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Constructive Function Theory in the Complex Plane through Potential Theory and Geometric Function Theory

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Abstract. This is a survey of some recent results concerning Bernstein-type inequalities for entire functions of the exponential type, harmonic majorants for classes of subharmonic functions, Phragmen-Lindelöf function, the Krein description of the Cartwright classes of entire functions with the finite logarithmic integral, structure of the Martin boundary of the Denjoy domains, smoothness properties of the Green function, etc. We generalize the classical Bernstein theorem concerning the constructive description of classes of functions uniformly continuous on the real line. Approximation of continuous bounded functions by entire functions of exponential type on an unbounded closed proper subset of the real line or on an unbounded quasismooth (in the sense of Lavrentiev) curve in the complex plane is studied. We discuss Totik's extension of the classical Bernstein theorem on polynomial approximation of piecewise analytic functions on a closed interval. The results are achieved by the application of methods and techniques of modern geometric function theory and potential theory.

Key Words and Phrases: harmonic functions; subharmonic functions; Green's function; Martin boundary; conformal invariants; capacity

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1. Positive harmonic functions on Denjoy domains (see [5]–[8]).

We denote by Denjoy domain an open subset Ω of the complex plane $\mathbf{C}$ whose complement $E := \overline{\mathbf{C}} \setminus \Omega$, where $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, is a subset of $\overline{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$, where $\mathbf{R}$ is the real axis. Throughout this section we rely on the following assumption: each point of E (including the point at infinity) is regular for the Dirichlet problem in Ω. Denote by $\mathcal{P}_\infty = \mathcal{P}_\infty(\Omega)$ the cone of positive harmonic functions on Ω which have vanishing boundary values at every point of $E \setminus \{\infty\}$. Independently, Levin [41], Ancona [3], and Benedicks [17] showed that either all functions in $\mathcal{P}_\infty$ are proportional or $\mathcal{P}_\infty$ is generated by two linearly independent (minimal) harmonic functions; that is, either dim $\mathcal{P}_\infty = 1$ or dim $\mathcal{P}_\infty = 2$, respectively. In other words, it means that the Martin boundary of Ω has either one or two "infinite" points.

The results in [3] and [17] are proved for positive harmonic functions in domain $\Omega \subset \mathbf{R}^n, n \geq 2$. We focus on the case $n = 2$ due to its extreme importance in the theory of entire functions, where positive harmonic functions and subharmonic functions in $\mathbf{C}$ which are non-positive on a subset of the real line were the subject of research significantly earlier.

Bernstein [19] showed that if an entire function f satisfies

$$\limsup_{|z|\to\infty} \frac{\log |f(z)|}{|z|} =: \sigma_f < \infty \tag{1}$$

and

$$|f(x)| \le 1, \quad x \in \mathbf{R},$$

then $|f'(x)| \le \sigma_f$ for any $x \in \mathbf{R}$.

In some extensions of the Bernstein theorem (see, for example, [49], [1], [40], [53], [2], [41], [42] and [43]) the entire function f satisfies (1) as well as a new condition

$$|f(x)| \le 1, \quad x \in E,$$

where $E \subset \mathbf{R}$ conforms to certain metric properties. Then the authors derived estimates on the growth of f in $\mathbf{C}$ of the form

$$|f(z)| \le (H_E(z))^{\sigma_f}, \quad z \in \mathbf{C},$$

where $H_E(z)$ is a "universal function" which does not depend on f.

We say that a subharmonic function u in $\mathbf{C}$ has finite degree $0 < \sigma < \infty$, if

$$\limsup_{|z|\to\infty} \frac{u(z)}{|z|} = \sigma.$$

We denote by $K_\sigma(E)$ the class of subharmonic in $\mathbf{C}$ functions of finite degree no larger than σ and non-positive on E. Let

$$v(z) = v(z, K_\sigma(E)) := \sup\{u(z) : u \in K_\sigma(E)\}, \quad z \in \mathbf{C},$$

be the subharmonic majorant of the class $K_\sigma(E)$. It is known that $v(z)$ is either finite everywhere on $\mathbf{C}$ or equal to $+\infty$ on $\mathbf{C} \setminus E$. The set E is said to be of type (α) in the former case, and of type (β) in the latter.

Theorem A. ([41], [42, Theorem 3.3], [43, Remark 1])

(a) Case (α) holds if and only if $\dim \mathcal{P}_\infty = 2$*;*

(b) case (β) holds if and only if $\dim \mathcal{P}_\infty = 1$.

There is a close connection between the dimension of $\mathcal{P}_\infty$ and the behavior of the Green function $g_\Omega(\cdot, z)$ for Ω with pole at $z \in \Omega$ (see [64], [50] or [52] for further details on logarithmic potential theory). Let $E^* := \mathbf{R} \setminus E$.

Theorem B. ([33, Theorem 3], [35, Section VIII A.2]) *Let*

$$U(z) = U(z, E) := \int_{E^*} g_\Omega(t, z)dt, \quad z \in \Omega.$$

Then

(a) $\dim \mathcal{P}_\infty = 1$ *if and only if* $U \equiv \infty$ *in* Ω;

(b) $\dim \mathcal{P}_\infty = 2$ *if and only if* U *is finite everywhere on* Ω.

The problem of finding a geometric description of E such that $\dim \mathcal{P}_\infty = 2$ or, equivalently, of E with the finite subharmonic majorants of classes $K_\sigma(E)$ attracted attention of a number of researches (see [2], [17], [33], [54], [31], [55] and [60]).

One of the basic results in this area is the following Benedicks' criterion. Let

$$R(x,r) := \left\{ z \in \mathbf{C} : |\Re z - x| < \frac{r}{2}, |\Im z| < \frac{r}{2} \right\}, \quad x \in \mathbf{R}, r > 0.$$

For an arbitrary fixed α with $0 < \alpha < 1$ and every $x \in \mathbf{R} \setminus \{0\}$, let $\beta_x(\cdot) = \beta_x(\cdot, \alpha, E)$ be the solution of the Dirichlet problem on $R(x, \alpha|x|) \setminus E$ with boundary values $\beta_x = 1$ on $\partial R(x, \alpha|x|)$ and $\beta_x = 0$ on $E \cap R(x, \alpha|x|)$.

Theorem C. ([17, Theorem 4]) *For every* α *with* $0 < \alpha < 1$,

(a) $\dim \mathcal{P}_\infty = 1$ *if and only if*

$$\int_{|x|\geq 1} \frac{\beta_x(x)dx}{|x|} = \infty;$$

(b) $\dim \mathcal{P}_\infty = 2$ *if and only if*

$$\int_{|x|\geq 1} \frac{\beta_x(x)dx}{|x|} < \infty.$$

Theorem C indicates that the dimension of $\mathcal{P}_\infty$ depends only on the geometry of E near infinity.

Theorems 1 and 2 below provide a natural and intrinsic characterization of E with a given $\dim \mathcal{P}_\infty$ in terms of the logarithmic capacity cap(S), $S \subset \mathbf{C}$, which appears most suitable for this theory. In these theorems we also connect the dimension of $\mathcal{P}_\infty$ with continuous properties of the Green function g_Ω in a neighborhood of infinity.

Theorem 1. *The following conditions are equivalent:*

(i) There exist points $a_j, b_j \in E$, $-\infty < j < \infty$ *such that*

$$b_{j-1} \leq a_j < b_j \leq a_{j+1}, \quad \lim_{j\to\pm\infty} a_j = \pm\infty, \tag{2}$$

$$\bigcup_{j=-\infty}^{\infty} (a_j, b_j) \supset E^*, \tag{3}$$

$$\inf_{-\infty<j<\infty} \frac{cap(E \cap [a_j, b_j])}{cap([a_j, b_j])} > 0,$$

$$\sum_{j=-\infty}^{\infty} \left(\frac{b_j - a_j}{|a_j| + 1} \right)^2 < \infty; \tag{4}$$

(ii) $\dim \mathcal{P}_\infty = 2$*;*

(iii) $\limsup_{\Omega \ni t \to \infty} g_\Omega(t,z)|t| < \infty$ *for any* $z \in \Omega$.

For particular results of this kind, see [54, Theorem 2], [55, Theorem 8], [43, Theorem 4] and [26, Theorem 1.11].

Notice that if $(a,\infty)\subset E^*$ or $(-\infty,a)\subset E^*$ for some $a\in\mathbf{R}$, then, by Theorem C, dim $\mathcal{P}_\infty=1$.

Theorem 2. *Let $E\cap(a,\infty)\neq\emptyset$ and $E\cap(-\infty,-a)\neq\emptyset$ for any $a>0$. The following conditions are equivalent:*

(i) There exist points $\{a_j,b_j\}_{j=-N}^{M}$, where $M+N=\infty$, such that $a_j,b_j\in E$,

$$b_{j-1}\le a_j<b_j\le a_{j+1},$$

$$\sup_j\frac{cap(E\cap[a_j,b_j])}{cap([a_j,b_j])}<1,$$

$$\sum_{j=-N}^{M}\left(\frac{b_j-a_j}{|a_j+b_j|+1}\right)^2=\infty;$$

(ii) dim $\mathcal{P}_\infty=1$*;*

(iii) $\limsup_{\Omega\ni t\to\infty}g_\Omega(t,z)|t|=\infty$ *for some $z\in\Omega$.*

Next, we state some metric tests which immediately follow from the theorems above and which use the one-dimensional Lebesgue measure (length) $|S|$ of a linear (Borel) set $S\subset\mathbf{R}$.

Remark 1. *Since*

$$cap([a_j,b_j])=\frac{b_j-a_j}{4}\quad\text{and}\quad cap(E\cap[a_j,b_j])\ge\frac{|E\cap[a_j,b_j]|}{4},$$

the existence of points $a_j,b_j\in E$, $-\infty<j<\infty$ satisfying (2)–(4) and the new inequality

$$\inf_{-\infty<j<\infty}\frac{|E\cap[a_j,b_j]|}{b_j-a_j}>0,$$

is sufficient for the validity of the parts (ii) and (iii) of Theorem 1 (cf. [53, Lemma 1], [17, Theorem 5] and [42, Theorem 3.8]).

Remark 2. *Let $E^*=\cup_{j=-N}^{M}(c_j,d_j)$, where $M+N=\infty$, be such that*

$$d_{j-1}<c_j<d_j<c_{j+1}.$$

In Theorem 2, taking the system of points $\{a_j,b_j\}$ to be the same as $\{c_j,d_j\}$, we derive that the condition

$$\sum_{j=-N}^{M}\left(\frac{d_j-c_j}{|c_j+d_j|+1}\right)^2=\infty$$

implies each of the parts (ii) and (iii) of Theorem 2 (cf. [17, Theorem 5], [33, Theorem 4] and [42, Theorem 3.7]).

Remark 3. *Let*

$$\theta_E(t) := |E^* \cap [-t,t]|, \quad t > 0.$$

The condition

$$\int_1^\infty \frac{\theta_E^2(t)dt}{t^3} < \infty$$

yields each of the parts (ii) and (iii) of Theorem 1 (cf. [54, Theorem 4], [31, Theorem 1], [55, Theorem 13 and Theorem 17] and [63, Theorem 2.2 and Corollary 4.1]).

Remark 4. *Let $\theta(t), t \geq 1$ be any increasing function such that*

$$0 < \theta(t) \leq 2t, \quad t \geq 1,$$

$$\int_1^\infty \frac{\theta^2(t)dt}{t^3} = \infty.$$

Then, there exists E such that

$$\theta_E(t) \leq \theta(t), \quad t > 2,$$

and parts (ii) and (iii) of Theorem 2 hold (cf. [55, Section 4.4], [31, Theorem 2] and [63, Corollary 4.2]).

Remark 5. *The continuous properties of the Green function g_Ω at boundary points are of independent interest in potential theory (see, for example, [24], [45], [47], [22], [26], [63] and references therein). Using conformal invariance of the Green function and the linear transformation*

$$w = \frac{d_0 - c_0}{2z - c_0 - d_0},$$

where (c_0, d_0) is any of the finite components of E^*, we can rephrase the part (i)⇔(iii) of Theorem 1 in the following equivalent form.

Let $F \subset \mathbf{R}$ be a regular compact set with the complement $G := \overline{\mathbf{C}} \setminus F$, and let $g_G(\cdot) = g_G(\cdot, \infty)$ be the Green function of G with pole at infinity. We assume that $F \subset [-1,1] =: I$ and $\pm 1, 0 \in F$. Let $F^* := \mathbf{R} \setminus F$. The equivalence (i)⇔(iii) in Theorem 1 can be restated as follows: *The following conditions are equivalent:*

(i') There exist points $a_j, b_j \in F$, $-\infty < j < \infty$ such that

$$-1 \leq a_{-1} < b_{-1} \leq a_{-2} < b_{-2} \leq \ldots < 0 < \ldots \leq a_1 < b_1 \leq a_0 < b_0 \leq 1,$$

$$\bigcup_{j=-\infty}^{\infty} (a_j, b_j) \supset F^* \cap I,$$

$$\inf_{-\infty < j < \infty} \frac{\operatorname{cap}(F \cap [a_j, b_j])}{\operatorname{cap}([a_j, b_j])} > 0,$$

$$\sum_{j=-\infty}^{\infty} \left(\frac{b_j - a_j}{a_j}\right)^2 < \infty;$$

(iii')
$$\limsup_{G\ni z\to 0}\frac{g_G(z)}{|z|}<\infty. \tag{5}$$

The monotonicity of the Green function yields

$$g_G(z)\geq g_{\overline{\mathbf{C}}\setminus I}(z),\quad z\in\mathbf{C}\setminus I,$$

that is, if F has the "highest density" at 0, then g_G has the "highest smoothness" at the origin. In particular,
$$g_G(iy)\geq g_{\overline{\mathbf{C}}\setminus I}(iy)>\frac{y}{2},\quad 0<y<1,$$

i.e.,
$$\limsup_{G\ni z\to 0}\frac{g_G(z)}{|z|}\geq\frac{1}{2}>0.$$

In this regard, Remark 5 describes the metric properties of F such that g_G has the "highest smoothness" at 0 (see the recent remarkable results by Carleson and Totik [26, Theorem 1.11] as well as Carroll and Gardiner [25] for another description of sets F whose Green's function possesses the property (5)).

2. On Approximation of continuous functions by entire functions on subsets of the real line (see [10], [14]).

For a closed unbounded set $E\subset\mathbf{C}$, denote by $BC(E)$ the class of (complex-valued) functions which are bounded and continuous on E. Let E_σ be the class of entire functions of exponential type at most $\sigma>0$ and let

$$A_\sigma(f,E):=\inf_{g\in E_\sigma}||f-g||_{C(E)},\quad f\in BC(E),$$

where $||\cdot||_{C(E)}$ means the uniform norm over E.

The classical Bernstein direct and inverse theorems (see [62, pp. 257, 340]) describe the relations between the smoothness of $f\in BC(\mathbf{R})$ and the rate of decrease of $A_\sigma(f,\mathbf{R})$ as $\sigma\to\infty$. In particular, from Bernstein's results it follows that for $f\in BC(\mathbf{R})$ and $0<\alpha<1$,
$$A_\sigma(f,\mathbf{R})=O(\sigma^{-\alpha})\quad\text{as }\sigma\to\infty \tag{6}$$

if and only if
$$\omega_{f,\mathbf{R}}(\delta)=O(\delta^\alpha)\quad\text{as }\delta\to+0, \tag{7}$$

where
$$\omega_{f,\mathbf{R}}(\delta):=\sup_{\substack{x_1,x_2\in\mathbf{R}\\|x_1-x_2|\leq\delta}}|f(x_2)-f(x_1)|,\quad\delta>0.$$

The main objective of this section is to extend Bernstein's results to the case where the function is considered on a proper subset of $\mathbf{R}$. The result by Brudnyi [23] for $E=\mathbf{R}\setminus(-1,1)$, discussed in [62, pp. 269-273], can be viewed as a first step in this direction.

We mostly focus on the case where the number of components of E is infinite. Some aspects of this problem are considered in the recent papers by Shirokov [57], [58].

Let

$$d(A,B) := \operatorname{dist}(A,B) = \inf_{z\in A, \zeta\in B} |z-\zeta|, \quad A, B \subset \mathbf{C},$$

and let $|B|$ denote, as before, the length of $B \subset \mathbf{C}$.

Unless otherwise stated, we denote by $C, C_1, C_2, \ldots$ positive constants that are either absolute or depend on E only.

The set $E^* := \mathbf{R} \setminus E$ consists of a finite or infinite number of disjoint open intervals $J_j = (a_j, b_j)$. In the remainder of this section we assume that if the number of J_js is infinite then E possesses the following two properties: for any j under consideration,

$$|J_j| \le C_1, \tag{8}$$

$$\sum_{k\ne j} \left(\frac{|J_k|}{d(J_k, J_j)} \right)^2 \le C_2. \tag{9}$$

We use the following examples to illustrate the forthcoming results and constructions. The examples show that the number of "holes" J_j can be infinite.

Example 1. Let

$$d_{l-1} < c_l < d_l < c_{l+1}, \quad l = 0, \pm 1, \pm 2, \ldots,$$

be such that

$$d_l - c_l \ge C_3, \quad c_{l+1} - d_l \le C_4.$$

Then, the set

$$E_1 = \cup_{l=-\infty}^{\infty} [c_l, d_l]$$

satisfies (8) and (9).

Example 2. A direct computation shows that the set $E_2 = \mathbf{R} \setminus E_2^*$, where

$$E_2^* = \cup_{j=-\infty}^{\infty} \cup_{k=2}^{\infty} \quad \{(2j + 2^{-k}(1-k^{-1}), 2j + 2^{-k}) \\ \cup (2j - 2^{-k}, 2j - 2^{-k}(1-k^{-1}))\},$$

also satisfies (8) and (9).

In the case of polynomial approximation of continuous functions on a finite interval $[a,b] \subset \mathbf{R}$, the special role of the endpoints a and b is well-known. An elegant idea, suggested in [27], is to introduce a new modulus of continuity by using a distance between the points on $[a,b]$ that is not Euclidean. In the case of entire function approximation on E, the endpoints of J_j also play a special role. We capture this effect by making use of a special distance between points of E in the definition of the modulus of continuity of a function $f \in BC(E)$. This distance is defined as follows. Let $\mathbf{H} := \{z : \Im z > 0\}$ be the upper half-plane. According to Levin [42] there exist vertical intervals $J_j' = (u_j, u_j + iv_j]$, $u_j \in \mathbf{R}, v_j > 0$ and a conformal mapping

$$\phi : \mathbf{H} \to \mathbf{H}_E := \mathbf{H} \setminus (\cup_j J_j')$$

normalized by $\phi(\infty)=\infty, \phi(i)=i$ such that ϕ can be extended continuously to $\overline{\mathbf{H}}$ and it satisfies the boundary correspondence $\phi(J_j)=J_j'$. For $x_1,x_2\in E$ such that $x_1<x_2$ set

$$\rho_E(x_1,x_2)=\rho_E(x_2,x_1):=\text{diam }\phi([x_1,x_2]),$$

where

$$\text{diam }B:=\sup_{z,\zeta\in B}|z-\zeta|,\quad B\subset\mathbf{C}.$$

In spite of its definition via the conformal mapping, the behavior of ρ_E can be characterized in purely geometrical terms as follows. According to (9),

$$d(J_j,E^*\setminus J_j)\ge C_5|J_j|,\quad C_5=C_2^{-1/2}.$$

Let the constant C be fixed such that $0<C<\min(1,C_5/2)$. For any component J_j of E^*, denote by $\tilde J_j$ the open interval with the same center as J_j and length $(1+C)|J_j|$. For $x_1,x_2\in E$ such that $x_1<x_2$ consider the function

$$\tau_E(x_1,x_2)=\tau_E(x_2,x_1)=\tau_{E,C}(x_1,x_2)$$

$$:=\begin{cases}\left(\dfrac{|J_j|}{d([x_1,x_2],J_j)}\right)^{1/2}(x_2-x_1), & \text{if } x_1,x_2\in\tilde J_j \text{ for some } j \text{ and}\\ & x_2-x_1<d([x_1,x_2],J_j),\\ |J_j|^{1/2}(x_2-x_1)^{1/2}, & \text{if } x_1,x_2\in\tilde J_j \text{ for some } j \text{ and}\\ & d([x_1,x_2],J_j)\le x_2-x_1\le\frac{C}{2}|J_j|,\\ x_2-x_1, & \text{otherwise.}\end{cases}$$

Theorem 3. *For $x_1,x_2\in E$,*

$$\frac{1}{C_6}\tau_E(x_1,x_2)\le\rho_E(x_1,x_2)\le C_6\tau_E(x_1,x_2),$$

where $C_6=C_6(E,C)>1$.

Notice that according to Theorem 3,

$$\rho_E(x_1,x_2)\ge C_7|x_2-x_1|,\quad x_1,x_2\in E.$$

The main result of this section is the following analogue of (6)-(7): for $f\in BC(E)$ and $0<\alpha<1$,

$$A_\sigma(f,E)=O(\sigma^{-\alpha})\quad\text{as }\sigma\to\infty\tag{10}$$

if and only if

$$\omega^*_{f,E}(\delta)=O(\delta^\alpha)\quad\text{as }\delta\to+0,\tag{11}$$

where

$$\omega^*_{f,E}(\delta):=\sup_{\substack{x_1,x_2\in E\\ \rho_E(x_1,x_2)\le\delta}}|f(x_2)-f(x_1)|\quad(\delta>0).$$

The statement (10)-(11) follows immediately from the direct Theorem 4 and the inverse Theorem 5 below.

Let $\omega(\delta), \delta > 0$ be a function of modulus of continuity type, i.e., a positive nondecreasing function with $\omega(+0) = 0$ such that

$$\omega(t\delta) \leq 2t\omega(\delta), \quad \delta > 0, t > 1.$$

Denote by $BC^*_\omega(E)$ the class of functions $f \in BC(E)$ satisfying

$$\omega^*_{f,E}(\delta) \leq \omega(\delta), \quad \delta > 0.$$

Theorem 4. *For $f \in BC^*_\omega(E)$ and $\sigma \geq 1$,*

$$A_\sigma(f, E) \leq C_8 \left(\frac{||f||_{C(E)}}{\sigma} + \omega\left(\frac{1}{\sigma}\right)\right).$$

Theorem 5. *Let $f \in BC(E)$ and let*

$$A_\sigma(f, E) \leq \omega\left(\frac{1}{\sigma}\right), \quad \sigma \geq 1.$$

Then for $x_1, x_2 \in E$,

$$|f(x_2) - f(x_1)| \leq C_9\, \Omega(\rho_E(x_1, x_2)),$$

where

$$\Omega(\delta) := \delta \left(||f||_{C(E)} + \int_\delta^1 \frac{\omega(t)}{t^2} dt \right), \quad 0 < \delta \leq \frac{1}{2},$$

and $\Omega(\delta) := \Omega(1/2)$ for $\delta > 1/2$.

3. Polynomial approximation of piecewise analytic functions on a compact subset of the real line (see [9]).

Let $E \subset \mathbf{R}$ be a compact set and let $\mathbf{P}_n$ be the set of all (real) polynomials of degree at most $n \in \mathbf{N} := \{1, 2, \ldots\}$. For any continuous function $f : E \to \mathbf{R}$, denote by $\mathcal{E}_n(f, E)$ the error of the best uniform approximation to f on E by polynomials from $\mathbf{P}_n$, i.e.,

$$\mathcal{E}_n(f, E) := \inf_{p \in \mathbf{P}_n} ||f - p||_{C(E)}.$$

The classical theory due to Bernstein (see [18], [20], [21]) states that for any $x_0 \in (-1, 1)$ and $\alpha > 0$, where α is not an even integer, there exists a finite nonzero limit

$$\lim_{n \to \infty} n^\alpha \mathcal{E}_n(|x - x_0|^\alpha, [-1, 1]).$$

A natural question as to what happens to the best polynomial approximations for a general set E and a point $x_0 \in E$ is investigated in monographs [65] and [63] where the reader can also find further references. This section is intended as an attempt to derive general estimates for $\mathcal{E}_n(|x-x_0|^\alpha, E)$ which, in particular, imply the following recent remarkable results by Totik [63, Chapter 10].

Define the density function for E at x_0 as

$$\Theta_E(t,x_0) := |[x_0-t, x_0+t] \setminus E|, \quad t>0.$$

Theorem D. *([63, Theorem 10.1]) If*

$$\int_0^1 \frac{\Theta_E(t,x_0)^2}{t^3} dt < \infty,$$

then for $\alpha > 0$, where α is not an even integer, we have

$$\liminf_{n\to\infty} n^\alpha \mathcal{E}_n(|x-x_0|^\alpha, E) > 0. \tag{12}$$

Conversely, if $0 \le \Theta(t) \le t$ is an increasing function on $[0,1]$ with

$$\int_0^1 \frac{\Theta(t)^2}{t^3} dt = \infty, \tag{13}$$

then there exists a compact set $E \subset [-1,1]$ such that

$$\Theta_E(t,0) \le \Theta(t), \quad 0 < t \le 1, \tag{14}$$

$$\lim_{n\to\infty} n^\alpha \mathcal{E}_n(|x|^\alpha, E) = 0. \tag{15}$$

Theorem E. *([63, Corollary 10.4]) There exists a compact set $E \subset [-1,1]$ with $|E| = 0$ which, for any $\alpha > 0$, where α is not an even integer, satisfies (12) with $x_0 = 0$.*

We consider E as a set in the complex plane $\mathbf{C}$ and use the notions of potential theory in the plane (see [50], [52] for details). Let E be of positive (logarithmic) capacity, i.e., $\mathrm{cap}(E) > 0$ and let $g_{\overline{\mathbf{C}}\setminus E}(z) = g_{\overline{\mathbf{C}}\setminus E}(z,\infty), z \in \overline{\mathbf{C}} \setminus E$ be the Green function of $\overline{\mathbf{C}} \setminus E$ with pole at infinity, where $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ is the extended complex plane. Let for $\delta > 0$ and $x \in E$,

$$E_\delta := \{\zeta \in \mathbf{C} \setminus E : g_{\overline{\mathbf{C}}\setminus E}(z) = \delta\},$$

$$d_\delta(x) := \mathrm{dist}(x, E_\delta) := \inf_{\zeta \in E_\delta} |\zeta - x|.$$

Following [61], we say that E is c-dense at $x_0 \in E$ if

$$\liminf_{t\to 0^+} \frac{\mathrm{cap}(E \cap [x_0-t, x_0+t])}{\mathrm{cap}([x_0-t, x_0+t])} > 0.$$

Theorem 6. *Let E be c-dense at x_0. Then for any $\alpha > 0$, where α is not an integer,*

$$\liminf_{n\to\infty} \frac{\mathcal{E}_n(|x-x_0|^\alpha, E)}{d_{1/n}(x_0)^\alpha} > 0. \tag{16}$$

The exceptional role of integer values of α in Theorem 6 is clear from the example of $E = [0,1]$ and $x_0 = 0$.

Theorem 7. *Let $E \cap [x_0, x_0+1]$ and $E \cap [x_0-1, x_0]$ be c-dense at x_0. Then, for any odd integer $\alpha > 0$, (16) holds.*

Let us mention three consequences of the above theorems. Consider the condition

$$\limsup_{\mathbf{C}\setminus E \ni z \to x_0} \frac{g_{\overline{\mathbf{C}}\setminus E}(z)}{|z-x_0|} < \infty. \tag{17}$$

The geometry of E that satisfies (17) is well-known. We refer the reader to [26], [63], [25], [8] and the many references therein for a comprehensive survey of this subject.

Theorem D and [63, Theorems 2.2 and 3.1] make it conceivable that (12) and (17) are equivalent.

Since (17) yields that $E \cap [x_0, x_0+1]$ and $E \cap [x_0-1, x_0]$ are c-dense at x_0 (see [26, Corollary 1.12] and [6, Theorem 1]) and

$$\liminf_{\delta\to 0^+} \frac{d_\delta(x_0)}{\delta} > 0,$$

Theorems 6 and 7 imply the following partial justification of the above conjecture.

Corollary 1. *For any $\alpha > 0$, where α is not an even integer, (17)$\Rightarrow$(12).*

Corollary 2. *Combining Corollary 1, [26, Corollary 1.12], and [63, Theorem 2.2], we obtain the first part of Theorem D.*

Corollary 3. *Corollary 1, [26, Corollary 1.12], and [7, Theorem 2] yield the existence of a compact set $E \subset [-1,1]$ of a vanishing Hausdorff dimension which for any $\alpha > 0$, where α is not an even integer, satisfies (12) with $x_0 = 0$ (cf. Theorem E).*

Next, we derive some estimates of $\mathcal{E}_n$ from above. According to [63, Theorem 10.5], the only nontrivial unknown case is where E consists of an infinite number of components. Let $E = [a,b] \setminus \cup_{j=1}^{\infty}(a_j, b_j)$, where (a_j, b_j) are mutually disjoint subintervals of (a,b). For any (a_j, b_j) we introduce a broken line l_j which successively joins points

$$a_j, \quad \frac{a_j+b_j}{2} - i\frac{b_j-a_j}{2} \quad \text{and } b_j,$$

and consider a Jordan arc $L := E \cup (\cup_{j=1}^{\infty} l_j)$. Let for $\delta > 0$ and $x \in E$,

$$L_\delta := \{\zeta \in \mathbf{C} : g_{\overline{\mathbf{C}}\setminus L}(z) = \delta\}, \quad \rho_\delta(x) := \operatorname{dist}(x, L_\delta).$$

Theorem 8. *Let $x_0 \in E \cap (a,b)$. For any $\alpha > 0$,*

$$\limsup_{n\to\infty} \frac{\mathcal{E}_n(|x-x_0|^\alpha, E)}{\rho_{1/n}(x_0)^\alpha} < \infty.$$

Corollary 4. *Let E and x_0 be as in Theorem 8. If for some $\alpha > 0$*

$$\limsup_{n\to\infty} n^\alpha \mathcal{E}_n(|x-x_0|^\alpha, E) > 0,$$

then

$$\sum_{j=1}^{\infty} \left(\frac{b_j - a_j}{\frac{b_j+a_j}{2} - x_0} \right)^2 < \infty. \tag{18}$$

Corollary 4 is an immediate consequence of Theorem 8 and the following statement: *Let E and x_0 be as in Theorem 8. The condition*

$$\limsup_{n\to\infty} n\rho_{1/n}(x_0) > 0$$

implies (18).

It is worth pointing out that Corollary 4 is a slightly weaker version of the conjectured implication (12)$\Longrightarrow$(17). We mean that the conditions (17) and (18) are somewhat close which can be seen from the following description of sets satisfying (17).

Theorem 9. *([8, p. 88]) Let E and x_0 be as in Theorem 8. The condition (17) is equivalent to the existence of points $c_j, d_j \in E, -\infty < j < \infty$ such that*

$$a \le c_{-1} < d_{-1} \le c_{-2} < d_{-2} \le \cdots < x_0 < \cdots \le c_1 < d_1 \le c_0 < d_0 \le b,$$

$$[a,b] \setminus E \subset \bigcup_{j=-\infty}^{\infty} (c_j, d_j),$$

$$\inf_{-\infty<j<\infty} \frac{cap(E \cap [c_j, d_j])}{cap([c_j, d_j])} > 0, \tag{19}$$

$$\sum_{j=-\infty}^{\infty} \left(\frac{d_j - c_j}{\frac{d_j+c_j}{2} - x_0} \right)^2 < \infty. \tag{20}$$

Considering the systems of intervals $\{(a_j, b_j)\}$ in Corollary 4 and $\{(c_j, d_j)\}$ in Theorem 9 as special coverings of the complement set $[a,b]\setminus E$, we see that they satisfy the analogous properties (18) and (20), respectively. The difference between them is expressed by (19).

In particular, Corollary 4 implies the second part of Theorem D. Indeed, let the function Θ be as in Theorem D. Following [63, (3.10)], consider the set

$$E = [-1,0] \bigcup \bigcup_{k=1}^{\infty} [2^{-k} + 4^{-k}(\Theta(2^{-k}) - \Theta(2^{-k-1})), 2^{-k+1}]. \tag{21}$$

Then, $\Theta_E(t,0)$ satisfies (13) and (14) (see [63, Section 3.1]) and for the components (a_j, b_j) of $[-1,1] \setminus E$ we have

$$\sum_{j=1}^{\infty} \left(\frac{b_j - a_j}{b_j + a_j} \right)^2 = \infty$$

(see [8, p. 122]). Therefore, by virtue of Corollary 4, we obtain (15).

4. On approximation by entire functions on an unbounded quasismooth curve (see [13]).

The starting point of our discussion is the classical Bernstein theorem which goes back to the beginning of the 20th century (see [62, p. 257]): For $f \in BC(\mathbf{R})$ and $\sigma > 0$,

$$A_\sigma(f, \mathbf{R}) \le C_1\, \omega_f \left(\frac{1}{\sigma}\right), \tag{22}$$

where $C_1 > 0$ is a constant and

$$\omega_f(\delta) := \sup_{\substack{x_1, x_2 \in \mathbf{R} \\ |x_2 - x_1| \le \delta}} |f(x_2) - f(x_1)|, \quad \delta > 0$$

is the modulus of continuity of f on the real line $\mathbf{R}$.

Looking at this result as a common ground of real and complex analysis, it is natural to consider $\mathbf{R}$ as a particular case of an unbounded closed set in $\mathbf{C}$. A comprehensive survey of the related results concerning polynomial approximation on compact sets in $\mathbf{C}$ can be found in monographs [59], [61], [29], [11]. The approximation on unbounded sets reveals new phenomena and requires new techniques to overcome the difficulties. Recently Shirokov [56]-[58] has adapted a technique from polynomial approximation theory to solve some problems concerning the approximation by entire functions.

We extend the Bernstein theorem to the case of functions bounded and continuous on a Jordan curve L with $\infty \in L$ and two basic restrictions on its geometry.

First, we always assume that L is quasismooth in the sense of Lavrentiev (see [48, p. 163]), i.e., that there exists a constant $C_2 = C_2(L) \ge 1$ with the property

$$|L(z_1, z_2)| \le C_2 |z_1 - z_2|, \quad z_1, z_2 \in L, \tag{23}$$

where $L(z_1, z_2)$ is the finite subarc of L between z_1 and z_2.

To introduce the second restriction we need some notation: Let $G_\pm$ be the two connected components of $\mathbf{C} \setminus L$. Let $\mathbf{H} = \mathbf{H}_+ := \{\tau : \Im\tau > 0\}$ and let $\mathbf{H}_- := \mathbf{C} \setminus \overline{\mathbf{H}}_+$. Denote by $\Phi_\pm : G_\pm \to \mathbf{H}_\pm$ a conformal mapping normalized by the condition $\Phi_\pm(\infty) = \infty$. We extend $\Phi_\pm$ continuously up to L and denote by $\Psi_\pm := \Phi_\pm^{-1}$ the inverse mapping. In what follows, we assume that there exists sufficiently large positive constant $C_3 = C_3(L)$ such that

$$\frac{1}{C_3} \le \frac{|\Phi_\pm(z)| + 1}{|z| + 1} \le C_3, \quad z \in \overline{G_\pm}. \tag{24}$$

The geometry of curves L satisfying (23) and (24) is comprehensively studied in [38], [51], [39]. We formulate a sufficient condition which immediately follows from the Ahlfors fundamental inequalities in the form proved in [32]. For more details, we refer the reader to [12, pp. 32–36]. Let $h(r), r \ge 0$, be a positive nonincreasing function satisfying

$$\int_1^\infty \frac{h(r)}{r} dr < \infty.$$

If there exist constants $C_4 > 1$ and $0 \le \theta_0 < \pi$ with the property that

$$\{\zeta \in L : |\zeta| \ge C_4\} \subset \{re^{i\theta} : r \ge C_4, |\theta - \theta_0| \le h(r)\} \bigcup \{re^{i\theta} : r \ge C_4, |\theta - \theta_0 - \pi| \le h(r)\},$$

then L satisfies (24).

The following example shows that the restriction (24) is crucial for the nature of

$$\lim_{\sigma\to\infty} A_\sigma(f, L) = 0, \quad f \in BC(L). \tag{25}$$

Let

$$S_\theta := \{x : x \le 0\} \cup \{xe^{i\theta\pi} : x > 0\}, \quad 0 < \theta < 1,$$

$$f_\theta(z) := \begin{cases} 1, & \text{if } z \in S_\theta, |z| \ge 1, \\ |z|, & \text{if } z \in S_\theta, |z| \le 1. \end{cases}$$

Since $|\Phi_\pm(z)| = |z|^{1/(1\mp\theta)}$, the curve S_θ does not satisfy (24).

We claim that

$$\lim_{\sigma\to\infty} A_\sigma(f_\theta, S_\theta) > 0. \tag{26}$$

To obtain a contradiction, suppose that (25) holds with $f = f_\theta$ and $L = S_\theta$. Let $e_{\sigma,\theta} \in E_\sigma$ satisfy

$$\lim_{\sigma\to\infty} ||f_\theta - e_{\sigma,\theta}||_{C(S_\theta)} = 0.$$

By the Phragmén-Lindelöf theorem [35, p. 25], for any fixed θ the set $\{e_{\sigma,\theta}\}$ is uniformly bounded (i.e., is a normal family) in

$$G_\theta := \{re^{i\phi} : r > 0, \theta < \phi < \pi\}.$$

Therefore, there exists a sequence $e_{\sigma_k,\theta}$, where $\sigma_k \to \infty$ as $k \to \infty$, which converges in G_θ to the bounded analytic function g_θ whose boundary values coincide with f_θ. Since by a uniqueness theorem for analytic functions and the structure of f_θ this is impossible, we have a contradiction which proves (26).

Similar to polynomial approximation on bounded Jordan arcs (see [29, Chapter IX], [11, Chapter 5]), in the case of approximation by entire functions the geometric characteristics of a curve L, the rate of approximation of a function $f \in BC(L)$, and its local smoothness properties interact in a complicated way.

We formulate two direct theorems which can be related to known results for polynomial approximation. The first one represents the "optimal" estimate of the rate of approximation of f in terms of its modulus of continuity

$$\omega_f(\delta) := \sup_{\substack{z_1, z_2 \in L \\ |z_2 - z_1| \le \delta}} |f(z_2) - f(z_1)|, \quad \delta > 0.$$

For $z \in L$ and $\delta > 0$, let

$$L_\delta^\pm := \{\zeta \in G_\pm : \Im\Phi_\pm(\zeta) = \pm\delta\},$$

$$d^{\pm}_{\delta}(z) := d(z, L^{\pm}_{\delta}), \quad d^{*}_{\delta}(z) := \max_{\pm} d^{\pm}_{\delta}(z),$$

where, as before,

$$d(z,S) := \mathrm{dist}(z,S), \quad z \in \mathbf{C},\ S \subset \mathbf{C}.$$

Theorem 10. *Let $f \in BC(L)$. For any $\sigma \geq 1$ there exists $e_\sigma \in E_\sigma$ with the property that*

$$|f(z) - e_\sigma(z)| \leq C_5 \left(\omega_f \left(d^{*}_{1/\sigma}(z) \right) + \frac{||f||_{C(L)}}{\sigma} \right), \quad z \in L, \tag{27}$$

where $C_5 = C_5(L) > 0$.

Since $A_\sigma(f,L) = 0$ for constant functions f, the inequality (27) is trivial in this case. The structure of (27) resembles the estimates of polynomial approximation on bounded Jordan arcs (see [34], [37], [16], [4], [44]).

Since for $z \in L$ and $\delta > 0$,

$$d^{*}_{\delta}(z) \geq C_6 \delta, \quad C_6 = C_6(L) > 0$$

the first term in the right-hand side of (27) is the main one.

The next theorem shows that the first term in the right-hand side of (27) cannot be improved, even locally. For $z \in \mathbf{C}$ and $\delta > 0$, let

$$D(z,\delta) := \{\zeta : |\zeta - z| < \delta\}.$$

Theorem 11. *Let L satisfy*

$$\sup_{z \in L} d^{*}_{1}(z) \leq C_7 = C_7(L). \tag{28}$$

Then for any $\sigma \geq 1$ there exists $f^\sigma \in BC(L)$ satisfying the following properties:

(i) $||f^\sigma||_{C(L)} \leq 1$;

(ii) $\omega_{f^\sigma}(\delta) \leq \delta, \delta > 0$;

(iii) For any $C > 0$ there exists a constant $C_8 = C_8(C,L) > 0$ such that for any $z \in L$ and $e_\sigma \in E_\sigma$ with

$$|f^\sigma(\zeta) - e_\sigma(\zeta)| \leq C\, d^{*}_{1/\sigma}(\zeta), \quad \zeta \in L$$

we have

$$\sup_{\zeta \in L \cap D(z, d^{*}_{1/\sigma}(z))} \frac{|f^\sigma(\zeta) - e_\sigma(\zeta)|}{d^{*}_{1/\sigma}(\zeta)} \geq C_8.$$

Note that (28) yields that there exist positive constants $\varepsilon = \varepsilon(L)$ and $C_9 = C_9(L)$ such that for $z \in L$ and $0 < \delta < 1$

$$d^{*}_{\delta}(z) \leq C_9 \delta^{\varepsilon}.$$

Not dwelling in more detail on the geometric nature of (28), we only mention the following simple sufficient condition for its validity: If there exist constants $0 \leq \theta_0 < \pi$ and $C_{10} > 0$ with the property that

$$L \subset \{z : |\Im z e^{-i\theta_0}| \leq C_{10}\},$$

then L satisfies (28).

The second direct theorem concerns the "optimal" estimate of $A_\sigma(f,L)$. Let $f_\pm(x) := f\circ\Psi_\pm(x), x\in\mathbf{R}$ and let

$$\mu_f(\delta) := \sum_{\pm}\omega_{f_\pm}(\delta), \quad \delta>0.$$

Theorem 12. *For any nonconstant function $f\in BC(L)$ and $\sigma\geq 1$ the inequalities*

$$\begin{aligned} A_\sigma(f,L) &\leq C_{11}\left(\mu_f\left(\frac{1}{\sigma}\right)+\frac{||f||_{C(L)}}{\sigma}\right)\\ &\leq C_{11}\left(1+2\frac{||f||_{C(L)}}{\mu_f(1)}\right)\mu_f\left(\frac{1}{\sigma}\right), \end{aligned} \tag{29}$$

hold with $C_{11}=C_{11}(L)>0$.

For $L=\mathbf{R}$ (29) essentially coincides with (22). The structure of the right-hand side of (29) resembles the estimates of the best polynomial approximation on Faber sets (cf. [36], [30], [28], [46]) and the best rational approximation on bounded quasiconformal curves (cf. [15]). It turns out that the estimate (29) is of right order also for curves with corners. To formulate the corresponding result we introduce some notation.

A bounded Jordan curve is called Dini-smooth (cf. [48, p. 48]) if it is smooth and if the angle $\gamma(s)$ of the tangent, considered in terms of the arclength s, satisfies

$$|\gamma(s_2)-\gamma(s_1)|<g(s_2-s_1), \quad s_2>s_1,$$

where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x}dx<\infty.$$

We call a Jordan arc Dini-smooth if it is a subarc of some Dini-smooth curve.

Let $L=L_\alpha, 0<\alpha<1$ be such that there exist a point $z_0\in L$ and sufficiently small constant $\varepsilon>0$ satisfying the following property: $L\cap D(z_0,2\varepsilon)$ consists of two Dini-smooth arcs joining at z_0 under the angle $\alpha\pi$ with respect to G_+. We choose β with $0<\beta<1$ and $\beta/\alpha\notin\mathbf{N}$. We introduce the function $(z-z_0)^{\beta/\alpha}$ by fixing a branch of the power function which is analytic in G_-. For $z\in L$ we consider the function $f(z)=f_\beta(z)$ which coincides with $(z-z_0)^{\beta/\alpha}$ on $L\cap D(z_0,\varepsilon)$ and is extended to L continuously by two constants.

According to the metric properties of a conformal mapping of the domain with a piecewise Dini-smooth boundary (see [48, Chapter 3]), we have for $0<\delta<\delta_0(L,\beta,\varepsilon)$

$$\frac{1}{C_{12}}\delta^\beta\leq\omega_{f_+}(\delta)\leq C_{12}\delta^\beta, \tag{30}$$

$$\omega_{f_-}(\delta)\leq C_{13}\delta^\gamma, \tag{31}$$

where $C_j=C_j(L,\beta,\varepsilon)>1, j=12,13$ and $\gamma:=\min(1,\beta/\alpha)>\beta$.

Theorem 13. *Under the above assumptions we have*

$$A_\sigma(f,L) \geq C_{14}\sigma^{-\beta}, \quad \sigma \geq 1, \tag{32}$$

where $C_{14} = C_{14}(L,\beta,\varepsilon) > 0$.

Comparing (29) with (30)-(32) we see that for the locally piecewise smooth curve L the estimate of $A_\sigma(f,L)$ given in Theorem 12 cannot be improved.

In the proofs of Theorems 10 and 12 we use the same construction of the approximating entire functions which can be described as follows. Let $\Omega_\pm \subset \mathbf{C}$ be domains bounded by the quasismooth curves $\Gamma^\pm(\ni \infty)$. Denote by $\phi_\pm : G_\pm \to \Omega_\pm$ a conformal mapping satisfying $\phi_\pm(\infty) = \infty$. By the same symbol $\phi_\pm$ we denote the continuous extension of $\phi_\pm$ to L. Let $\psi_\pm := \phi_\pm^{-1}$ and let for $w \in \mathbf{C}$ and $\delta > 0$

$$\Gamma_\delta^\pm := \{\xi \in \Omega_\pm : \Im\Phi_\pm \circ \psi_\pm(\xi) = \pm\delta\},$$

$$\rho_\delta^\pm(w) := d(w, \Gamma_\delta^\pm).$$

For $f \in BC(L)$ we consider the functions

$$f^\pm(w) := f \circ \psi_\pm(w), \quad w \in \Gamma^\pm.$$

Let

$$\omega_{f^\pm}(\delta) := \sup_{\substack{w_1,w_2\in\Gamma^\pm \\ |w_2-w_1|\leq\delta}} |f^\pm(w_2) - f^\pm(w_1)|, \quad \delta > 0$$

be the modulus of continuity of $f^\pm$ on $\Gamma^\pm$ and let

$$\mu_f(z,\delta) := \sum_\pm \omega_{f^\pm}(\rho_\delta^\pm(\phi_\pm(z))), \quad z \in L, \delta > 0.$$

Theorem 14. *Let $f \in BC(L)$ and let either $\Omega_\pm = G_\pm$ or $\Omega_\pm = \mathbf{H}_\pm$. For any $\sigma \geq 1$ there exists $e_\sigma \in E_\sigma$ such that*

$$|f(z) - e_\sigma(z)| \leq C_{15}\left(\mu_f\left(z, \frac{1}{\sigma}\right) + \frac{||f||_{C(L)}}{\sigma}\right), \quad z \in L,$$

where $C_{15} = C_{15}(L) > 0$.

Let $\Omega_\pm = G_\pm$ in Theorem 14, i.e., $\Gamma^\pm = L$, then we obtain Theorem 10. For $\Omega_\pm = \mathbf{H}_\pm$ in Theorem 14, i.e., $\Gamma^\pm = \mathbf{R}$, we obtain the first inequality in (29). It is well-known that for $t > 1$ and $\delta > 0$ the module of continuity satisfies

$$\omega_{f_\pm}(t\delta) \leq (t+1)\omega_{f_\pm}(\delta)$$

and, consequently,

$$\mu_f(1) \leq 2\sigma\mu\left(\frac{1}{\sigma}\right), \quad \sigma \geq 1,$$

which implies the second inequality in (29).

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