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On Hardy type inequality in variable exponent Lebesgue space $L^{p(\cdot)}(0,l)$

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Abstract. We study a variable exponent Hardy type inequality

$$\left\|x^{-1}Hf\right\|_{L^{p(\cdot)}(0,l)} \leq C\left\|f\right\|_{L^{p(\cdot)}(0,l)}; f \geq 0.$$

in the norms of variable exponent Lebesgue spaces $L^{p(\cdot)}(0,l)$. In terms of regularity conditions for $p:(0,l)\to(1,\infty)$, we derive necessary and sufficient conditions for this inequality to hold for all $f \in L^{p(\cdot)}(0,l)$.

Key Words and Phrases: Hardy operator, Hardy type inequality, variable exponent, weighted inequality
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1. Introduction

In this paper, we study a variable exponent Hardy type inequality

$$\left\|x^{-1}Hf\right\|_{L^{p(\cdot)}(0,l)} \leq C\left\|f\right\|_{L^{p(\cdot)}(0,l)} \quad ; f \geq 0, \tag{1}$$

where $Hf(x)=\int\limits_0^x f(t)dt$, $l>0$, the constant $C>0$ depends on l and the function p. This topic was a subject of recent works [2], [4], [5], [8], [9], [10], [11],[12], [13], [14], [15]. According to those works (see, e.g. [4], [9], [10], [13]), the condition

$$A := \limsup_{x\to 0}\left|p(x)-p(0)\right|\ln\frac{1}{x} < \infty, \tag{2}$$

is sufficient for the inequality (1) to hold if the $p:(0;l)\to[1;\infty)$ is a measurable function on $(0,l)$ and is separate from zero and infinity. In Theorem 1, we prove some extension of this result, where the condition

$$B := \limsup_{x\to 0}\left|p(x)-p\left(\frac{x}{2}\right)\right|\ln\frac{1}{x} < \infty, \tag{3}$$

is assumed for the exponent function.

In [9], an example of exponent function p was constructed such that the inequality (1) and the condition (2) are violated for that simultaneously. Though, that example indicates essentiality of condition (2) for the exponent p, it does not proves, in general, its necessity. In this paper, we prove that the condition (3) is necessary if the function p increases near the origin (see, Theorem 2).

It is not difficult to show that no condition of logariphmic type is needed for the exponent when the exponent p is non increasing near the origin.

From the trivial inequality

$$\left|p(x)-p\left(\frac{x}{2}\right)\right|\ln\frac{1}{x}\leq|p(x)-p\left(0\right)|\ln\frac{1}{x}+\left|p\left(\frac{x}{2}\right)-p\left(0\right)\right|\ln\frac{1}{x},$$

it follows that the condition (3) is weaker (2). For example, the function

$$p\left(x\right)=p\left(0\right)+\frac{C_1}{\sqrt{\ln\frac{l}{2x}}},$$

satisfies to condition (3) but does not satisfy to the condition (2).

In our results, we prove sufficiency of condition (3) for sufficiently large values of $p\left(0\right)$. In other words, we assert that the condition (2) is necessary and sufficient if the value $p\left(0\right)$ is sufficiently large (or equivalently, B is sufficiently small). One can suppose about the case if the condition (2) satisfied but (3) does not. I connection, we claim the following assertion that prevents this case in some sense.

Proposition 1. Let $A<\infty$ and the limit $b:=\lim\limits_{x\to0}\left|p\left(x\right)-p\left(\frac{x}{2}\right)\right|\ln\frac{1}{x}$ exists, then $b=0$.

Proof. Let $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty}\left[\,p(\frac{x_n}{2})-p(0)\right]\ln\frac{2}{x_n}=A$. Tending $n\to\infty$ in the identity

$$\left[p(x_n)-p\left(0\right)\right]\ln\frac{1}{x_n}=\left[p(x_n)-p\left(\frac{x_n}{2}\right)\right]\ln\frac{1}{x_n}$$

$$+\left[p\left(\frac{x_n}{2}\right)-p\left(0\right)\right]\left(\ln\frac{2}{x_n}\right)\left(\frac{\ln\frac{1}{x_n}}{\ln\frac{2}{x_n}}\right),$$

we infer $A=A+\lim\limits_{n\to\infty}\left[p(x_n)-p\left(\frac{x_n}{2}\right)\right]\ln\frac{1}{x_n}$, i.e. $\lim\limits_{n\to\infty}\left[p(x_n)-p\left(\frac{x_n}{2}\right)\right]\ln\frac{1}{x_n}=0$. Now using the existence of limit $\lim\limits_{x\to0}\left|p\left(x\right)-p\left(\frac{x}{2}\right)\right|\ln\frac{1}{x}$, we find $b=0$.

Also in the Theorems 3 and 4 we prove other necessity conditions for case of more general exponents p. In Theorem 5, we obtain some necessary and sufficient condition on boundedness of Hardy's operator. Note, the necessity of condition $p(0)>1$ was considered in [9].

We refer to [3] and references therein for full description of variable exponent Lebesgue spaces and boundedness of classical integral operators there. Also there arise a new extension or refinement for such operators in different function spaces (see e.g. [16], [7]).

2. Notations

As to the basic properties of spaces $L^{p(.)}$, we refer to [6]. Throughout this paper, it is assumed that $p(x)$ is a measurable function in $(0,l)$, taking its values from the interval $[1,\infty)$ with $p^{+} = \sup\{p(x) : x \in (0,l)\} < \infty$. The space of functions $L^{p(.)}(0,l)$ is introduced as the class of measurable functions $f(x)$ in $(0,l)$ which have a finite $I_{p(.)}(f) = \int\limits_0^l |f|^{p(x)}\,dx$ modular. A norm in $L^{p(.)}(0,l)$ is given in the form

$$\|f\| = \left\{\lambda > 0 : I_{p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

For $1 < p^{-}$, $p^{+} < \infty$ the space $L^{p(.)}(0,l)$ is a reflexive Banach space.

Denote by $\Lambda(B)$ a class of measurable functions $f : (0,l) \to \mathbb{R}$ satisfying the condition (3). For the function $1 < p(x) < \infty$ $p\prime(x)$ denotes the conjugate function of $p(x)$, $\frac{1}{p(x)} + \frac{1}{p\prime(x)} = 1$. We denote by $C, C_1, C_2, ...$ various positive constants whose values may vary at each appearance. $B(x,r)$ denotes a one-dimensional ball with center at x and radius $r > 0$, i.e. $B(x,r) = (x-r, x+r)$. We write $u \sim v$ if there exist positive constants C_1, C_2 such that $C_1 u(x) \le v(x) \le C_1 u(x)$. By χ_E we denote the characteristic function of the set E.

3. Main Results

In this paper, following main results are obtained.

Theorem 1. *Let the function $p(x)$ be nondecreasing on some little neighborhood of zero and measurable on $(0,l)$ and such that $1 < p^{-} \le p(x) \le p^{+} < \infty$. Then for the inequality (1) to hold it is sufficient that $p \in \Lambda(B)$ with*

$$B < p(0)(p(0)-1). \tag{4}$$

Theorem 2. *Let $1 < p^{-} \le p(x) \le p^{+} < \infty$ be a nondecreasing function on $(0,l)$. Then for the inequality (1) to hold it is necessary that $p \in \Lambda(B)$ by some $B \ne \infty$.*

Also the following two theorems take place on necessity condition for the inequality (1).

Theorem 3. *Let* $1 < p^- \leq p(x) \leq p^+ < \infty$ *be a nondecreasing function on* $(0,l)$. *Then for the inequality* (1) *to hold it is necessary that*

$$\left\| x^{-1} \right\|_{p(.);(a,l)} \leq Ca^{-\frac{1}{p'(a)}};\ a \in (0,l). \tag{5}$$

Theorem 4. *Let* $1 < p^- \leq p(x) \leq p^+ < \infty$ *be a measurable function on* $(0,l)$. *Then for the inequality* (1) *to hold it is necessary that*

$$\left\| x^{-1} \right\|_{p(.);(a,l)} \leq C_1 a^{-\frac{1}{\overline{p}'_a}};\ a \in (0,l), \tag{6}$$

where $\frac{1}{\overline{p}'_a} := \frac{1}{a} \int\limits_0^a \frac{dy}{p'(y)}$.

Theorem 5. *Let* $1 < p^- \leq p(x) \leq p^+ < \infty$ *be a nondecreasing function on* $(0,l)$. *Then for the inequality*

$$\left\| x^{-1} Hf \right\|_{L^{\frac{p(.)}{\varepsilon}}(0,l)} \leq C \left\| f \right\|_{L^{\frac{p(.)}{\varepsilon}}(0,l)} \ ;\ f \geq 0,$$

to hold by some $\varepsilon \in (0,1)$ *it is necessary and sufficient that* $p \in \Lambda(B)$ *by some* $B \neq \infty$.

4. Proof of Main Results

Proof of Theorem 1.

Let $f(x) \geq 0$ be a measurable function such that $\|f\|_{L^{p(\cdot)}(0,l)} \leq 1$. Then

$$I_{p(.)}\left(f\right) \leq 1. \tag{7}$$

In order to prove Theorem 1 we have to prove

$$\left\| x^{-1} Hf \right\|_{L^{p(\cdot)}(0,l)} \leq C_1.$$

This inequality follows from the inequality

$$I_{p(.)}\left(x^{-1} Hf\right) \leq C_2. \tag{8}$$

By Minkowski inequality, for $L^{p(.)}$ norms, we get

$$\left\|x^{-1}Hf\right\|_{L^{p(\cdot)}(0,l)} \leq \left\|x^{-1}Hf\right\|_{L^{p(\cdot)}(0,\delta)} + \left\|x^{-1}Hf\right\|_{L^{p(\cdot)}(\delta,l)}$$
$$:= i_1 + i_2,$$

where δ is such that the condition (4) provides a similar condition in $(0,\delta)$:

$$B < p(x)\,(p(x)-1)\,.$$

The estimate near zero (i_1)***.***

By triangle property of $p(.)$ norms, we have the inequalities

$$i_1 \leq \left\| x^{-\frac{1}{p(x)}-\frac{1}{p'(x)}} \sum_{n=0}^{\infty} \int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)dt \right\|_{L^{p(\cdot)}(0,\delta)}$$

$$\leq \sum_{n=0}^{\infty} \left\| x^{-\frac{1}{p(x)}-\frac{1}{p'(x)}} \int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)dt \right\|_{L^{p(\cdot)}(0,\delta)} . \tag{9}$$

Denote $B_{x,n} = (2^{-n-1}x, 2^{-n}x]$ and $p_{x,n} = \inf\{p(t) : t \in B_{x,n}\}; n = 1, 2, ...$ Put $\varphi(t) = t^{\frac{1}{p(t)}}$. Since the condition (4) holds, it is not difficult to show that the function $\varphi(t)$ satisfies ∇_2 condition: $\exists 0 < \eta < 1, \varphi(\eta s) < \frac{1}{2}\varphi(s), s \in (0,\delta)$. Therefore [1], there exists an $\alpha = \alpha(\delta) \in (0,1)$ such that

$$\frac{\varphi(s)}{s^{\alpha}} \leq C\frac{\varphi(r)}{r^{\alpha}},\ 0 < s < r < \delta. \tag{10}$$

Then by (10) we have

$$\frac{\varphi(t)}{t^{\alpha}} \leq C\frac{\varphi(x)}{x^{\alpha}}, \tag{11}$$

where t is a point in $B_{x,n}, 0 < x < \delta$ and the constant C does not depend on n.

By using inequality (11) and $2^{-n-1}x < t < 2^{-n}x$ we have the estimates

$$t^{\frac{1}{p(t)}} = t^{\alpha} t^{\frac{1}{p(t)}-\alpha} \leq Ct^{\alpha} x^{\frac{1}{p(x)}-\alpha} \leq C2^{-n\alpha} x^{\frac{1}{p(x)}}.$$

Hence

$$x^{-\frac{1}{p(x)}} \leq C2^{-n\alpha} t^{-\frac{1}{p(t)}}$$

Therefore, and due to Holder's inequality, for $x \in B(0,\delta)$, we get

$$x^{-\frac{1}{p(x)}-\frac{1}{p'(x)}} \sum_{n=0}^{\infty} \int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)dt$$

$$\leq C2^{-n\alpha}x^{-\frac{1}{p(x)}}t^{-\frac{1}{p(t)}}\int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)dt$$

$$\leq C2^{-n\alpha}x^{-\frac{1}{p(x)}}t^{-\frac{1}{p(t)}}\left(\int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}^{-}}dt\right)^{\frac{1}{p_{x,n}^{-}}}\left(2^{-n}x\right)^{\frac{1}{\left(p_{x,n}^{-}\right)'}}. \tag{12}$$

It follows from the condition $p \in \Lambda(B)$ that

$$\left(2^{-n}x\right)^{\frac{1}{\left(p_{x,n}^{-}\right)'}} \leq 2^{-\frac{1}{\left(p_{x,n}^{-}\right)'}}t^{\frac{1}{\left(p_{x,n}^{-}\right)'}} \leq C_1 t^{\frac{1}{p'(t)}}, \tag{13}$$

where C depends only p, δ.

Let us demonstrate the details of (13) . There exists a point $y \in B_{x,n}$ such that $p_{x,n}^{-} \sim p(y)$. Obviously, the point y depends on x, n. Hence $t^{\frac{1}{\left(p_{x,n}^{-}\right)'}} \leq t^{\frac{1}{p'(y)}}$. By virtue of $2^{-n-1}x < y < 2^{-n}x$ we have $\frac{t}{2} < y < 2t$. Therefore, and by virtue of the condition (3), $t^{\frac{1}{p'(y)}} \sim t^{\frac{1}{p'(t)}}$.

Combining (12) and (13) we get

$$x^{-\frac{1}{p(x)}-\frac{1}{p'(x)}}\sum_{n=0}^{\infty}\int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)dt \leq C2^{-n\alpha}x^{-\frac{1}{p(x)}}\left(\int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}^{-}}dt\right)^{\frac{1}{p_{x,n}^{-}}}, \tag{14}$$

where $0 < x < \delta$, $n = 1, 2, ...$and the constant C_2 does not depend on n, x.

Simultaneously

$$\int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}^{-}}dt \leq \int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)^{p(t)}\chi_{\{f(t)\geq 1\}}dt + \int\limits_{2^{-n-1}x}^{2^{-n}x} dt \leq 1 + 2^{-n}\delta \leq C_3.$$

By the last inequality and (14), we have

$$I_{p(.);(0,\delta)}\left(x^{-\frac{1}{p(x)}-\frac{1}{p'(x)}}\int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)dt\right) \leq C_4 2^{-n\alpha p^{-}}\int\limits_{0}^{\delta} x^{-1}\left(\int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}^{-}}dt\right)^{\frac{p(x)}{p_{x,n}^{-}}}dx$$

$$\leq C_4 C_3^{\frac{p^{+}}{p^{-}}-1}2^{-n\alpha p^{-}}\int\limits_{0}^{\delta} x^{-1}\left(\int\limits_{2^{-n-1}x}^{2^{-n}x}\left(f(t)^{p(t)}+1\right)dt\right)dx,$$

which, due to Fubini's theorem, yields

$$\le C_4 C_3^{\frac{p^+}{p^-}-1} 2^{-n\alpha p^-} \ln 2 \int\limits_0^{2^{-n}\delta} \left(\int\limits_{2^{-n-1}x}^{2^{-n}x} x^{-1} dx \right) \left(f(t)^{p(t)} + 1 \right) dt$$

$$= C_5 2^{-n\alpha p^-} \ln 2 \int\limits_0^{2^{-n}\delta} \left(f(t)^{p(t)} + 1 \right) dt \le C_6 2^{-n\alpha p^-}. \tag{15}$$

Therefore

$$\left\| x^{-\frac{1}{p(x)} - \frac{1}{p'(x)}} \int\limits_{2^{-n-1}x}^{2^{-n}x} f(t)dt \right\|_{L^{p(\cdot)}(0,\delta)} \le C 2^{-\frac{n\alpha p^-}{p^+}}$$

By (15) and (9), we get

$$i_1 \le C \sum_{n=0}^{\infty} 2^{-\frac{n\alpha p^-}{p^+}} \le C_1. \tag{16}$$

The estimate of $\boldsymbol{i_2}$. We have

$$i_2 \le \left\| x^{-1} H f \right\|_{L^{p(\cdot)}(\delta,l)}$$

$$\le \left(\int\limits_0^l f(t)dt \right) \left\| x^{-1} \right\|_{L^{p(\cdot)}(\delta,l)}, \tag{17}$$

from which by virtue of Holder's inequality, for $p(x)$-norms by the condition (7) we obtain

$$\int\limits_0^l f(t)dt \le \|f\|_{L^{p(\cdot)}(0,l)} \|1\|_{L^{p'(\cdot)}(0,l)} = C_1.$$

From (17) and the last inequality we infer

$$i_2 \le C. \tag{18}$$

Now using (18) and (16) from (9) derive the estimate (8).

This completes the proof of Theorem 1

Proof of Theorems 2, 3 and 4.

Let $a \in (0,l)$ be a fixed number. Put $f(x) = a^{-\frac{1}{p(x)}}\ \chi_{(0,a)}(x)$ in the inequality (1). We have $I_{p(.);(0,l)}\,(f) = 1$. Then $\|f\|_{p(.);(0,l)} \leq 1$. It follows from the inequality (1) that $I_{p(.);(0,l)}\left(\frac{1}{x}Hf\right) \leq C$. Therefore

$$\int\limits_a^1 x^{-p(x)} \left(\int\limits_0^a a^{-\frac{1}{p(y)}}dy\right)^{p(x)} dx \leq C. \tag{19}$$

Using Young's inequality, we have

$$a^{1-\frac{1}{a}\int\limits_0^a \frac{dy}{p(y)}} \leq \int\limits_0^a a^{-\frac{1}{p(y)}}dy.$$

Now, using this inequality from (19) we infer

$$\int\limits_a^1 \left(\frac{a}{x}\right)^{p(x)} a^{-\frac{p(x)}{a}\int\limits_0^a \frac{dy}{p(y)}} dx \leq C. \tag{20}$$

Denote $\frac{1}{\overline{p}_a} = \frac{1}{a}\int\limits_0^a \frac{dy}{p(y)}$ then $\frac{1}{\overline{p}'_a} = \frac{1}{a}\int\limits_0^a \frac{dy}{p'(y)}$. It follows from the inequality (20) that

$$\int\limits_a^1 \left(\frac{x^{-1}}{a^{-\frac{1}{\overline{p}'_a}}}\right)^{p(x)} dx \leq C. \tag{21}$$

According to (21):

$$\left\|x^{-1}\right\|_{p(.);(a,l)} \leq C_1 a^{-\frac{1}{\overline{p}'_a}},$$

or

$$\left\|x^{-1}\right\|_{p(.);(a,l)} \leq C_1 a^{-\frac{1}{a}\int\limits_0^a \frac{dy}{p'(y)}}. \tag{22}$$

This proves Theorem 4.

Since p is monotone, we have $\frac{1}{a}\int\limits_0^a \frac{dy}{p'(y)} \leq \frac{1}{p'(a)}$. Then from (22) we find

$$\left\|x^{-1}\right\|_{p(.);(a,l)} \leq C_1 a^{-\frac{1}{p'(a)}}. \tag{23}$$

This proves Theorem 3.

We conjecture that the condition (23) is a really necessary condition. But unfortunately, we can not prove its sufficiency.

Because we shall deduce the condition (3) from (23). In this way, we have

$$\left\|x^{-1}\right\|_{p(.);(a,l)} \geq \left\|x^{-1}\right\|_{p(.);(2a,3a)} \geq \frac{1}{3a}\left\|1\right\|_{p(.);(2a,3a)}. \tag{24}$$

Using increment of the function p, we have

$$1 \geq \int_{2a}^{3a} \left(\frac{1}{\|1\|_{p(.);(2a,3a)}}\right)^{p(x)} dx \geq \int_{2a}^{3a} \left(\frac{1}{\|1\|_{p(.);(2a,3a)}}\right)^{p(2a)} dx \geq$$

$$\geq a\left(\frac{1}{\|1\|_{p(.);(2a,3a)}}\right)^{p(2a)}.$$

This means that

$$\|1\|_{p(.);(2a,3a)} \geq a^{\frac{1}{p(2a)}}. \tag{25}$$

From (25), (24) and (23) we find

$$\left(\frac{1}{a}\right)^{\frac{1}{p(a)}-\frac{1}{p(2a)}} \leq C_1,$$

or

$$\left[\frac{1}{p(a)} - \frac{1}{p(2a)}\right] \ln\frac{1}{a} \leq C_2.$$

This implies the condition (3) by the constant $B = p(0)^2 CC_1$, where C is the constant in the inequality (1), C_1 does nor depends on the p.

This completes the proof of Theorem 2.

Proof of Theorem 5.

Sufficiency. Put $p_1(x) = \frac{p(x)}{\varepsilon}$. The condition $p \in \Lambda(B)$ imply $p_1 \in \Lambda(\frac{B}{\varepsilon})$ and the condition (4) for p_1 implies $0 < \varepsilon < \frac{p(0)^2}{B+p(0)}$. Choosing ε such and applying Theorem 1 with the exponent p_1, we finish the sufficiency part.

Necessity. Let the inequality (1) holds by some $\varepsilon \in (0,1)$ and the exponent $p_1(x) = \frac{p(x)}{\varepsilon}$. Then, using Theorem 2, we find $p_1 \in \Lambda(B)$, where B_1 depends on p, ε.

This completes the proof of Theorem 5.

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