *[16] Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f : A \bigcup B \to A \bigcup B$ satisfies:*
(1) $fA \subseteq B$ and $fB \subseteq A$,
(2) $d(fx, fy) \le kd(x, y)$ forall $x \in A$ and $y \in B$ where $k \in (0, 1)$.
Then f has a unique fixed point in $A \bigcap B$.

These types of maps are nonself mappings on subsets of metric spaces. They bear a close association with the proximity point problem. References [15, 30, 31] are some examples of these type of works.

The purpose of this paper is to apply the control function mentioned above to establish a cyclic Kannan type fixed point result in generalized Menger spaces. Our result extends an existing result and is illustrated with an example.

The following function is utilized in our theorem:

Definition 11. ***Ψ-function [5]***
A function $\psi : [0,1] \times [0,1] \to [0,1]$ is said to be a Ψ-function if

1. *ψ-is monotone increasing and continuous,*
2. *$\psi(x,x) > x$ for all $0 < x < 1$,*
3. *$\psi(1,1) = 1, \psi(0,0) = 0$.*

An example of Ψ-function:
$\psi(x,y) = \frac{p\sqrt{x}+q\sqrt{y}}{p+q}$, p and q are positive numbers.

2. Main Result

Theorem 2. *Let (X, F, Δ) be a complete generalized Menger space, where Δ is the 3rd order minimum t-norm given by $\Delta(\alpha, \beta, \gamma) = \min\{\alpha, \beta, \gamma\}$ and let there exist two non-empty closed subsets A and B of X such that the mapping $T : A \bigcup B \to A \bigcup B$ satisfies the following conditions :*

(i) $TA \subseteq B$ and $TB \subseteq A$ (2.1)

(ii) $F_{Tx,Ty}(\phi(t)) \geq \psi(F_{x,Tx}(\phi(\frac{t_1}{a})), F_{y,Ty}(\phi(\frac{t_2}{b})))$ (2.2)

for all $x \in A$ and $y \in B$ where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$, ψ is a Ψ-function and ϕ is a Φ-function. Then $A \bigcap B$ is nonempty and T has a unique fixed point in $A \bigcap B$.

Proof. Let x_0 be any arbitrary point of A. We now construct a sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}$, for all positive integers $n \geq 1$.
Then, by (2.1), we obtain

$$x_{2n} = T^{2n}x_0 \in A \text{ and } x_{2n+1} = T^{2n+1}x_0 \in B \text{ for all integers } n \geq 0. \tag{2.3}$$

Now, for $t, t_1, t_2 > 0$, with $t = t_1 + t_2$ and taking n be even we have

$$F_{x_{n+1},x_n}(\phi(t)) = F_{Tx_n,Tx_{n-1}}(\phi(t)) \geq \psi(F_{x_n,Tx_n}(\phi(\frac{t_1}{a})), F_{x_{n-1},Tx_{n-1}}(\phi(\frac{t_2}{b}))) =$$

$$(since x_n \in A, x_{n-1} \in B)$$

$$= \psi(F_{x_n,x_{n+1}}(\phi(\frac{t_1}{a})), F_{x_{n-1},x_n}(\phi(\frac{t_2}{b}))) = \psi(F_{x_{n+1},x_n}(\phi(\frac{t_1}{a})), F_{x_n,x_{n-1}}(\phi(\frac{t_2}{b}))).$$

Let $t_1 = \frac{at}{a+b}$, $t_2 = \frac{bt}{a+b}$ and $c = a + b$, (2.5)
then obviously we have $0 < c < 1$.

Then we have from (2.4)

$$F_{x_{n+1},x_n}(\phi(t)) \geq \psi(F_{x_{n+1},x_n}(\phi(\frac{t}{c})), F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))).$$

Again for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$ and taking n be odd we have

$$F_{x_{n+1},x_n}(\phi(t)) = F_{Tx_n,Tx_{n-1}}(\phi(t)) = F_{Tx_{n-1},Tx_n}(\phi(t)) \geq$$
$$\geq \psi(F_{x_{n-1},Tx_{n-1}}(\phi(\frac{t_1}{a})), F_{x_n,Tx_n}(\phi(\frac{t_2}{b})))$$
$$(since x_{n-1} \in A, x_n \in B)$$
$$= \psi(F_{x_{n-1},x_n}(\phi(\frac{t_1}{a})), F_{x_n,x_{n+1}}(\phi(\frac{t_2}{b}))) = \psi(F_{x_n,x_{n-1}}(\phi(\frac{t_1}{a})), F_{x_{n+1},x_n}(\phi(\frac{t_2}{b}))).$$

Taking t, t_1, t_2 and c as in (2.5) we have from (2.7)

$$F_{x_{n+1},x_n}(\phi(t)) \geq \psi(F_{x_n,x_{n-1}}(\phi(\frac{t}{c})), F_{x_{n+1},x_n}(\phi(\frac{t}{c}))).$$

We now claim that for all $t > 0$

$$F_{x_{n+1},x_n}(\phi(\frac{t}{c})) \geq F_{x_n,x_{n-1}}(\phi(\frac{t}{c})).$$

If possible, let for some $t > 0$

$$F_{x_{n+1},x_n}(\phi(\frac{t}{c})) < F_{x_n,x_{n-1}}(\phi(\frac{t}{c})).$$

Then we have from (2.6) and (2.8)

$$F_{x_{n+1},x_n}(\phi(t)) \geq \psi(F_{x_{n+1},x_n}(\phi(\frac{t}{c})), F_{x_{n+1},x_n}(\phi(\frac{t}{c}))) > F_{x_{n+1},x_n}(\phi(\frac{t}{c})) \geq F_{x_{n+1},x_n}(\phi(t)),$$

which is a contradiction, since $0 < c < 1$, ϕ is strictly increasing and F is non-decreasing.

Therefore, for all $t > 0$

$$F_{x_{n+1},x_n}(\phi(\frac{t}{c})) \geq F_{x_n,x_{n-1}}(\phi(\frac{t}{c})).$$

Hence, using (2.9), we have from (2.6) and (2.8)

$$F_{x_{n+1},x_n}(\phi(t)) \geq \psi(F_{x_n,x_{n-1}}(\phi(\frac{t}{c})), F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))) \geq F_{x_n,x_{n-1}}(\phi(\frac{t}{c})).$$

By (2.10), we have for all positive integers n

$$F_{x_{n+1},x_n}(\phi(t)) \geq F_{x_n,x_{n-1}}(\phi(\frac{t}{c})).$$

By repeated application of (2.10), we have after n steps

$$F_{x_{n+1},x_n}(\phi(t)) \geq F_{x_1,x_0}(\phi(\frac{t}{c^n})).$$

Therefore,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(\phi(t)) = 1$$

for all $t > 0$. By virtue of property of ϕ and F, we can choose $s > 0$ such that $s > \phi(t)$. Thus, the above limit implies that for all $s > 0$

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(s) = 1.$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\epsilon) \leq 1 - \lambda.$$

We take $m(k)$ corresponding to $n(k)$ to be the smallest integer satisfying (2.13), so that

$$F_{x_{m(k)-1},x_{n(k)}}(\epsilon) > 1 - \lambda.$$

We now claim that

$$F_{x_{m(k)-2},x_{n(k)}}(\epsilon) > 1 - \lambda.$$

If possible, let for $\epsilon > 0$

$$F_{x_{m(k)-2},x_{n(k)}}(\epsilon) \leq 1 - \lambda,$$

which contradicts the fact that $m(k)$ is the smallest integer satisfying (2.13).

Hence,

$$F_{x_{m(k)-2},x_{n(k)}}(\epsilon) > 1 - \lambda.$$

If $\epsilon_1 < \epsilon$, then we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_1) \leq F_{x_{m(k)},x_{n(k)}}(\epsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $m(k) > n(k) > k$ and satisfying (2.13), (2.14), (2.15) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)) \leq 1 - \lambda,$$

$$F_{x_{m(k)-1},x_{n(k)}}(\phi(\epsilon_2)) > 1 - \lambda,$$

and

$$F_{x_{m(k)-2},x_{n(k)}}(\phi(\epsilon_2)) > 1-\lambda.$$

By the properties of ϕ, we can choose $\rho_1 > 0$ and $\rho_2 > 0$ such that $\rho_1 + \rho_2 < \phi(\epsilon_2)$.

Again by (2.12), we have for sufficiently large k

$$F_{x_{m(k)},x_{m(k)-1}}(\rho_1) > 1-\lambda,$$

and

$$F_{x_{m(k)-1},x_{m(k)-2}}(\rho_2) > 1-\lambda.$$

As F is left continuous, we have

$$F_{x_{m(k)-2},x_{n(k)}}(\phi(\epsilon_2)-\rho_1-\rho_2) > 1-\lambda.$$

Now from (2.16) we have

$$1-\lambda \geq F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2))$$

$$\geq \Delta(F_{x_{m(k)},x_{m(k)-1}}(\rho_1), F_{x_{m(k)-1},x_{m(k)-2}}(\rho_2), F_{x_{m(k)-2},x_{n(k)}}(\phi(\epsilon_2)-\rho_1-\rho_2)) > \Delta(1-\lambda, 1-\lambda, 1-\lambda)$$

(using (2.19), (2.20), (2.21))

$$= 1-\lambda,$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence.

Since X is a complete, we have $x_n \to z \in X$ for $n \to \infty$. That is,

$$\lim_{n\to\infty} x_n = z.$$

The subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of $\{x_n\}$ also converge to z. Now $\{x_{2n}\} \subset A$ and A is closed. Therefore, $z \in A$. Similarly $\{x_{2n-1}\} \subset B$ and B is closed. Therefore, $z \in B$. Thus we have $z \in A \bigcap B$.

We now show that $Tz = z$. If possible, let $0 < F_{z,Tz}(\phi(t)) < 1$ for some $t > 0$. By virtue of the property of ϕ, we can choose $\xi_1, \xi_2, t_1, t_2 > 0$ such that $\phi(t) = \xi_1 + \xi_2 + \phi(t_1 + t_2)$ with $\phi(\frac{t_1}{a}) > \phi(t)$ and $\phi(\frac{t_2}{b}) > \phi(t)$. This is possible since $0 < a, b < 1$.

Now we can get two possible cases.

Case-I Taking n be even so that $x_n \in A$. As $z \in A \bigcap B \subset B$, we have

$$\begin{aligned}
F_{z,Tz}(\phi(t)) &\geq \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), F_{x_{n+1},Tz}(\phi(t_1+t_2))) \\
&= \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), F_{Tx_n,Tz}(\phi(t_1+t_2))) \\
&\geq \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), \psi(F_{x_n,x_{n+1}}(\phi(\frac{t_1}{a})), F_{z,Tz}(\phi(\frac{t_2}{b})))) \\
&\geq \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), \psi(F_{x_n,x_{n+1}}(\phi(\frac{t_1}{a})), F_{z,Tz}(\phi(t)))).
\end{aligned}$$

By (2.12), (2.22) and (2.23), there exists a positive integer N_1 such that

$$F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), F_{x_n,x_{n+1}}(\phi(\frac{t_1}{a})) > F_{z,Tz}(\phi(t))$$

for all $n > N_1$.

Then we have from (2.23)

$$F_{z,Tz}(\phi(t)) > F_{z,Tz}(\phi(t)),$$

which is a contradiction.

Case-II Taking n be odd so that $x_n \in B$. As $z \in A \bigcap B \subset A$,
we have

$$F_{z,Tz}(\phi(t)) \geq \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), F_{x_{n+1},Tz}(\phi(t_1 + t_2)))$$

$$= \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), F_{Tz,Tx_n}(\phi(t_1 + t_2)))$$

$$\geq \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), \psi(F_{z,Tz}(\phi(\frac{t_1}{a})), F_{x_n,x_{n+1}}(\phi(\frac{t_2}{b}))))$$

$$\geq \Delta(F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), \psi(F_{z,Tz}(\phi(t)), F_{x_n,x_{n+1}}(\phi(\frac{t_2}{b})))).$$

By (2.12), (2.22) and (2.24), there exists a positive integer N_2 such that

$$F_{z,x_n}(\xi_1), F_{x_n,x_{n+1}}(\xi_2), F_{x_n,x_{n+1}}(\phi(\frac{t_2}{b})) > F_{z,Tz}(\phi(t))$$

for all $n > N = \max\{N_1, N_2\}$.

Then we have from (2.24),

$$F_{z,Tz}(\phi(t)) > F_{z,Tz}(\phi(t)),$$

which is a contradiction.

Combining Case-I and Case-II we have $F_{z,Tz}(\phi(t)) = 1$ for all $t > 0$ which implies that $z = Tz$.

For uniqueness, let z and u be two fixed points in $A \bigcap B$. Therefore, for all $t > 0$,

$$F_{z,u}(\phi(t)) = F_{Tz,Tu}(\phi(t))$$

$$\geq \psi(F_{z,Tz}(\phi(\frac{t_1}{a})), F_{u,Tu}(\phi(\frac{t_2}{b})))$$

(for $t_1, t_2 > 0$ and $t_1 + t_2 = t$)

$$= \psi(F_{z,z}(\phi(\frac{t_1}{a})), F_{u,u}(\phi(\frac{t_2}{b})))$$

$= \psi(1, 1) = 1.$

By virtue of property of ϕ, we can assert that $z = u$. ◄

Remark 1. *The present work is an extension of the result in [5].*

Now we give the following example to illustrate our result.

Example 1. *Let* $X = \{x_1, x_2, x_3, x_4\}$, $A = \{x_2, x_3\}$ *and* $B = \{x_1, x_2, x_4\}$. *Here the t-norm* $\Delta(a, b, c) = min(a, b, c)$, *that is* Δ *is the 3-rd order minimum t-norm and let* $F_{x,y}(t)$ *be defined as*

$$F_{x_1,x_2}(t) = F_{x_2,x_1}(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 0.70, & \text{if } 0 < t < 6, \\ 1, & \text{if } t \ge 6. \end{cases}$$

$$F_{x_1,x_3}(t) = F_{x_3,x_1}(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 0.90, & \text{if } 0 < t \le 3, \\ 1, & \text{if } t > 3. \end{cases}$$

$$F_{x_1,x_4}(t) = F_{x_4,x_1}(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 0.80, & \text{if } 0 < t \le 4, \\ 1, & \text{if } t > 4. \end{cases}$$

$$F_{x_2,x_3}(t) = F_{x_3,x_2}(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 0.95, & \text{if } 0 < t \le 3, \\ 1, & \text{if } t > 3. \end{cases}$$

$$F_{x_2,x_4}(t) = F_{x_4,x_2}(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 0.80, & \text{if } 0 < t \le 4, \\ 1, & \text{if } t > 4. \end{cases}$$

$$F_{x_3,x_4}(t) = F_{x_4,x_3}(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 0.70, & \text{if } 0 < t < 6, \\ 1, & \text{if } t \ge 6. \end{cases}$$

It is easy to verify that (X, F, Δ) *is a complete generalized Menger space. If we define* $T : A \bigcup B \to A \bigcup B$ *as follows:* $Tx_1 = x_2, Tx_2 = x_2, Tx_3 = x_2, Tx_4 = x_3$, *then it will satisfy all the conditions of the Theorem 2.1 where*

$$\phi(t) = \begin{cases} \sqrt{t}, & \text{if } t > 0, \\ 0, & \text{if } t \le 0, \end{cases}$$

$$\psi(x, y) = \frac{\sqrt{x} + \sqrt{y}}{2},$$

$a = 0.20$, $b = 0.75$ *and* x_2 *is the unique fixed point of* T *in* $A \bigcap B$.

In this example (X, F, Δ_1) *is not a Menger space for any choice of t-norm* Δ_1 *as can be seen from the fact that*

$$F_{x_3,x_4}(5) \not\geq \Delta_1(F_{x_3,x_2}(.5), F_{x_2,x_4}(4.5))$$

for any t-norm Δ_1.

This shows that generalized Menger spaces are effective generalization of generalized metric spaces.

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Binayak S. Choudhury
Department of Mathematics, Bengal Engineering and Science University, Shibpur Howrah-711103, India.
E-mail: binayak12@yahoo.co.in

Krishnapada Das
Department of Mathematics, Bengal Engineering and Science University, Shibpur Howrah-711103, India.
E-mail: kestapm@yahoo.co.in

Samir Kumar Bhandari
Department of Mathematics, Bengal Engineering and Science University, Shibpur Howrah-711103, India.
E-mail: skbhit@yahoo.co.in

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