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Orthogonal Polynomials with Respect to the Form $u = \lambda x^{-1}v + \delta_0 + ((u)_1 - \lambda)\,\delta_0'$

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Abstract. In this paper, we study properties of the form (linear functional) $u = \lambda x^{-1}v + \delta_0 + (\lambda - (u)_1)\delta_0'$, where v is a regular form. We give a necessary and sufficient condition for the regularity of the form u. The coefficients of the three-term recurrence relation, satisfied by the corresponding sequence of orthogonal polynomials, are given explicitly. A study of the semi-classical character of the founded families is done. We conclude with some illustrative examples.

Key Words and Phrases: orthogonal polynomials; integral representations; semi-classical forms

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1. Introduction

The semi-classical forms are a natural generalization of the classical forms (Hermite, Laguerre, Jacobi, and Bessel). Since the system corresponding to the problem of determining all the semi-classical forms of class $s \geq 1$ becomes non-linear, the problem was only solved when $s = 1$ and for some particular cases [2,4,9]. Thus, several authors use different processes in order to obtain semi-classical forms of class $s \geq 1$. For instance, let v be a regular form and let us define a new form u by the relation $D(x)u = A(x)v$, where $A(x)$ and $D(x)$ are non-zero polynomials. When $D(x) = 1$, v is positive-definite and $A(x)$ is a positive polynomial, Christoffel [7] has proved that u is still a positive-definite form. This result has been generalized in [8]. The cases $A(x) = \lambda \neq 0$ and $D(x) = x - c, x^2, x^3, x^4$ were treated in [13,14,16,18], where it was shown that under certain regularity conditions the form u is still regular. Moreover, if v is semi-classical, then u is also semi-classical; see also [1,3,5,19]. When $A(x) = D(x)$, u is obtained from v by adding finitely many mass points and their derivates [11,12] and when $A(x)$ and $D(x)$ have no non-trivial common factor, it was found a necessary and sufficient condition for u to be regular in [10]. When $A(x)$ and $D(x)$ are of degree equal to one, an extensive study of the form u has been carried out in [20].

In this paper, we study the form $u = \lambda x^{-1}v + \delta_0 + (\lambda - (u)_1)\delta_0', (u)_1 \neq \lambda$ which is equivalent to the above problem with $D(x) = x^2$ and $A(x) = \lambda x$. For $(u)_1 \neq \lambda$, this

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situation has never been studied before.
The second section is devoted to the preliminary results and notations used in the sequel. In the third section, an explicit necessary and sufficient condition for the regularity of the new form is given. We obtain the coefficients of the three-term recurrence relation satisfied by the new family of orthogonal polynomials. In the fourth section, the stability of the semi-classical families is proved. Finally, in the last section, we give detailed study of some examples.

2. Preliminary results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle, n \geq 0$, the moments of v. For any form v and any polynomial h, we let v', hv, δ_0 and $x^{-1}v$ be the forms defined by: $\langle v', f\rangle := -\langle v, f'\rangle$, $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle \delta_0, f\rangle := f(0)$ and $\langle x^{-1}v, f\rangle := \langle v, \theta_0 f\rangle$ where, in general, $\big(\theta_c f\big)(x) = \dfrac{f(x)-f(c)}{x-c}$, $f \in \mathcal{P}, c \in \mathbb{C}$.
Then, it is straightforward to prove that for $f, g \in \mathcal{P}$ and for $v \in \mathcal{P}'$ we have

$$x^{-1}(xv) = v - (v)_0\delta_0, \tag{1}$$

$$x(x^{-1}v) = v, \tag{2}$$

$$x^{-1}\delta_0 = -\delta_0^{'}, \tag{3}$$

$$(fv)^{'} = f^{'}v + fv^{'}, \tag{4}$$

$$(\theta_0(fg))(x) = g(x)(\theta_0 f)(x) + f(0)(\theta_0 g)(x). \tag{5}$$

A form v is called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$ ($\deg S_n \leq n$) such that $\langle v, S_nS_m\rangle = r_n\delta_{n,m}$, $r_n \neq 0$, $n \geq 0$.

Then $\deg S_n = n, n \geq 0$ and we can always suppose each S_n is monic. In such a case, the sequence $\{S_n\}_{n\geq 0}$ is unique. It is said to be the sequence of monic orthogonal polynomials with respect to u. In the sequel, it will be denoted as MOPS. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [6])

$$\begin{aligned} &S_{n+2}(x) = (x-\xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x)\ , \quad n \geq 0\ ,\\ &S_1(x) = x - \xi_0\ , \qquad S_0(x) = 1\ . \end{aligned} \tag{6}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $n \geq 0$. By convention, we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the first order associated sequence for the sequence $\{S_n\}_{n\geq 0}$ satisfying the recurrence relation

$$\begin{aligned} &S_{n+2}^{(1)}(x) = (x-\xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x), \quad n \geq 0,\\ &S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1, \quad (S_{-1}^{(1)}(x) = 0). \end{aligned} \tag{7}$$

Another important representation of $S_n^{(1)}(x)$ is (see [6])

$$S_n^{(1)}(x) := \left\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x - \zeta} \right\rangle. \tag{8}$$

Also, let $\{S_n(.,\mu)\}_{n\geq 0}$ be the co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [6]

$$S_n(x,\mu) = S_n(x) - \mu S_{n-1}^{(1)}(x), \quad n \geq 0. \tag{9}$$

The form u is called normalized if $(u)_0 = 1$. In this paper, we suppose that the forms are normalized.

3. The study of $u = \lambda x^{-1}v + \delta_0 + ((u)_1 - \lambda)\,\delta_0^{'}$

3.1. General case

Let v be a regular form and $\{S_n\}_{n\geq 0}$ be the corresponding MOPS. For $\lambda \in \mathbb{C} - \{0\}$, we can define a new form $u \in \mathcal{P}'$ by the relation

$$u = \lambda x^{-1}v + \delta_0 + ((u)_1 - \lambda)\,\delta_0^{'}. \tag{10}$$

Equivalently, from (1), (2) and (10) we have

$$x^2 u = \lambda x v. \tag{11}$$

The form u depends on two arbitrary parameters $(u)_1$ and λ.

The case $(u)_1 = \lambda$ was treated in [18] so henceforth, we assume $\lambda \neq (u)_1$.

If we suppose that the form v has the following integral representation:

$\langle v,\, f\rangle = \int_{-\infty}^{+\infty} V(x) f(x)\,dx, \quad f \in \mathcal{P}$, with $(v)_0 = \int_{-\infty}^{+\infty} V(x)\,dx = 1,$

where V is a locally integrable function with rapid decay and continuous at the point $x = 0$, then the form u is represented by

$$\langle u,\, f\rangle = \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x} f(x)\,dx + \left\{1 - \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x}\,dx\right\} f(0) + ((u)_1 - \lambda) f^{'}(0), \tag{12}$$

where

$$P \int_{-\infty}^{+\infty} \frac{V(x)}{x} f(x)\,dx = \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{-\epsilon} \frac{V(x)}{x} f(x)\,dx + \int_{\epsilon}^{+\infty} \frac{V(x)}{x} f(x)\,dx \right\}.$$

Suppose u is regular, and let $\{Q_n\}_{n\geq 0}$ be its corresponding MOPS. It satisfies

$$\begin{array}{l} Q_{n+2}(x) = (x - \beta_{n+1}) Q_{n+1}(x) - \gamma_{n+1} Q_n(x)\,, \quad n \geq 0\,, \\ Q_1(x) = x - \beta_0\,, \qquad Q_0(x) = 1\,. \end{array} \tag{13}$$

It follows from (11) that the sequence $\{Q_n\}_{n\geq 0}$, when it exists, satisfies the following finite-type relations [15, p 301, Proposition 2.1.]:

$$\begin{aligned} &xQ_{n+2}(x) = S_{n+3}(x) + c_{n+2}S_{n+2}(x) + b_{n+1}S_{n+1}(x) + a_nS_n(x)\ , \quad n \geq 0 \\ &xQ_1(x) = S_2(x) + c_1S_1(x) + b_0\ , \quad xQ_0(x) = S_1(x) + c_0\ , \end{aligned} \tag{14}$$

with $(a_n,\, b_n,\, c_n) \in \mathbb{C} - \{0\} \times \mathbb{C}^2$.

Moreover, the sequence $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to u if and only if

$$\langle u,\, Q_{n+1}(x)\rangle = 0, \quad \langle u,\, xQ_{n+2}(x)\rangle = 0, \quad n \geq 0, \quad \langle u,\, xQ_1(x)\rangle \neq 0\ , \tag{15}$$

since the other orthogonality condition comes from (14).

Substituting x by 0 in (14), we get

$$c_{n+2}S_{n+2}(0) + b_{n+1}S_{n+1}(0) + a_nS_n(0) = -S_{n+3}(0), n \geq 0, \tag{16}$$

$$c_1S_1(0) + b_0 = -S_2(0), \tag{17}$$

$$c_0 = \xi_0. \tag{18}$$

Subtracting (16)-(17) from (14), we obtain after dividing by x (for $n \geq 0$)

$$Q_{n+2}(x) = (\theta_0S_{n+3})(x) + c_{n+2}(\theta_0S_{n+2})(x) + b_{n+1}(\theta_0S_{n+1})(x) + a_n(\theta_0S_n)(x), \tag{19}$$

$$Q_1(x) = (\theta_0S_2)(x) + c_1. \tag{20}$$

From (14)-(15) and (19)-(20), we have

$$\begin{cases} 0 = \langle u,\, Q_{n+2}(x)\rangle \\ \quad = \langle u,\, (\theta_0S_{n+3})(x)\rangle + c_{n+2}\langle u,\, (\theta_0S_{n+2})(x)\rangle + b_{n+1}\langle u,\, (\theta_0S_{n+1})(x)\rangle \\ \qquad\quad + a_n\langle u,\, (\theta_0S_n)(x)\rangle, \quad n \geq 0, \\ 0 = \langle u,\, xQ_{n+2}(x)\rangle \\ \quad = \langle u,\, S_{n+3}(x)\rangle + c_{n+2}\langle u,\, S_{n+2}(x)\rangle + b_{n+1}\langle u,\, S_{n+1}(x)\rangle + a_n\langle u,\, S_n(x)\rangle, \quad n \geq 0, \end{cases} \tag{21}$$

$$\begin{cases} 0 = \langle u,\, Q_1(x)\rangle = \langle u,\, (\theta_0S_2)(x)\rangle + c_1, \\ 0 \neq \langle u,\, xQ_1(x)\rangle = \langle u,\, S_2(x)\rangle + c_1\langle u,\, S_1(x)\rangle + b_0. \end{cases} \tag{22}$$

The determinant of the system defined by (16) and (21) is

$$\Delta_n := \begin{vmatrix} S_{n+2}(0) & S_{n+1}(0) & S_n(0) \\ \langle u,\, S_{n+2}\rangle & \langle u,\, S_{n+1}\rangle & \langle u,\, S_n\rangle \\ \langle u,\, \theta_0S_{n+2}\rangle & \langle u,\, \theta_0S_{n+1}\rangle & \langle u,\, \theta_0S_n\rangle \end{vmatrix}, \quad n \geq 0, \tag{23}$$

Then the above system is equivalent to

$$\Delta_n a_n = -\Delta_{n+1}, \quad n \geq 0 \tag{24}$$

$$\Delta_n b_{n+1} = -\begin{vmatrix} S_{n+2}(0) & S_{n+3}(0) & S_n(0) \\ \langle u,\, S_{n+2}\rangle & \langle u,\, S_{n+3}\rangle & \langle u,\, S_n\rangle \\ \langle u,\, \theta_0 S_{n+2}\rangle & \langle u,\, \theta_0 S_{n+3}\rangle & \langle u,\, \theta_0 S_n\rangle \end{vmatrix}, \; n \geq 0, \tag{25}$$

$$\Delta_n c_{n+2} = -\begin{vmatrix} S_{n+3}(0) & S_{n+1}(0) & S_n(0) \\ \langle u,\, S_{n+3}\rangle & \langle u,\, S_{n+1}\rangle & \langle u,\, S_n\rangle \\ \langle u,\, \theta_0 S_{n+3}\rangle & \langle u,\, \theta_0 S_{n+1}\rangle & \langle u,\, \theta_0 S_n\rangle \end{vmatrix}, \; n \geq 0. \tag{26}$$

Proposition 1. *The form u is regular if and only if $\Delta_n \neq 0$, $n \geq 0$.*

Proof. Necessity. If $\{Q_n\}_{n\geq 0}$ is orthogonal, then $a_n \neq 0$, $n \geq 0$. This implies $\Delta_n \neq 0$, $n \geq 0$. Assume the contrary, i.e. there exists an $n_0 \geq 1$ such that $\Delta_{n_0} = 0$. Then it follows from (24) that $\Delta_0 = 0 = -\langle u,\, xQ_1(x)\rangle \neq 0$, which is a contradiction.
Sufficiency. Using (11) we get that the condition $\langle u,\, Q_1(x)\rangle = 0$ is satisfied for

$$c_1 = \xi_1 + \xi_0 - (u)_1. \tag{27}$$

Then, from (16)-(17) and (27) we have

$$\begin{aligned} c_0 &= \xi_0, \\ b_0 &= \rho_1 - \xi_0((u)_1 - \xi_0). \end{aligned} \tag{28}$$

From (22) and (17) we have
$\langle u,\, xQ_1(x)\rangle = \langle u,\, (\theta_0 S_2)(x)\rangle(S_1(0) - \langle u,\, S_1(x)\rangle) + \langle u,\, S_2(x)\rangle - S_2(0) = \Delta_0 \neq 0.$
We had just proved that the initial conditions (17)-(18) and (22) are satisfied. Further, the system defined by (16) and (21) is a Cramer system whose solution is given by (24)-(26).◀

Proposition 2. *We may write for $n \geq 0$*

$$\beta_n = \xi_{n+1} + c_n - c_{n+1}, \tag{29}$$

$$\gamma_1 = \Delta_0, \quad \gamma_2 = \lambda\frac{a_0}{\Delta_0}, \quad \gamma_{n+3} = \frac{a_{n+1}}{a_n}\rho_{n+1}, \tag{30}$$

$$\gamma_{n+2} - \rho_{n+3} = b_{n+1} - b_{n+2} + c_{n+2}(\xi_{n+2} - \xi_{n+3} + c_{n+3} - c_{n+2}), \tag{31}$$

$$c_{n+1}\gamma_{n+2} - c_{n+2}\rho_{n+2} = a_n - a_{n+1} + b_{n+1}(\xi_{n+1} - \xi_{n+3} + c_{n+3} - c_{n+2}), \tag{32}$$

$$b_n\gamma_{n+2} - b_{n+1}\rho_{n+1} = a_n(\xi_n - \xi_{n+3} + c_{n+3} - c_{n+2}). \tag{33}$$

Proof. After multiplication of (13) for $n \to n+1$ by x, we substitute xQ_{k+1} by $S_{k+2} + c_{k+1}S_{k+1} + b_k S_k + a_{k-1}S_{k-1}$ with $k = n+2,\, n+1,\, n$ and we apply the recurrence relation (6). Then the comparison of the coefficients of S_{n+3} and S_{n-1} (resp. S_{n+2}, S_{n+1} and S_n)yields (29)-(30) for $n \geq 1$ by taking into account the expression for a_n from (24)(resp. (31)-(33) by taking into account the expression for β_{n+2} from (29)).

Multiplying (13) by x with $n = 0$ and using the same proceeding, we prove (29) for $n = 1$.

From (14) and (18) we obtain (29) for $n = 0$.

By (13), we have $\gamma_1 = \langle u,\, xQ_1\rangle = \Delta_0$ and $\gamma_2 = \frac{\langle u, x^2Q_2\rangle}{\langle u, xQ_1\rangle} = \frac{\langle u, x^2Q_2\rangle}{\Delta_0}$.

So, using (11), (18) and (24), we get $\gamma_2 = -\lambda\dfrac{\Delta_1}{\Delta_0^2}$. ◀

The computation of Δ_n

As was seen above, it is important to have an explicit expression for Δ_n, $n \geq 0$.

From (8) and (10) we have

$$\langle u,\, S_{n+1}\rangle = \lambda S_n^{(1)}(0) + S_{n+1}(0) + \Theta S'_{n+1}(0), \quad n \geq 0, \tag{34}$$

$$\langle u,\, xS_{n+1}(x)\rangle = \Theta S_{n+1}(0), \quad n \geq 0, \tag{35}$$

$$\langle u,\, \theta_0 S_{n+1}\rangle = \lambda (S_n^{(1)})'(0) + S'_{n+1}(0) + \frac{1}{2}\Theta S''_{n+1}(0), \quad n \geq 0, \tag{36}$$

with

$$\Theta = (u)_1 - \lambda. \tag{37}$$

Using (6) and (23), we obtain

$$\Delta_n = \begin{vmatrix} 0 & S_{n+1}(0) & S_n(0) \\ \langle u,\, xS_{n+1}(x)\rangle & \langle u,\, S_{n+1}(x)\rangle & \langle u,\, S_n(x)\rangle \\ \langle u,\, S_{n+1}(x)\rangle & \langle u,\, (\theta_0 S_{n+1})(x)\rangle & \langle u,\, (\theta_0 S_n)(x)\rangle \end{vmatrix}, \quad n \geq 0, \tag{38}$$

that is

$$\begin{aligned}\Delta_n = -S_{n+1}(0) &\begin{vmatrix} \langle u,\, xS_{n+1}(x)\rangle & \langle u,\, S_n(x)\rangle \\ \langle u,\, S_{n+1}(x)\rangle & \langle u,\, (\theta_0 S_n)(x)\rangle \end{vmatrix} \\ +S_n(0) &\begin{vmatrix} \langle u,\, xS_{n+1}(x)\rangle & \langle u,\, S_{n+1}(x)\rangle \\ \langle u,\, S_{n+1}(x)\rangle & \langle u,\, (\theta_0 S_{n+1})(x)\rangle \end{vmatrix}.\end{aligned} \tag{39}$$

Therefore, using (34)-(36) we obtain

$$\begin{aligned}\Delta_n = {} & \lambda S_{n+1}(0,-\lambda)\left(S_{n+1}(0)S^{(1)}_{n-1}(0) - S_n(0)S^{(1)}_n(0)\right) + \\ & +\Theta\left(S_n(0)X_{n,n+1}(0) - S_{n+1}(0)X_{n,n}(0) - \lambda S^{(1)}_n(0)Y_{n,n}(0)\right), \quad n \geq 0\end{aligned} \tag{40}$$

with (for $n, m \geq 0$)

$$\begin{aligned}X_{n,m}(0) = {} & \lambda\left(S_{n+1}(0)(S^{(1)}_{m-1})'(0) - S^{(1)}_{m-1}(0)S'_{n+1}(0)\right) \\ & +\tfrac{1}{2}\Theta\left(S_{n+1}(0)S''_m(0) - 2S'_{n+1}(0)S'_m(0)\right),\end{aligned} \tag{41}$$

$$Y_{n,m}(0) = S_m(0)S'_{n+1}(0) - S_{n+1}(0)S'_m(0). \tag{42}$$

Moreover, if the form u is regular we have from (25), (26) and (6)

$$b_{n+1} = \rho_{n+2} - \begin{vmatrix} S_{n+2}(0) & 0 & S_n(0) \\ \langle u,\, S_{n+2}(x)\rangle & \langle u,\, xS_{n+2}(x)\rangle & \langle u,\, S_n(x)\rangle \\ \langle u,\, (\theta_0 S_{n+2})(x)\rangle & \langle u,\, S_{n+2}(x)\rangle & \langle u,\, (\theta_0 S_n)(x)\rangle \end{vmatrix} \Delta_n^{-1}, \tag{43}$$

$$c_{n+2} = \xi_{n+2} - \begin{vmatrix} 0 & S_{n+1}(0) & S_n(0) \\ \langle u,\, xS_{n+2}(x)\rangle & \langle u,\, S_{n+1}(x)\rangle & \langle u,\, S_n(x)\rangle \\ \langle u,\, S_{n+2}(x)\rangle & \langle u,\, (\theta_0 S_{n+1})(x)\rangle & \langle u,\, (\theta_0 S_n)(x)\rangle \end{vmatrix} \Delta_n^{-1}. \tag{44}$$

Then, using (34)-(36) and (41)-(42), the last equations become for $n \geq 0$

$$\begin{aligned}b_{n+1} = {} & \rho_{n+2} + \Delta_n^{-1}\Big\{\lambda S_{n+2}(0,-\lambda)\left(S_{n+2}(0)S^{(1)}_{n-1}(0) - S_n(0)S^{(1)}_{n+1}(0)\right) \\ & +\Theta\left(S_n(0)X_{n+1,n+2}(0) - S_{n+2}(0)X_{n+1,n}(0) - \lambda S^{(1)}_{n+1}(0)Y_{n+1,n}(0)\right)\Big\}\end{aligned} \tag{45}$$

and

$$\begin{aligned}c_{n+2} = {} & \xi_{n+2} + \Delta_n^{-1}\Big\{\lambda S_{n+2}(0,-\lambda)\left(S_n(0)S^{(1)}_n(0) - S_{n+1}(0)S^{(1)}_{n-1}(0)\right) \\ & +\Theta\left(S_{n+1}(0)X_{n+1,n}(0) - S_n(0)X_{n+1,n+1}(0) + \lambda S^{(1)}_{n+1}(0)Y_{n,n}(0)\right)\Big\}.\end{aligned} \tag{46}$$

Remark 1. *1. In fact, using the well known identity (see [6], page 86)*

$$S_n(0)S^{(1)}_n(0) - S_{n+1}(0)S^{(1)}_{n-1}(0) = \prod_{\mu=0}^{n}\rho_\mu := \tau_n \neq 0,\ n \geq 0, \tag{47}$$

and the confluent Christoffel-Darboux formula (see [6], page 23)

$$S_n(0)S'_{n+1}(0)-S_{n+1}(0)S'_n(0)=\tau_n\sum_{\nu=0}^{n}\frac{S_\nu^2(0)}{\tau_\nu},\ n\geq 0. \tag{48}$$

we can rewrite (40) for $n\geq 0$ as follows:

$$\Delta_n=-\lambda\tau_nS_{n+1}(0,-\lambda)+\Theta\Big\{S_n(0)X_{n,n+1}(0)-S_{n+1}(0)X_{n,n}(0)-\lambda\tau_nS_n^{(1)}(0)\sum_{\nu=0}^{n}\frac{S_\nu^2(0)}{\tau_\nu}\Big\}. \tag{49}$$

2. *Taking $(u)_1=\lambda$ in (49), we can easily obtain the result given in [18]. Indeed, the form $u=\lambda x^{-1}v+\delta_0$ is regular if and only if $S_{n+1}(0,-\lambda)\neq 0$, $n\geq 0$.*

3. *From (6) and (48) we get*

$$S_n(0)S'_{n+2}(0)-S_{n+2}(0)S'_n(0)=S_{n+1}(0)S_n(0)-\xi_{n+1}\tau_n\sum_{\nu=0}^{n}\frac{S_\nu^2(0)}{\tau_\nu},\ \ n\geq 0, \tag{50}$$

and from (8) and (47), we have

$$S_{n+2}(0)S_{n-1}^{(1)}(0)-S_n(0)S_{n+1}^{(1)}(0)=\tau_n\xi_{n+1},\ \ n\geq 0. \tag{51}$$

The interest of the last two results will be shown further on.

3.2. Particular case: v is symmetric

We recall that a form v is called symmetric if $(v)_{2n+1}=0, n\geq 0$. The conditions $(v)_{2n+1}=0, n\geq 0$ are equivalent to the fact that the corresponding MOPS $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (6) with $\xi_n=0, n\geq 0$ [6].
In the sequel, the form v will be supposed regular and symmetric.

Proposition 3. *When the form v is symmetric, the form u (defined by (10)) is regular if and only if*

$$d_n:=\lambda+\Theta\Lambda_n\neq 0,\ \ n\geq 0 \tag{52}$$

where

$$\Lambda_n=1+\sum_{\nu=0}^{n-1}\prod_{k=0}^{\nu}\frac{\rho_{2k+1}}{\rho_{2k+2}},\ ,\ n\geq 0.\qquad\left(\sum_{\nu=0}^{-1}=0\right) \tag{53}$$

For the proof we use the following lemmas:

Lemma 1. [14] *When $\{S_n\}_{n\geq 0}$ given by (6) is symmetric we have*

$$S_{2n+2}(0)=(-1)^{n+1}\prod_{\nu=0}^{n}\rho_{2\nu+1},\qquad S_{2n+1}(0)=0,\qquad n\geq 0,$$

$$S^{(1)}_{2n+2}(0)=(-1)^{n+1}\prod_{\nu=0}^{n}\rho_{2\nu+2},\qquad S^{(1)}_{2n+1}(0)=0,\qquad n\geq 0,$$

$$(S^{(1)}_{2n})'(0)=0,\qquad n\geq 0.$$

Lemma 2. *When $\{S_n\}_{n\geq 0}$ given by (6) is symmetric we have*

$$S'_{2n}(0)=0,\qquad S''_{2n+1}(0)=0,\quad n\geq 0, \tag{54}$$

$$S'_{2n+1}(0)=(-1)^n\Lambda_n\prod_{\nu=0}^{n}\rho_{2\nu},\quad n\geq 0. \tag{55}$$

Proof. We have $S_n(-x)=(-1)^nS_n(x), n\geq 0$ because u is a symmetric form (see [6] Theorem 4.3). As a consequence, we easily deduce (54).

We prove (55) by induction. For $n=0$, from (6) we obtain $S_1'(0)=1$. Therefore, (55) is valid for $n=0$. Now, suppose (55) for $0\leq m\leq n$. Then, taking $n\longrightarrow 2n+2$ in (6) with $\xi_n=0$ from Lemma 1 we obtain

$$S'_{2n+3}(0)=(-1)^{n+1}\prod_{\nu=0}^{n}\rho_{2\nu+1}+(-1)^{n+1}\Lambda_n\prod_{\nu=0}^{n+1}\rho_{2\nu}.$$

Therefore,

$$S'_{2n+3}(0)=(-1)^{n+1}\left(\prod_{\nu=0}^{n+1}\rho_{2\nu}\right)\left\{\prod_{\nu=0}^{n}\frac{\rho_{2\nu+1}}{\rho_{2\nu}}+\Lambda_n\right\}=(-1)^{n+1}\Lambda_{n+1}\prod_{\nu=0}^{n+1}\rho_{2\nu}.$$

Hence we have (55) for $n\geq 0$.◀

Proof. (of Proposition 3.) Following Lemma 1 and Lemma 2, we have for (41)-(42)

$$X_{2n,2n+1}(0)=-S'_{2n+1}(0)\left(\lambda S^{(1)}_{2n}(0)+\Theta S'_{2n+1}(0)\right),\ X_{2n+1,2n+1}(0)=0,\ n\geq 0.$$

$$Y_{2n,2n}(0)=S_{2n}(0)S'_{2n+1}(0),\ n\geq 0.$$

Then, (49) becomes

$$\begin{aligned}\Delta_{2n}&=-S_{2n}(0)\left(\lambda S^{(1)}_{2n}(0)+\Theta S'_{2n+1}(0)\right)^2,\ n\geq 0,\\ \Delta_{2n+1}&=\lambda S^2_{2n+2}(0)S^{(1)}_{2n}(0),\ n\geq 0.\end{aligned} \tag{56}$$

Using the Lemma 1 and Lemma 2, we obtain

$$\begin{aligned} \Delta_{2n} &= (-1)^{n+1}\tau_{2n}d_n^2\prod_{\nu=0}^{n}\rho_{2\nu},\ n\geq 0,\\ \Delta_{2n+1} &= \lambda(-1)^{n}\tau_{2n+1}\prod_{\nu=0}^{n}\rho_{2\nu+1},\ n\geq 0. \end{aligned} \tag{57}$$

Therefore, u is regular if and only if $d_n \neq 0,\ n \geq 0$. ◄

In such conditions, from (24), (27)-(28), (45)-(46) and (50)-(51) we get

$$a_{2n} = \lambda\rho_{2n+1}\Omega_n d_n^{-2},\ a_{2n+1} = -\left(\rho_{2n+2}d_{n+1}\right)^2\left(\lambda\Omega_n\right)^{-1},\ \ n\geq 0, \tag{58}$$

$$b_0 = \rho_1,\ c_0 = 0,\ c_1 = -(u)_1, \tag{59}$$

$$b_{2n+1} = \rho_{2n+2} + \Theta\Omega_n d_n^{-1},\ b_{2n+2} = \rho_{2n+3},\ \ n\geq 0, \tag{60}$$

$$c_{2n+2} = \lambda\Omega_n d_n^{-2},\ c_{2n+3} = -\rho_{2n+2}d_{n+1}d_n(\lambda\Omega_n)^{-1},\ \ n\geq 0, \tag{61}$$

$$\beta_0 = (u)_1,\ \beta_{2n+2} = c_{2n+2} - c_{2n+3},\ \ n\geq 0, \tag{62}$$

$$\beta_1 = -(u)_1 - \frac{\rho_1\lambda}{(u)_1^2},\ \beta_{2n+3} = c_{2n+3} - c_{2n+4},\ \ n\geq 0, \tag{63}$$

$$\gamma_1 = -(u)_1^2,\ \gamma_{2n+3} = -\left(d_{n+1}d_n\rho_{2n+2}\lambda^{-1}\Omega_n^{-1}\right)^2,\ \ n\geq 0, \tag{64}$$

$$\gamma_2 = -\left(\frac{\rho_1\lambda}{(u)_1^2}\right)^2,\ \gamma_{2n+4} = -\left(\lambda\Omega_{n+1}d_{n+1}^{-2}\right)^2,\ \ n\geq 0, \tag{65}$$

with $\Omega_n = \displaystyle\prod_{\nu=0}^{n}\frac{\rho_{2\nu+1}}{\rho_{2\nu}}, n \geq 0$.

Remark 2. *As an immediate consequence of (52), we have: if v is symmetric and positive definite form and $(u)_1 = \lambda + 1$, then u is regular for every $\lambda \in \mathbb{C}-]-\infty, 0]$.*

4. The semi-classical case

Let us recall that a form v is called semi-classical when it is regular and there exist two polynomials Φ and Ψ such that (see [17]*):*

$$(\Phi v)' + \Psi v = 0\ ,\quad \deg(\Psi) \geq 1\ ,\quad \Phi\ ismonic. \tag{66}$$

The class of the semi-classical form v is $s = \max\big(\deg\Psi - 1, \deg\Phi - 2\big)$ if and only if the following condition is satisfied:

$$\prod_{c}\left(|\Phi'(c) + \Psi(c)| + \left|\langle v, \theta_c\Psi + \theta_c^2\Phi\rangle\right|\right) > 0\ , \tag{67}$$

where c belongs to the set of zeros of Φ [18].

In the sequel, we will assume v is a semi-classical form of class s satisfying (66). Multiplying (66) by λx and using (4), we obtain

$$(\lambda x\Phi v)' + \big(\lambda x\Psi - \lambda\Phi\big)v = 0\,. \tag{68}$$

So, if $\Phi(0) = 0$, by using (11) we obtain $\big(x^2\Phi u\big)' + \big(x^2\Psi - x\Phi\big)u = 0$, and, if $\Phi(0) \neq 0$, multiplying (68) by x and using (11) we get

$$\Big(\tilde{\Phi}u\Big)' + \tilde{\Psi}u = 0 \tag{69}$$

with

$$\begin{cases} \tilde{\Phi}(x) = x^3\Phi(x)\,, \\ \tilde{\Psi}(x) = x\big(x^2\Psi(x) - 2x\Phi(x)\big)\,. \end{cases} \tag{70}$$

It is clear that if the form u is regular, then it is also semi-classical, and the class $\tilde{s}$ of u is at most $s+3$.

Proposition 4. *The class of u depends only on the zero $x = 0$.*

For the proof we use the following Lemma:

Lemma 3. *For all zeros $a \neq 0$ of Φ we have*

$$\tilde{\Psi}(a) + \tilde{\Phi}'(a) = a^3(\Psi(a) + \Phi'(a)), \tag{71}$$

and

$$\langle u,\, \theta_a\tilde{\Psi} + \theta_a^2\tilde{\Phi}\rangle = a(\Theta + a)(\Psi(a) + \Phi'(a)) + \lambda a^2\left\langle v,\, \theta_a\Psi + \theta_a^2\Phi\right\rangle. \tag{72}$$

Proof. Let a be a zero of Φ and $\Phi_a(x) = (\theta_a\Phi)(x)$. Then, from (70) we have

$$\tilde{\Phi}(x) = x^3(x-a)\Phi_a(x). \tag{73}$$

Using the definition of the operator θ_a, it is easy to prove that for $f \in \mathcal{P}$ we have

$$\theta_0(\theta_a f) = \theta_a(\theta_0 f). \tag{74}$$

Therefore, from (11) and (73)-(74), we get

$$\left\langle u,\, \theta_a^2\tilde{\Phi}\right\rangle = \lambda\left\langle v,\, (\theta_a(\zeta^2)\Phi_a(\zeta)))(x)\right\rangle + a(a+\Theta)\Phi'(a).$$

Now, taking $g(x) = x^2$ and $f(x) = \Phi_a(x)$ in (5), we get

$$\left\langle v,\, (\theta_a(\zeta^2\Phi_a))(x)\right\rangle = \langle v,\, (x+a)\Phi_a(x)\rangle + a^2\left\langle v,\, \theta_a^2\Phi\right\rangle.$$

Then

$$\left\langle u,\, \theta_a^2\tilde{\Phi}\right\rangle = \lambda\langle v,\, (x+a)\Phi_a(x)\rangle + \lambda a^2\left\langle v,\, \theta_a^2\Phi\right\rangle + a(a+\Theta)\Phi'(a). \tag{75}$$

Using the same proceeding as we did to obtain (75), we easily prove that

$$\left\langle u,\, \theta_a\tilde{\Psi}\right\rangle = \lambda a^2 \left\langle v,\, \theta_a\Psi\right\rangle + \lambda \left\langle v,\, (x+a)\Psi(x) - 2x\Phi_a\right\rangle + a(a+\Theta)\Psi(a). \tag{76}$$

Thus, from (75)-(76) we obtain

$$\left\langle u,\, \theta_a\tilde{\Psi} + \theta_a^2\tilde{\Phi}\right\rangle = a(\Theta+a)(\Psi(a)+\Phi^{'}(a)) + \lambda a^2\langle v,\, \theta_a\Psi + \theta_a^2\Phi\,\rangle + \lambda\langle v,\, -\Phi + (x+a)\Psi\rangle\,.$$

But from (66) we have

$$\left\langle (\Phi v)^{'} + \Psi v,\, (x+a)\right\rangle = 0. \tag{77}$$

Then the last equation becomes (72).

From (70) we have $\tilde{\Phi}'(a) = a^3\Phi'(a)$ and $\tilde{\Psi}(a) = a^3\Psi(a)$. Hence, (71) holds.◄

Proof. (*of Proposition 4.*) Let a be a zero of Φ such that $a \neq 0$.
If $\Psi(a)+\Phi^{'}(a) = 0$, using (72) we have $\left\langle u,\, \theta_a\tilde{\Psi} + \theta_a^2\tilde{\Phi}\right\rangle \neq 0$, since v is semi-classical and so satisfies (67).
If $\Psi(a)+\Phi^{'}(a) \neq 0$, then from (71) we obtain $\tilde{\Psi}(a)+\tilde{\Phi}^{'}(a) \neq 0$.
In any case, we cannot simplify by $x-a$.◄

Proposition 5. *Let v be a semi-classical form of class s satisfying (66). Then the form u given by (10) is also semi-classical of class $\tilde{s}$ satisfying (69). Moreover,*

1. $\tilde{s} = s+3$ *if* $\Phi(0) \neq 0$*;*

2. $\tilde{s} = s+2$ *if* $\Phi(0)$ *and* $\Psi(0) \neq 0$*;*

3. $\tilde{s} = s+1$ *if* $\Phi(0) = 0,\ \Psi(0) = 0$ *and* $\lambda\langle v,\, \theta_0\Psi + \theta_0^2\Phi\rangle + \Phi^{'}(0) + \Theta(\Psi^{'}(0) + \frac{1}{2}\Phi^{''}(0)) \neq 0$.

Proof. From (70) we have $\tilde{\Psi}(0)+\tilde{\Phi}^{'}(0) = 0$ and $\langle u,\, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi}\rangle = -\Theta\Phi(0)$.
Since $\Theta \neq 0$, we have the following:

1. If $\Phi(0) \neq 0$, then it is not possible to simplify by x according to standard criterion (67), which means that the class of u is $\tilde{s} = s+3$.

2. If $\Phi(0) = 0$, then it is possible to simplify by x. Therefore, u fulfils (69) with

$$\tilde{\Phi}(x) = x^2\Phi(x),\ \tilde{\Psi}(x) = x(x\Psi(x) - \Phi(x)). \tag{78}$$

Here we have

$$\tilde{\Phi}^{'}(0) + \tilde{\Psi}(0) = 0,\ \langle u,\, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi}\rangle = \Theta\Psi(0).$$

If $\Psi(0) \neq 0$, then it is not possible to simplify by x and the class of u is $\tilde{s} = s + 2$.

3. If $\Phi(0) = \Psi(0) = 0$, we can simplify (69)-(78) by x. We obtain

$$\tilde{\Phi}(x) = x\Phi(x),\ \tilde{\Psi}(x) = x\Psi(x). \tag{79}$$

Thus, we have

$$\tilde{\Phi}^{'}(0) + \tilde{\Psi}(0) = 0,\ \langle u,\, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi}\rangle = \lambda\langle v,\, \theta_0\Psi + \theta_0^2\Phi\rangle + \Phi^{'}(0) + \Theta(\Psi^{'}(0) + \frac{1}{2}\Phi^{''}(0)).$$

If $\lambda\langle v,\, \theta_0\Psi + \theta_0^2\Phi\rangle + \Phi^{'}(0) + \Theta(\Psi^{'}(0) + \frac{1}{2}\Phi^{''}(0)) \neq 0$, then it is not possible to simplify, which means that the class of u is $\tilde{s} = s + 1$.

▲

5. Illustrative Examples

1. We study the problem (10) with $v = \mathcal{H}(\mu)$ where $\mathcal{H}(\mu)$ is the Generalized Hermite form. In this case, the form v is symmetric. This form has the following integral representation [6]:

$$\langle v,\, f\rangle = \frac{1}{\Gamma(\mu + \frac{1}{2})}\int_{-\infty}^{+\infty} e^{-x^2}|x|^{2\mu}f(x)dx,\ \mathfrak{Re}\mu > -\frac{1}{2}.$$

Therefore,

$$P\int_{-\infty}^{+\infty} \frac{e^{-x^2}|x|^{2\mu}}{x}dx = 0.$$

Thus, using (12) we obtain the following integral representation of u:

$$\langle u,\, f\rangle = \frac{\lambda}{\Gamma(\mu + \frac{1}{2})}P\int_{-\infty}^{+\infty} sgn(x)|x|^{2\mu-1}e^{-x^2}f(x)dx + f(0) + \Theta f^{'}(0).$$

The form v is semi-classical of class $s = 1$, it satisfies (66) with [2]

$$\Phi(x) = x, \quad \Psi(x) = 2x^2 - (2\mu + 1), \quad \mu \neq 0. \tag{80}$$

The sequence $\{S_n\}_{n\geq 0}$ fulfils (6) with [16]

$$\xi_n = 0, \quad \rho_{2n+1} = n + \mu + \frac{1}{2}, \quad \rho_{2n+2} = n + 1, \quad 2\mu \neq -2n - 1, \quad n \geq 0. \tag{81}$$

First, we study the regularity of the form u.

We have from (81)

$$\frac{\rho_{2k+1}}{\rho_{2k+2}} = \frac{k+\mu+\frac{1}{2}}{k+1}.$$

Then

$$\prod_{k=0}^{\nu} \frac{\rho_{2k+1}}{\rho_{2k+2}} = \frac{\Gamma(\nu+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+2)} = \frac{1}{(\nu+1)\Gamma(\mu+\frac{1}{2})} h_\nu,$$

with

$$h_n = \frac{\Gamma(n+\mu+\frac{3}{2})}{\Gamma(n+1)},\ n \geq 0$$

and

$$h_{n+1} = \frac{n+\mu+\frac{3}{2}}{n+1} h_n,\ n \geq 0.$$

Therefore,

$$h_{n+1} - h_n = \frac{\mu+\frac{1}{2}}{n+1} h_n,\ n \geq 0,$$

and, consequently, from the above results we obtain

$$\sum_{\nu=0}^{n-1} \prod_{k=0}^{\nu} \frac{\rho_{2k+1}}{\rho_{2k+2}} = \frac{1}{\Gamma(\mu+\frac{3}{2})} \sum_{\nu=0}^{n-1} (h_{\nu+1} - h_\nu) = \frac{1}{\Gamma(\mu+\frac{3}{2})} \frac{\Gamma(n+\mu+\frac{3}{2})}{\Gamma(n+1)} - 1,\ n \geq 1.$$

Finally, (53) and (52) become respectively

$$\Lambda_n = \frac{\Gamma(n+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2})\Gamma(n+1)},\ n \geq 0 \tag{82}$$

and

$$d_n = \left(1 - \frac{\Gamma(n+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2})\Gamma(n+1)}\right)\lambda + (u)_1 \frac{\Gamma(n+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{3}{2})\Gamma(n+1)},\ n \geq 0. \tag{83}$$

Then, u is regular for every $\lambda \neq 0$ and $(u)_1$ such that

$$\lambda^{-1}(u)_1 \neq 1 - \frac{\Gamma(\mu+\frac{3}{2})\Gamma(n+1)}{\Gamma(n+\mu+\frac{3}{2})},\ n \geq 0. \tag{84}$$

Using (81) and (83), we obtain for (58)-(61)

$$a_{2n} = \frac{\lambda(n+\mu+\frac{1}{2})(\mu+\frac{1}{2})\Lambda_n}{d_n^2},\ a_{2n+1} = -\frac{(n+1)^2 d_{n+1}^2}{\lambda(\mu+\frac{1}{2})\Lambda_n},\ n \geq 0, \tag{85}$$

$$b_0 = \mu + \frac{1}{2},\ b_{2n+1} = n+1+\frac{((u)_1-\lambda)(\mu+\frac{1}{2})\Lambda_n}{d_n},\ b_{2n+2} = n+\mu+\frac{3}{2},\ n \geq 0, \tag{86}$$

$$c_0 = 0,\ c_1 = -(u)_1,\ c_{2n+2} = \frac{\lambda(\mu+\frac{1}{2})\Lambda_n}{d_n^2},\ c_{2n+3} = -\frac{d_n d_{n+1}(n+1)}{\lambda(\mu+\frac{1}{2})\Lambda_n},\ n \geq 0. \tag{87}$$

Therefore, we have for (62)-(65)

$$\beta_0 = (u)_1,\ \beta_{2n+2} = \frac{\lambda(\mu+\frac{1}{2})\Lambda_n}{d_n^2} + \frac{d_n d_{n+1}(n+1)}{\lambda(\mu+\frac{1}{2})\Lambda_n},\ n \geq 0,$$
$$\beta_1 = -(u)_1 - \frac{\lambda(\mu+\frac{1}{2})}{(u)_1^2},\ \beta_{2n+3} = -\frac{d_n d_{n+1}(n+1)}{\lambda(\mu+\frac{1}{2})\Lambda_n} - \frac{\lambda(\mu+\frac{1}{2})\Lambda_{n+1}}{d_{n+1}^2},\ n \geq 0, \tag{88}$$

$$\gamma_1 = -(u)_1^2,\ \gamma_{2n+3} = -\left(\frac{d_n d_{n+1}(n+1)}{\lambda(\mu+\frac{1}{2})\Lambda_n}\right)^2,\ n \geq 0,$$
$$\gamma_2 = -\left(\frac{\lambda(\mu+\frac{1}{2})}{(u)_1^2}\right)^2,\ \gamma_{2n+4} = -\left(\frac{\lambda(\mu+\frac{1}{2})\Lambda_{n+1}}{d_{n+1}^2}\right)^2,\ n \geq 0. \tag{89}$$

Finally, from Proposition 5, (81) and (78), we obtain that the form u is semi-classical of class $\tilde{s} = 3$ and fulfils the functional equation (69) with

$$\tilde{\Phi}(x) = x^3,\ \tilde{\Psi}(x) = 2x^2(x^2 - \mu - 1). \tag{90}$$

2. We study the problem (10) with $v = \mathcal{J}(-\frac{1}{2}, \frac{1}{2})$ where $\mathcal{J}$ is the Jacobi form. In this case, the form v is not symmetric. This form has the following integral representation:

$$\langle v,\, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x), \text{ with [8] } V(x) = \tfrac{1}{\pi}Y(1-x^2)\sqrt{\tfrac{1-x}{1+x}} \quad \text{where } Y(x) = \begin{cases} 1, & x > 0; \\ 0, & x \leq 0. \end{cases}$$

Therefore,

$$P\int_{-1}^{1} \frac{1-x}{x\sqrt{1-x^2}}dx = -\pi.$$

Thus, using (12), we obtain the following integral representation of u:

$$\langle u,\, f\rangle = \frac{\lambda}{\pi}P\int_{-1}^{1} \frac{1}{x}\sqrt{\frac{1-x}{1+x}} + (1+\lambda)f(0) + \Theta f'(0).$$

The form v is classical (semi-classical of class $s = 0$), it satisfies (66) with [3]

$$\Phi(x) = x^2 - 1,\ \Psi(x) = -2x - 1. \tag{91}$$

The sequence $\{S_n\}_{n\geq 0}$ fulfils (6) with [3]

$$\xi_0 = \frac{1}{2},\ \xi_{n+1} = 0,\ \rho_{n+1} = \frac{1}{4},\ n \geq 0. \tag{92}$$

First, we study the regularity of the form u.

From (6), (7) and (92), we can obtain by induction

$$S_{2n}(0) = \frac{(-1)^n}{4^n},\ S_{2n+1}(0) = \frac{(-1)^{n+1}}{2^{2n+1}},\ n \geq 0,$$

$$S^{(1)}_{2n}(0)=\frac{(-1)^n}{4^n},\ S^{(1)}_{2n+1}(0)=0,\ n\geq 0,$$

$$S'_{2n+1}(0)=\frac{(-1)^n(n+1)}{4^n},\ S'_{2n+2}(0)=\frac{(-1)^{n+1}(n+1)}{2^{2n+1}},\ n\geq 0,$$

$$S''_{2n+2}(0)=\frac{(-1)^n(n+1)(n+2)}{4^n},\ S''_{2n+3}(0)=\frac{(-1)^{n+1}(n+1)(n+2)}{2^{2n+1}},\ n\geq 0,$$

$$(S^{(1)}_{2n})'(0)=0,\ (S^{(1)}_{2n+1})'(0)=\frac{(-1)^n(n+1)}{4^n},\ n\geq 0.$$

Then, from (41) we get for $n\geq 0$

$$X_{2n,2n+1}(0)=-(n+1)\frac{(3n+2)(u)_1-3n\lambda}{2^{4n+1}},$$

$$X_{2n,2n}(0)=n\frac{(n+3)\lambda-(n+1)(u)_1}{4^{2n}},$$

$$X_{2n+1,2n+2}(0)=-(n+1)\frac{(3n+4)(u)_1-(3n+2)\lambda}{2^{4n+3}},$$

$$X_{2n+1,2n+1}(0)=(n+1)\frac{(n+2)(u)_1-n\lambda}{4^{2n+1}}.$$

Therefore, from (49) and the above results we have for $n\geq 0$

$$\begin{aligned}\Delta_{2n}&=\tfrac{(-1)^n}{2^{6n+1}}\left(\lambda-2\lambda^2-\Theta\left(2(n+1)(2n-1)\Theta+4(n+1)(u)_1\right)\right),\\ \Delta_{2n+1}&=\frac{(-1)^n\lambda}{4^{3n+2}}\left(\lambda+\Theta\left(2(n+1)(2n+1)\Theta+4(n+1)(u)_1\right)\right).\end{aligned}\tag{93}$$

In particular, we have: if $(\lambda,(u)_1)\in(i\mathbb{R})\times(i\mathbb{R})$, then $\Delta_n\neq 0$, $n\geq 0$, since $\Im\Delta_{2n}=-i\lambda\frac{(-1)^n}{2^{6n+1}}\neq 0$ and $\Re\Delta_{2n+1}=\frac{(-1)^n\lambda^2}{4^{3n+2}}\neq 0$.

Using (50)-(51), (92), we obtain for (24), (27)-(28) and (45)-(46)

$$a_n=-\frac{\Delta_{n+1}}{\Delta_n},\ n\geq 0,$$

$$b_0=\frac{1}{2}(1-(u)_1),\ b_{2n+2}=\frac{1}{4}+\frac{(-1)^n\Theta\left(4(n+1)\Theta+2(u)_1\right)}{\Delta_{2n+1}4^{3n+3}},\ n\geq 0,$$

$$b_{2n+1}=\frac{1}{4}+\frac{(-1)^n\Theta\left(-(2n+1)\Theta-(u)_1\right)}{\Delta_{2n}4^{3n+1}},\ n\geq 0,$$

$$c_0=\frac{1}{2},\ c_{2n+2}=-\frac{(-1)^n\left(\lambda+2\Theta(2n+1)\left(n\Theta+(u)_1\right)\right)}{\Delta_{2n}4^{3n+1}},\ n\geq 0,$$

$$c_1=\frac{1}{2}-(u)_1,\ c_{2n+3}=\frac{(-1)^n\left(\lambda-2\lambda^2+\Theta\left(-2n(2n+3)\Theta-2(2n+3)(u)_1\right)\right)}{\Delta_{2n+1}2^{6n+5}},\ n\geq 0.$$

Therefore, we have for (29) and (30)

$$\beta_0 = (u)_1,\ \beta_1 = \frac{\lambda - 2(u)_1 \left(\lambda - (u)_1^2\right)}{\lambda - 2(u)_1^2}, \beta_{n+2} = c_{n+2} - c_{n+3},\ n \geq 0,$$

$$\gamma_1 = \frac{\lambda}{2} - (u)_1^2,\ \gamma_2 = -\lambda \frac{-3\Theta(\Theta + (u)_1)}{4\left(\lambda - 2(u)_1^2\right)^2},\ \gamma_{n+3} = \frac{\Delta_{n+2}\Delta_n}{4\Delta_{n+1}^2},\ n \geq 0.$$

Finally, from Proposition 5, (91) and (70), we obtain that the form u is semi-classical of class $\tilde{s} = 3$ and fulfils the functional equation (69) with

$$\tilde{\Phi}(x) = x^3(x^2 - 1),\ \tilde{\Psi}(x) = x^2(-4x^2 - x + 2).$$

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