Azerbaijan Journal of Mathematics
V. 2, No 2, 2012, July
ISSN 2218-6816

Potential Like Operators in the Theory of Boundary Value Problems in Non-Smooth Domains

Vladimir B. Vasilyev

Abstract. We consider a general elliptic pseudo differential equation in a wedge of codimension 2. Under existence of a special factorization for the operator symbol one can obtain an integral representation for the solution of the equation. It includes potential like operators. A priori estimates for the solution are given also.

Key Words and Phrases: pseudo differential equation; wave factorization; potential like operator

2010 Mathematics Subject Classifications: 47G40; 32W35

1. Introduction

In 90th, the author has introduced the concept of wave factorization of elliptic symbol [6],[9] for pseudo differential equations in model non-smooth domains. The solvability depends on the index of wave factorization, and looks different for some cases. So, particularly, if the index roughly speaking is negative, then for solvability of original equation one needs some solvability conditions on right hand side. For this purpose, we write a special integral representation for the solution, which permits to separate these solvability conditions explicitly. The integral representation includes certain special integral operators, which can be treated as potential like operators.

2. Equations and Factorization

We consider the equation

$$(Au_+)(x) = f_+(x),\ x \in W_+^a, \tag{1}$$

where A is a pseudo differential operator with symbol $A(\xi)$, $\xi \in \mathbf{R}^m$, satisfying the condition

$$c_1 \leq |A(\xi)(1+|\xi|)^{-\alpha}| \leq c_2,$$

c_1,c_2 are positive constants, and admitting the wave factorization with respect to the wedge $W_+^a = C_+^a \times \mathbf{R}^{m-2}$, $C_+^a = \{x \in \mathbf{R}^2\colon\ \mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2), \mathrm{x}_2 > a|x_1|, a > 0\}$.

So, we represent every $(x_1, x_2, ..., x_m) \in W_+^a$ in the form (x_1, x_2, x''), $x'' = (x_3, ..., x_m)$, $x'' \in \mathbf{R}^{m-2}$, $m \geq 3$. We seek for the solution $u_+ \in H^s(W_+^a)$.

By definition, $H^s(\mathbf{R}^m)$ is the space of distributions with the norm [2]

$$||u||_s^2 = \int\limits_{\mathbf{R}^m} |\tilde{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi < +\infty.$$

Let's denote by $H^s(W_+^a)$ the space of functions whose supports belong to $\overline{W_+^a}$, $H_0^s(W_+^a)$ is the space of distributions which admit continuation lf, $lf \in H^s(\mathbf{R}^m)$, $||f||_s^+ = \inf ||lf||_s$, where infimum is chosen among all continuations l.

We remind the definition of the wave factorization, because it is a key point for our forthcoming considerations.

Definition 1. *Wave factorization of elliptic symbol $A(\xi)$ with respect to the wedge W_+^a is defined as*

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ should satisfy the following assumptions for almost all $\xi'' \in \mathbf{R}^{m-2}$:

1) $A_{\neq}(\xi)$, $A_{=}(\xi)$ are defined on the whole of $\mathbf{R}^m$ except may be for the points $\{\xi \in \mathbf{R}^m\colon \mathrm{a}\xi_2 = |\xi_1|\}$;

2) $A_{\neq}(\xi)$, $A_{=}(\xi)$ admit an analytical continuation into radial tube domains $\mathrm{T}(\overset{*}{C}{}_{\pm}^a)$ over the cones $\overset{*}{C}{}_{\pm}^a$, respectively [10],

$$\overset{*}{C}{}_{+}^a = \left\{\xi \in \mathbf{R}^2 :\ a\xi_2 > |\xi_2|\right\}, \quad \overset{*}{C}{}_{-}^a = -\overset{*}{C}{}_{+}^a,$$

and satisfy the estimates

$$\left|A_{\neq}^{\pm 1}(\xi + i\tau)\right| \leq c(1 + |\xi| + |\tau|)^{\pm æ},$$

$$\left|A_{=}^{\pm 1}(\xi \pm i\tau)\right| \leq c(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \quad \tau \in \overset{*}{C}{}_{\pm}^a.$$

The number $æ \in \mathbf{R}$ is called the index of wave factorization. Generally speaking it may be a complex number, but the solvability picture is defined by its real part only.

Theorem 1. *[6, 9] If $æ - s = \delta$, $|\delta| < 1/2$, then the unique solution of pseudo differential equation(1) can be written with the help of the special integral operator G_2. This operator is defined by the formula*

$$(G_2 u)(\xi) = \frac{1}{2} u(\xi) + \lim_{\tau \to 0+} \int\limits_{\mathbf{R}^2} \frac{u(y_1, y_2, \xi'') dy_1 dy_2}{(\xi_1 - y_1)^2 - a^2(\xi_2 - y_2 + i\tau)^2},$$

and we have $\tilde{u}_+ = A_{\neq}^{-1} G_2 A_{=}^{-1} \tilde{lf}$ with a priori estimate $||u||_s \leq c||f||_{s-\alpha}^+$.

If $æ - s = n + \delta$, $n \in \mathbf{N}$, $|\delta| < \frac{1}{2}$, then the solution of equation (1) is non-unique and depends on n arbitrary functions from corresponding Sobolev-Slobodetskii spaces defined on wedge sides. It's possible to give different boundary conditions for uniquely identifying these arbitrary functions. The author tried to verify both classical boundary conditions (and particular, the Dirichlet and Neumann conditions) and other non-local (or integral) conditions, and found the unique solvability conditions for corresponding boundary value problems (see [7],[8]).

3. The Potentials

Here we will consider in more detail the case $æ - s = n + \delta$, $n \in \mathbf{Z}$, $n < 0$, $|\delta| < \frac{1}{2}$, and below we will use the following shortenings. For all points $x, \xi \in \mathbf{R}^m$ we will write $x = (x_1, x_2, x'')$, $\xi = (\xi_1, \xi_2, \xi'')$, and we will omit the dependence on the parameters x'', ξ''.

Further, because $s - \alpha > æ - \delta - \alpha$, then if $f \in H_0^{s-\alpha}(W_+^a)$ we have $f \in H^{æ-s-\alpha}(W_+^a)$. Following the previous result, there is the unique solution

$$\tilde{w}_+ = A_{\neq}^{-1} G_2 A_{=}^{-1} \tilde{lf}. \tag{2}$$

Let $A_{=}^{-1} \tilde{lf} = \tilde{g}$. We do change of variables and denote

$$(G_2 \tilde{g}) \left(\frac{x_2 + x_1}{2}, \frac{x_2 - x_1}{2a} \right) = \tilde{h}\,(x_1, x_2)\,,$$

$\tilde{g}\left(\frac{y_2+y_1}{2}, \frac{y_2-y_1}{2a}\right) = \tilde{g}_1\,(y_1, y_2)\,.$ Then, obviously,
$\tilde{h}(x_1 x_2) = \lim\limits_{\tau \to 0+} \int\limits_{\mathbf{R}^2} \frac{\tilde{g}(y_1, y_2) dy}{(x_1 - y_1 - ai\tau)(x_2 - y_2 + i\tau).}$

We decompose the kernel $(x_j - y_j + ib)^{-1}$ with the help of the following formula $(b = \pm a\tau)$:

$$\frac{1}{x_1 - y_1 + ib} = \frac{1}{(x_1 - x_2 - i + ib)(1 - \frac{y_1 - x_2 - i}{x_1 - x_2 - i + ib})} =$$

$$= \sum_{k=0}^{p-1} \frac{(y_1 - x_2 - i)^k}{(x_1 - x_2 - i + ib)^{k+1}} +$$

$$+ \frac{(y_1 - x_2 - i)^p}{(x_1 - x_2 - i + ib)^{p+1} \left(1 - \frac{y_1 - x_2 - i}{x_1 - x_2 - i + ib}\right)} =$$

$$= \sum_{k=0}^{p-1} \frac{\Lambda^k(y_1, x_2)}{\Lambda_b^k(x_1, x_2)} + \frac{\Lambda^p(y_1, x_2)}{\Lambda_b^p(x_1, x_2)(x_1 - y_1 + ib)},$$

where $\Lambda_b(x_1, x_2) \equiv x_1 - x_2 - i + ib$, $\Lambda(x_1, x_2) = x_1 - x_2 + i$.

Analogously,

$$\frac{1}{x_2-y_2+ib}=\sum_{r=0}^{q-1}\frac{\underline{\Lambda}^r(y_2,x_1)}{\underline{\Lambda}_b^{r+1}(x_2,x_1)}+\frac{\underline{\Lambda}^q(y_2,x_1)}{\underline{\Lambda}_b^q(x_2,x_1)(x_2-y_2+ib)},$$
$$\underline{\Lambda}_b(x_2,x_1)\equiv x_2-x_1+i+ib).$$

Then

$$\frac{1}{(x_1-y_1-ai\tau)(x_2-y_2+ai\tau)}=\sum_{k=0}^{p-1}\sum_{r=0}^{q-1}\frac{\Lambda^k(y_1,x_2)\underline{\Lambda}^r(y_2,x_1)}{\Lambda_{-a\tau}^{k+1}(x_1,x_2)\underline{\Lambda}_{a\tau}^{r+1}(x_2,x_1)}+$$
$$+\sum_{k=0}^{p-1}\frac{\Lambda^k(y_1,x_2)\underline{\Lambda}^q(y_2,x_1)}{\Lambda_{-a\tau}^{k+1}(x_1,x_2)\underline{\Lambda}_{a\tau}^{q}(x_2,x_1)(x_2-y_2+ai\tau)}+$$
$$+\sum_{r=0}^{q-1}\frac{\underline{\Lambda}^r(y_2,x_1)\Lambda^p(y_1,x_2)}{\Lambda_{-a\tau}^{p}(x_1,x_2)\underline{\Lambda}_{a\tau}^{r+1}(x_2,x_1)(x_1-y_1-ai\tau)}+$$
$$+\frac{\Lambda^7(y_1,x_2)\underline{\Lambda}^q(y_2,x_1)}{\Lambda_{-a\tau}^{p}(x_1,x_2)\underline{\Lambda}_{a\tau}^{q}(x_2,x_1)(x_1-y_1-ai\tau)(x_2-y_2+ai\tau)}.$$

Let $\tilde{g}_1(y_1,y_2)\in S(\mathbf{R}^m)$. We multiply the last equality by $\tilde{g}_1(y_1,y_2)$, integrate over y_1,y_2 and pass to the limit as $\tau\to 0+$. Then we have

$$\tilde{h}(x_1,x_2)=\sum_{k=0}^{p-1}\sum_{r=0}^{q-1}\Phi_{-\mathrm{k}-1,-\mathrm{r}-1}(x,x)\int_{\mathbf{R}^2}\Phi_{\mathrm{k},\mathrm{r}}(y,x)\tilde{g}_1(y)dy+$$
$$+\lim_{\tau\to 0+}\sum_{k=0}^{p-1}\Phi_{-\mathrm{k}-1,\mathrm{q}}(x,x)\int_{\mathbf{R}^2}\frac{\Phi_{\mathrm{k},\mathrm{q}}(y,x)\tilde{g}_1(y)dy}{\mathrm{x}_2-y_2+ia\tau}+$$
$$+\lim_{\tau\to 0+}\sum_{r=0}^{q-1}\Phi_{-\mathrm{p},-\mathrm{r}-1}(x,x)\int_{\mathbf{R}^2}\frac{\Phi_{\mathrm{p},\mathrm{r}}(y,x)\tilde{g}_1(y)dy}{\mathrm{x}_1-y_1-ia\tau}+$$
$$+\Phi_{-p,-q}(x,x)\lim_{\tau\to 0+}\int_{\mathbf{R}^2}\frac{\Phi_{\mathrm{p},\mathrm{q}}(y,x)\tilde{g}_1(y)dy}{(\mathrm{x}_1-y_1-ia\tau)(x_2-y_2+ia\tau)},$$

where $\Phi_{p,q}(y,x)\equiv\Lambda^p(y_1,x_2)\underline{\Lambda}^q(y_2,x_1)\equiv(y_1-x_2-i)^p(y_2-x_1+i)^q$ is a polynomial of order less or equal $p+q$ with respect to the variables x_1,x_2,y_1,y_2.

Let's write:

$$\tilde{h}=\sum_{k=0}^{p-1}\sum_{r=0}^{q-1}\Phi_{-k-1,-r-1}\Pi'\Phi_{k,r}\ \tilde{g}_1+\sum_{k=0}^{p-1}\Phi_{-k-1,-q}\Pi_+^{(2)}\Pi_1'\Phi_{k,q}\ \tilde{g}_1+$$
$$+\sum_{r=0}^{q-1}\Phi_{-p,-r-1}\Pi_-^{(1)}\Pi_2'\Phi_{p,r}\ \tilde{g}_1+\Phi_{-p,-q}G_2\Phi_{p,q}\ \tilde{g}_1, \tag{3}$$

where

$$(\Pi^{(1)}_{\pm}\tilde{g})(\zeta) = \lim_{\tau\to 0+}\int\limits_{-\infty}^{+\infty} \frac{\tilde{g}(\eta_1,\zeta_2)d\eta_1}{\eta_1-\zeta_1\pm i\tau},$$

$$(\Pi^{(2)}_{\pm}\tilde{g})(\zeta) = \lim_{\tau\to 0+}\int\limits_{-\infty}^{+\infty} \frac{\tilde{g}(\zeta_1,\eta_2)d\eta_2}{\eta_2-\zeta_2\pm i\tau},$$

$$(\Pi^{'}_{j}\tilde{g})(\zeta_{3-j}) = \int\limits_{-\infty}^{+\infty} \tilde{g}(\zeta_1,\zeta_2)d\zeta_j,\ j=1,2, \qquad \Pi^{'}\tilde{g} = \int\limits_{\mathbf{R}^2} \tilde{g}(\zeta_1,\zeta_2)d\zeta.$$

The last expansion was obtained under assumption $\tilde{g}_1 \in S(\mathbf{R}^m)$. It's easy to verify, that this formula will be valid for $\tilde{g}_1 \in \tilde{H}^{p+q-\delta}(\mathbf{R}^m)$. We need arguments like [2] and the density property for $S(\mathbf{R}^m)$ in $\tilde{H}^{p+q-\delta}(\mathbf{R}^m)$.

Let's return to the formula (3). Changing variables, taking $p+q=|n|$, and denoting

$$\tilde{w}_+\left(\frac{x_2+x_1}{2},\frac{x_2-x_1}{2a}\right) \equiv \tilde{W}_+(x_1,x_2),$$

$$A^{-1}_{\neq}\left(\frac{x_2+x_1}{2},\frac{x_2-x_1}{2a}\right) \equiv a^{-1}_{\neq}(x_1,x_2),$$

using (3) for (2) (so that $A^{-1}_{=}\tilde{l}f \in \tilde{H}^{|n|-\delta}(\mathbf{R}^m)$,)
we obtain the following decomposition:

$$\tilde{W}_+(x) = \sum_{k=0}^{p-1}\sum_{r=0}^{q-1} \frac{\tilde{P}_{k,r}(x)}{a_{\neq}(x)\Phi_{\mathrm{k+1,r+1}}(x,x)} +$$
$$+\sum_{k=0}^{p-1} \frac{\tilde{M}_{k,q}(x)}{\Phi_{\mathrm{k+1,q}}(x,x)a_{\neq}(x)} + \sum_{r=0}^{q-1} \frac{\tilde{N}_{p,r}(x)}{\Phi_{p,r+1}(x,x)a_{\neq}(x)} + \tilde{U}_+(x),$$

where $\tilde{P}_{k,r}(x) = \Pi^{'}\Phi_{k,r}\tilde{g}_1$, $\tilde{M}_{k,q}(x) = \Pi^{(2)}_{+}\Pi^{'}_{1}\Phi_{k,q}\tilde{g}_1$, $\tilde{N}_{p,q}(x) = \Pi^{(1)}_{-}\Pi^{'}_{2}\Phi_{k,r}\tilde{g}_1$, $\tilde{U}_+ = a^{-1}_{\neq}(x)\Phi_{-p,-q}(x,x)G'_2\Phi_{p,q}\tilde{g}_1$.

The function $\tilde{M}_{k,q}(x)$ has the form

$$\tilde{M}_{k,q}(x) = \lim_{\tau\to 0+}\int\limits_{\mathbf{R}^2} \frac{\Phi_{\mathrm{k,q}}(y,x)\tilde{g}_1(y)dy}{(\mathrm{x}_1-y_1-ia\tau)} = \sum_{|\sigma|+|\beta|=0}^{k+q} e_{\sigma,\beta}x^{\sigma} \lim_{\tau\to 0+}\int\limits_{\mathbf{R}^2} \frac{y^{\beta}\tilde{g}_1(y)dy}{\mathrm{x}_1-y_1-ia\tau},$$

where σ, β are multi-indices.

We consider more precisely

$$\lim_{\tau\to 0+}\int\limits_{\mathbf{R}^2} \frac{y^{\beta}\tilde{g}_1(y)dy}{\mathrm{x}_1-y_1-ia\tau} = \lim_{\tau\to 0+}\int_{-\infty}^{+\infty} \frac{y_1^{\beta_1}\tilde{g}_2(y_1)dy_1}{\mathrm{x}_1-y_1-ia\tau},$$

denoting

$$\tilde{g}_{\beta 2}(y_1) \equiv \int_{-\infty}^{+\infty} y_2^{\beta_2} \tilde{g}_1(y_1, y_2) dy_2.$$

The Cauchy integral can be written as follows:

$$\lim_{\tau\to 0+} \int_{-\infty}^{+\infty} \frac{y_1^{\beta_1} \tilde{g}_{\beta_2}(y_1) dy_1}{\mathrm{x}_1 - y_1 - ia\tau} = \sum_{j=0}^{l-1} \frac{i}{(x_1 - i)^{j+1}} \int_{-\infty}^{+\infty} (y_1 - i)^j y_1^{\beta_1} \tilde{g}_{\beta_2}(y_1) dy_1 +$$

$$\lim_{\tau\to 0+} \frac{1}{(x_1 - i)^l} \int_{-\infty}^{+\infty} \frac{(y_1 - i)^l y_1^{\beta_1} \tilde{g}_{\beta_2}(y_1) dy_1}{y_1 - x_1 - ia\tau},$$

and, consequently,

$$\lim_{\tau\to 0+} \int\limits_{\mathbf{R}^2} \frac{y^\beta \tilde{g}_{\beta_2}(y) dy}{x_1 - y_1 - ia\tau} = \sum_{j=0}^{l-1} \frac{i\lambda_j}{(x_1 - i)^{j+1}} + \tilde{g}_\beta(x_1),$$

where

$$\tilde{g}_\beta(x_1) = \frac{1}{(x_1 - i)^l} \lim_{\tau\to 0+} \int\limits_{\mathbf{R}^2} \frac{y^\beta (y_1 - i)^l \tilde{g}_1(y) dy}{x_1 - y_1 - ia\tau}, \lambda_j = \int_{R^2} (y_1 - i)^l y^\beta \tilde{g}_1(y) dy.$$

Let $l = |n| - |\beta| - 1$. Note that the last formula doesn't include the sum under$|\beta| = |n| - 1$. It is easily verified that $\tilde{g}_\beta(x_1) \in \tilde{H}^{|n|-\delta-|\beta|-\frac{1}{2}}$ $(\mathbf{R}_-)$ (see the a priori estimates below). Thus,

$$\sum_{k=0}^{p-1} \frac{\tilde{M}_{k,q}(x)}{\Phi_{\mathrm{k+1,q}}(x,x) a_{\neq}(x)} = \sum_{k=0}^{p-1} \sum_{|\sigma|+|\beta|=0}^{k+q} \frac{e_{\sigma,\beta} x^\sigma}{\Phi_{\mathrm{k+1,q}}(x,x) a_{\neq}(x)} \sum_{j=0}^{l-1} \frac{i\lambda j}{(x_1 - i)^{j+1}} +$$

$$\sum_{k=0}^{p-1} \sum_{|\sigma|+|\beta|=0}^{k+q} \frac{e_{\sigma,\beta} x^\sigma \tilde{g}_\beta(x_1)}{\Phi_{\mathrm{k+1,q}}(x,x) a_{\neq}(x)}.$$

Obviously, for the $\tilde{N}_{p,r}(x)$ we have analogous representation:

$$\sum_{r=0}^{q-1} \frac{\tilde{N}_{p,r}(x)}{\Phi_{\mathrm{p,r+1}}(x,x) a_{\neq}(x,x)} =$$

$$= \sum_{r=0}^{q-1} \sum_{|\gamma|+|t|=0}^{r+p} \frac{b_{\gamma,t} x^\gamma}{\Phi_{p,r+1}(x,x) a_{\neq}(x)} \sum_{j=0}^{l'-1} \frac{i\mu_j}{(x_2 - i)^{j+1}} +$$

$$+\sum_{r=0}^{q-1}\sum_{|\gamma|+|t|=0}^{r+p}\frac{b_{\gamma,t}x^{\gamma}\tilde{f}_t(x_2)}{\Phi_{p,r+1}(x,x)a_{\neq}(x)}.$$

Using the previous uniqueness theorem and a lemma on radial tube domains from [6, 9] we conclude

$$\tilde{U}_+(\xi_1-a\xi_2,\xi_1+a\xi_2)\in\tilde{H}^{|n|-\delta-\kappa}(W_+^a)=\tilde{H}^s(W_+^a).$$

Then we can write

$$\tilde{W}_+(x)=\tilde{U}_+(x)+\sum_{k=0}^{p-1}\sum_{r=0}^{q-1}\frac{\tilde{P}_{r,k}(x)}{a_{\neq}(x,x)\Phi_{k+1,\mathrm{r}+1}(x,x)}+$$
$$+\sum_{k=0}^{p-1}\sum_{|\sigma|+|\beta|=0}^{k+q}\frac{e_{\sigma,\beta}x^{\sigma}}{\Phi_{k+1,q}(x,x)a_{\neq}(x)}\sum_{j=0}^{l}\frac{i\lambda_j}{(x_1-i)^{j+1}}+$$
$$+\sum_{r=0}^{q-1}\sum_{|\gamma|+|t|=0}^{r+p}\frac{b_{\gamma,t}x^{\gamma}}{\Phi_{p,r+1}(x,x)a_{\neq}(x)}\sum_{j=0}^{l'}\frac{i\mu_j}{(x_2-i)^{j+1}}+$$
$$+\sum_{k=0}^{p-1}\sum_{|\sigma|+|\beta|=0}^{k+q}\frac{e_{\sigma,\beta}x^{\sigma}\tilde{g}_{\beta}(x_1)}{\Phi_{k+1,q}(x,x)a_{\neq}(x)}+\sum_{r=0}^{q-1}\sum_{|\gamma|+|t|=0}^{r+p}\frac{b_{\gamma,t}x^{\gamma}\tilde{f}_t(x_2)}{\Phi_{\mathrm{p},\mathrm{r}+1}(x,x)a_{\neq}(x)}. \tag{4}$$

Using G.Eskin's methods [2] one can verify that representation $\tilde{W}_+$ in the form (4) is unique; thus, the solution of equation (1) with the right hand side $f\in H_0^{s-\alpha}(W_+^a)$ is belonging to $H^s(W_+^a)$ if the following conditions hold:

$$\tilde{P}_{k,r}(x)\equiv 0,\ \ \tilde{g}_{\beta}(x_1)\equiv 0,\ \ \tilde{f}_t(x_2)\equiv 0,\lambda_j,\mu_j\equiv 0,$$

for all possible k, r, β, t, j.

The coefficients of the polynomial $\tilde{P}_{k,r}(x)$ (of orders notbgreater than $|n|-2$) are

$$c'_{\sigma,\beta}=c_{\sigma,\beta}\int_{\mathbf{R}^2}y^{\beta}\tilde{g}_1(y)dy$$

if

$$\Phi_{\mathrm{k,r}}(x,y)=\sum_{|\sigma|+|\beta|=0}^{k+r}c_{\sigma,\beta}x^{\sigma}y^{\beta}.$$

According to Fourier transform properties, the condition $c'_{\sigma,\beta}=0$ is equivalent to the following one:

$$\left.\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}\frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}}g_1\right|_{x=0}=0,\ \ |\beta|=0,1,...,|n|-2.$$

The condition $\lambda_j = 0$ is equivalent to

$$\frac{\partial^{j+\beta_1}}{\partial x_{x_1}^{j+\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} g_1 \Big|_{x=0} = 0.$$

The similar assertion is true for μ_j.

The conditions $\tilde{g}_\beta(x_1) = 0$, $\tilde{f}_t(x_2) = 0$ are equivalent to the following ones:

$$\lim_{\tau\to 0+} \int\limits_{\mathbf{R}^2} \frac{y^\beta (y_1 - i)^l \tilde{g}_1(y) dy}{x_1 - y_1 - ia\tau} = 0,$$

$$\lim_{\tau\to 0+} \int\limits_{\mathbf{R}^2} \frac{y^t (y_2 - i)^k \tilde{g}_1(y) dy}{x_2 - y_2 - ia\tau} = 0.$$

And, according to Fourier transform and Cauchy type integral properties, we obtain:

$$\frac{\partial^{\beta_1+l}}{\partial x_1^{\beta_1+l}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} g_1 \Big|_{\substack{x_1 \le 0 \\ x_2 = 0}} = 0,$$

$$\frac{\partial^{t_1}}{\partial x_1^{t_1}} \frac{\partial^{t_2+k}}{\partial x_2^{t_2+k}} g_1 \Big|_{\substack{x_2 \ge 0 \\ x_1 = 0}} = 0.$$

Let's summarize these results.

Theorem 2 (Main Theorem)**.** *Let* $æ - s = n + \delta,\ n \in \mathbf{Z},\ n < 0,\ |\delta| < \frac{1}{2}$.

Then for arbitrary right hand side $f \in H_0^{s-\alpha}(W_+^a)$ *there exists a unique solution of the equation* (1)*, and its Fourier transform is represented in the form* (4)*, where one needs to substitute* $x_1 = \xi_1 - a\xi_2, x_2 = \xi_1 + a\xi_2$.

The function $U_+(ay_1 - y_2, ay_1 + y_2) \in H^s(W_+^a)$, $g_\beta(ay_1 - y_2), f_\beta(ay_1 + y_2) \in H^{|n|-\delta-|\beta|-\frac{1}{2}}(B_\pm)$, *respectively,* $B_+ = \{(y_1, y_2)\colon \mathrm{a}y_1 - y_2 \le 0\,, \mathrm{a}y_1 + y_2 = 0\}$,$B_- = \{(y_1, y_2)\colon \mathrm{a}y_1 - y_2 = 0\,, \mathrm{a}y_1 + y_2 \ge 0\}$, $|\beta| = 0, 1, ..., |n| - 1$, *and the constants* $\lambda_j, \mu_j, j = 0, 1, ..., l$, *in the representation* (4) *are defined uniquely.*

The equation (1) *has a solution from* $H^s(W_+^a)$ *if the following conditions hold:*

$$\left(\frac{1}{a}\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)^{\beta_1} \left(\frac{1}{a}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)^{\beta_2} A_{=}^{-1} l f(y) \Big|_{y=0} = 0,$$

$$\left(\frac{1}{a}\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)^{\beta_1} \left(\frac{1}{a}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)^{\beta_2} A_{=}^{-1} l f(y) \Big|_{\substack{ay_1 - y_2 \le 0 \\ ay_1 + y_2 = 0}} = 0,$$

$$\left(\frac{1}{a}\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_2}\right)^{\beta_1}\left(\frac{1}{a}\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)^{\beta_2}A_{=}^{-1}lf(y)\Bigg|_{\substack{ay_1-y_2=0\\ ay_1+y_2\geq 0}}=0.$$

The following a priori estimates hold:

$$||u_+||_s\leq c||f||^+_{s-\alpha},$$

$$[g_\beta]_{n_\beta}\leq c||f||^+_{s-\alpha},\ [f_\beta]_{n_\beta}\leq c||f||^+_{s-\alpha},$$

$$n_\beta=|n|-\delta-|\beta|-1/2,|\beta|=0,1,...,|n|-1,|c'_{\sigma,\beta}|\leq c||f||^+_{s-\alpha},$$

$$|\lambda_j|\leq c||f||^+_{s-\alpha},\ |\mu_j|\leq c||f||^+_{s-\alpha},\ |\beta|=0,1,...,|n|-2,j=0,1,...,l.$$

Proof. We need to prove a priori estimates only. Using standard theorem from [2] on boundedness of pseudo differential operators, we have

$$||u_+||_s=||U_+||_s=||\tilde{U}_+||_s=$$
$$=||a_{\neq}^{-1}\Phi^{-1}_{\mathrm{p,q}}G_2\Phi_{\mathrm{p,q}}A_{=}^{1}l\tilde{f}||_s\leq c||\Phi^{-1}_{\mathrm{p,q}}G_2\Phi_{\mathrm{p,q}}A_{=}^{-1}l\tilde{f}||_{s-æ}\leq$$
$$\leq c||G_2\Phi_{\mathrm{p,q}}A_{=}^{-1}l\tilde{f}||_{s-æ-n}\leq c||\Phi_{\mathrm{p,q}}A_{=}^{-1}l\tilde{f}||_{s-æ+n}\leq c||A_{=}^{-1}l\tilde{f}||_{s-æ}\leq$$
$$\leq c||l\tilde{f}||_{s-\alpha}\leq c||lf||_{s-\alpha}\leq c||f||^+_{s-\alpha}.$$

Further, if, for example, $0<\delta<1/2$, then

$$[g_\beta]_{n_\beta}=\left[(x_1-i)^{-l}\Pi_-^{(2)}\Pi_2'y^\beta(y_1-i)^lA_+^{-1}\widetilde{lf}\right]_{n_\beta}\leq$$

$$\leq\left[\Pi_-^{(1)}\Pi_2'y^\beta(y_1-i)^lA_+^{-1}\widetilde{lf}\right]_{|n|-\delta-|\beta|-1/2-l}\leq,$$

because we have

$$\leq 0<|n|-\delta-|\beta|-1/2-l=1/2-\delta<1/2\leq$$

$$\leq c\left[\Pi_2'y^\beta(y_1-i)A_+^{-1}\widetilde{lf}\right]_{1/2-\delta}\leq$$

(the restriction on hyper-plane theorem [2])

$$\leq c||y^\beta(y_1-i)^lA_{=}^{-1}l\tilde{f}]_{1-\delta}\leq ...\leq c||f||^+_{s-\alpha}.$$

Finally,

$$|c'_{\sigma,\beta}|\leq c\int_{\mathbf{R}^2}y^\beta\tilde{g}_1(y)dy|\leq$$

$$\leq c\int\limits_{\mathbf{R}^2}(1+|y|)^{|\beta|}|A_{=}^{-1}(y)|\cdot|l\tilde{f}(y)|dy \leq c\int\limits_{\mathbf{R}^2}(1+|y|)^{|\beta|+æ-\alpha}|l\tilde{f}(y)|dy =$$

$$= c\int\limits_{\mathbf{R}^2}(1+|y|)^{|\rho|+æ-s}(1+|y|)^{s-\alpha}|l\tilde{f}(y)|dy \leq$$

(the Cauchy inequality)

$$\leq c\;(\int\limits_{\mathbf{R}^2}(1+|y|)^{2(|\beta|+æ-s)}dy)^{1/2}||l\tilde{f}||_{s-\alpha}.$$

The integral is convergent if $2(|\beta|+æ-s) < -2$, i.e. $|\beta|-|n|+\delta < -1$. Thus, we have the needed estimate.

The λ_j, μ_j can be estimated analogously.◀

Remark 1. *Let's consider $\tilde{g}_\beta(x_1)$ in detail.*

The expression $y^\beta(y_1-i)^l\tilde{g}_1(y)$ obviously can be written in the form $P_k(y_1,y_2)\tilde{g}_1(y)$, where $P_k(y_1,y_2)$ is a polynomial of order $k=|\beta|+l$ of variables y_1, y_2. Then the integral

$$\int_{-\infty}^{+\infty} P_k(y_1,y_2)\tilde{g}_1(y)dy_2$$

means that we take (for inverse Fourier images) partial derivatives for g_1 and their restrictions on the line $y_2=0$. In other words, we take linear combination of partial derivatives in variable y_2 up to certain order with respect to one of the angle (wedge) side. Concerning other variable, the expression

$$\int\limits_{\mathbf{R}^2}\frac{y^\beta(y_1-i)^l\tilde{g}_1(y)dy}{x_1-y_1-ia\tau}$$

can be written as follows:

$$\int\limits_{\mathbf{R}^2}\frac{P_k(y_1,y_2)\tilde{g}_1(y)dy}{x_1-y_1-ia\tau} = \int\limits_{\mathbf{R}^2}\frac{\sum_{k_1+k_2=0}^{|k|}c_{k_1,k_2}y_1^{k_1}y_2^{k_2}\tilde{g}_1(y)dy}{x_1-y_1-ia\tau} =$$

$$= \sum_{k_1+k_2=0}^{|k|}c_{k_1,k_2}\int_{-\infty}^{+\infty}\frac{y_1^{k_1}\tilde{h}_1^{(k_2)}(y_1)dy_1}{x_1-y_1-ia\tau},$$

where

$$\tilde{h}_1^{(k_2)}(y_1) = \int_{-\infty}^{+\infty} y_2^{k_2}\tilde{g}_1(y_2)dy_2. \tag{5}$$

The functions

$$\int_{-\infty}^{+\infty} \frac{y_1^{k_1} \tilde{h}_1^{(k_2)}(y_1) dy_1}{x_1 - y_1 - ia\tau} \tag{6}$$

evidently can be treated as some potential like operators generated by right hand side of the equation (1)*.*

The notation (5) *in our opinion is convenient because applying inverse Fourier transform to* (5) *gives the following:*

$$\left. \frac{\partial^{k_2}}{\partial x_2^{k_2}} g_1 \right|_{x_2=0} \equiv h_1^{(k_2)}(x_1).$$

Now about integral (6)*. It is an ordinary Cauchy type integral by which one solves well-known classical Riemann problem for upper and lower hyper-planes. The boundary Riemann problem is a holomorphic functions theory problem. Since real and image parts of holomorphic functions are harmonic functions, then real and imaginary parts of Cauchy type integrals are (logarithmic) double and single layer potentials, respectively [3],[5],[4]. So, the terminology above is justified. The representation formula* (4) *shows that the solution has "principal" part and a linear combination of double and single layer potentials for traces of right hand side partial derivatives. In other words, the formula* (4) *is similar to integral representation for solution of the equation* (1)*, which corresponds to the case* $\text{æ} - s = n + \delta$, $n < 0$, $n \in \mathbf{Z}$, $|\delta| < \frac{1}{2}$*. (See also. G. Eskin [1]).*

Remark 2. *It would be very interesting to obtain essential multi-dimensional variants for such potential like operators, but there are some difficulties. The author hopes to explore this case in one of his forthcoming papers.*

References

[1] G Eskin. Conjugation problem for principal type equations with two variables. *Trans. Moscow Math.Soc.*, 21:245–292, 1970.

[2] G Eskin. *Boundary Value Problems for Elliptic Pseudo Differential Equations.* AMS, Providence, Rhode Island, 1981.

[3] F Gakhov. *Boundary Value Problems.* Pergamon Press, Oxford, 1966.

[4] S Mikhlin and S Proessdorf. *Singular Integral Operators.* Akademie-Verlag, Berlin, 1986.

[5] N Muskhelishvili. *Singular Integral Equations.* North Holland, Amsterdam, 1953.

[6] V Vasilyev. *Wave Factorization of Elliptic Symbols: Theory and Application. Introduction to the Boundary Value Problems in Non-Smooth Domains.* Kluwer Academic Publishers, Dordrecht-Boston-London, 2000.

[7] V Vasilyev. Boundary Value Problems for Elliptic Equation with Non-Local Boundary Condition. *Mathematical Modelling and Boundary Value Problems*, (3):45–48, 2008.

[8] V Vasilyev. Boundary Value Problems for Elliptic Equation with Non-Local Boundary Condition. II. *Mathematical Modelling and Boundary Value Problems*, (3):62–64, 2009.

[9] V Vasilyev. *Fourier Multipliers,Pseudo Differential Equations, Wave Factorization, Boundary Value Problems.* URSS, Moscow, 2010.

[10] V Vladimirov. *Methods of Functions Theory of Several Complex Variables*. Nauka, Moscow, 1964.

Vladimir B. Vasilyev
Lipetsk State Technical University, Moskovskaya 30, Lipetsk 398600, Russia
E-mail: vbv57@inbox.ru

Received 26 April 2012
Accepted 28 May 2012