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# On Retro Banach Frames of Type $P$

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**Abstract.** Retro Banach frames of type $P$ in Banach spaces have been introduced and studied. Necessary and sufficient conditions for existence of retro Banach frames of type $P$ are obtained. A characterization of retro Banach frames obtained from translation of retro Banach frames of type $P$ is given. Retro Banach frames of type $P$ in product spaces with suitable norm are discussed.

**Key Words and Phrases**: Frames; Banach frames; retro Banach frames

**2010 Mathematics Subject Classifications**: 42C15; 42C30

## 1. Introduction

In 1946, D. Gabor [9] introduced a fundamental approach to signal decomposition in terms of elementary signals. Duffin and Schaeffer in [6] in 1952, while addressing some deep problems in non-harmonic Fourier series, abstracted Gabor's method to define frames for Hilbert spaces. Later, in 1986, Daubechies, Grossmann and Meyer [5] found new applications to wavelets and Gabor transforms in which frames played an important role.

Today, frames play important roles in many applications in mathematics, science and engineering. In particular frames are widely used in sampling theory, wavelet theory, wireless communication, signal processing, image processing, differential equations, filter banks, geophysics, quantum computing, wireless sensor network, multiple-antenna code design and many more. Reason is that frames provides both great liberties in the design of vector space decompositions, as well as quantitative measure on the computability and robustness of the corresponding reconstructions. In the theoretical direction, powerful tools from operator theory and Banach spaces are being employed to study frames. For a nice and comprehensive survey on various types of frames, one may refer to [1] and the references therein.

 

Coifman and Weiss [4] introduced the notion of atomic decomposition for function spaces. Later, Feichtinger and Gröchenig [8] extended this idea to Banach spaces. This concept was further generalized by Gröchenig [10] who introduced the notion of Banach frames for Banach spaces. Casazza, Han and Larson [2] also carried out a study of atomic decompositions and Banach frames. Recently, various generalization of frames in Banach spaces have been introduced and studied. One of such notion namely, retro Banach frames in Banach spaces introduced and studied in [13]. Retro Banach frames were further studied in [14, 15, 16].

In this paper retro Banach frames of type $P$ in Banach spaces have been introduced and studied. Example and counter example regarding existence of retro Banach frames of type $P$ are given. Necessary and sufficient conditions for existence of retro Banach frames of type $P$ are obtained. A characterization of a retro Banach frames obtained from translation of retro Banach frames of type $P$ is given. Further we discuss retro Banach frames of type $P$ in product spaces with suitable norm.

## 2. Preliminaries

Throughout this paper $E$ will be denoted an infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $E^*$ the conjugate space of $E$. For a sequence $\{x_n\} \subset E$, $[x_n]$ denotes the closure of linear span of $\{x_n\}$ in the norm topology of $E$.

**Definition 1** ([13]). *Let $E$ be a Banach space and $E^*$ its conjugate space. Let $(E^*)_d$ be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a* retro Banach frame *for $E^*$ with respect to $(E^*)_d$ if:*

*(i) $\{f(x_n)\} \in (E^*)_d$, for all $f \in E^*$,*

*(ii) there exist positive constants $A$ and $B$ with $0 < A \leq B < \infty$ such that*

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \quad \text{for all } \ f \in E^*, \tag{1}$$

*(iii) $T$ is a bounded linear operator such that $T(\{f(x_n)\}) = f$, for all $f \in E^*$.*

*The positive constants $A$ and $B$ are called* lower *and* upper retro frame bounds *of the retro Banach frame $(\{x_n\}, T)$, respectively. They are not unique. The operator $T : (E^*)_d \to E^*$ is called the* reconstruction operator *(or the* pre-frame operator*) and the inequality (2.1) is called the* retro frame inequality*.*

*The retro Banach frame $(\{x_n\}, T)$ is called* tight *if $A = B$ and* normalized tight *if $A = B = 1$. If removal of one $x_j$ render the collection $\{x_n\}_{n\neq j}$ no longer a retro Banach frame for $E^*$, then $(\{x_n\}, T)$ is called an* exact retro Banach frame*.*

**Lemma 1.** *Let $E$ be a Banach space and $\{f_n\} \subset E^*$ be a sequence such that $\{x \in E : f_n(x) = 0$, for all $\in \mathbb{N}\} = \{0\}$. Then $E$ is linearly isometric to the Banach space $X = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_X = \|x\|_E$, $x \in E$.*

**Lemma 2** ([13])**.** *Let $(\{x_n\}, T)$ be a retro Banach frame for $E^*$. Then $(\{x_n\}, T)$ is exact if and only if $x_n \notin [x_i]_{i \neq n}$ for all $n \in \mathbb{N}$.*

*Proof.* Suppose that $(\{x_n\}, T)$ is exact. Fix $n \in \mathbb{N}$. Then there exists no reconstruction operator $T_0$ such that $(\{x_i\}_{i \neq n}, T_0)$ is a retro Banach frame for $E^*$. Therefore, by using Lemma 1, $[x_i]_{i \neq n} \neq E$. Hence $x_n \notin [x_i]_{i \neq n}$ for all $n \in \mathbb{N}$.

Conversely, let $x_n \notin [x_i]_{i \neq n}$ for all $n \in \mathbb{N}$ and let $(\{x_n\}, T)$ be not exact. Then, there exists a positive integer $m_0$ and a reconstruction operator $T_0$ such that $(\{x_i\}_{i \neq m_0}, T_0)$ is a retro Banach frame for $E^*$. Thus, by using retro frame inequality for $(\{x_i\}_{i \neq m_0}, T_0)$, we obtain $[x_i]_{i \neq m_0} = E$. This gives $x_{m_0} \in [x_i]_{i \neq m_0}$, a contradiction.◄

**Remark 1.** *Let $(\{x_n\}, T)$ be an exact retro Banach frame for $E^*$. Then there exists a sequence $\{f_n\} \subset E^*$, called the* admissible sequence *to $(\{x_n\}, T)$, such that*

$$f_n(x_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \text{ for all } n, m \in \mathbb{N}. \end{cases}$$

**Theorem 1** ([7], p. 609)**.** *Let $T$ be a compact operator in a complex Banach space $X$, and let $\lambda$ be a fixed non-zero complex number. Then, the non-homogenous equations*

$$(\lambda I - T)x = y, \tag{2}$$

$$(\lambda I - T^*)f = g, \tag{3}$$

*have a unique solution for any $y \in X$ or $g \in X^*$ if and only if each of the homogenous equations*

$$(\lambda I - T)x = 0, \tag{4}$$

$$(\lambda I - T^*)f = 0, \tag{5}$$

*has zero as the solution. Furthermore, if one of the homogenous equations has a non-zero solution, then they both have the same finite number of linearly independent solutions. In this case the equation (2.2) and (2.3) have solutions if and only if $y$ and $g$ are orthogonal to all the solutions of (2.4) and (2.5), respectively. Moreover, the general solution for (2.2) is found by adding a particular solution of (2.2) to the general solution of (2.4).*

## 3. Retro Banach frames of Type $P$

**Definition 2.** *A retro Banach frame* $(\{x_n\}, T)$ *for* $E^*$ *is said to be of* type $P$ *if it is exact and there exists a functional* $\Psi \in E^*$ *such that* $\Psi(x_n) = 1$, *for all* $n \in \mathbb{N}$.

The functional $\Psi \in E^*$ is called the *associated functional* of $(\{x_n\}, T)$.
Towards the existence of retro Banach frames of type $P$ we have following example.

**Example 1.** *Let* $E = \ell^1$ *and* $\{e_n\} \subset E$ *be the sequence of canonical unit vectors.*

*(a) Define sequences* $\{x_n\} \subset E$, $\{f_n\} \subset E^*$:

$$\left.\begin{array}{l} x_1 = e_1 \\ x_n = (-1)^{n+1} e_1 + e_n, \quad n = 2, 3, \dots \end{array}\right\},$$

$$\left.\begin{array}{l} f_1(x) = \xi_1 + \sum_{j=2}^{\infty} (-1)^j \xi_j \\ f_n(x) = \xi_n, \quad n = 2, 3, \dots \end{array}\right\}, x = \{\xi_j\} \in E.$$

*Then, by Lemma 1, there exists a reconstruction operator* $T : (E^*)_d = \{\{f(x_n)\} : f \in E^*\} \to E^*$ *such that* $(\{x_n\}, T)$ *is an exact retro Banach frame for* $E^*$ *with respect to* $(E^*)_d$ *and with bounds* $A = B = 1$. *Also,* $\Psi = (1, 2, 0, 2, 0, \dots) \in E^*$ *is such that* $\Psi(x_n) = 1$, *for all* $n \in \mathbb{N}$. *Hence* $(\{x_n\}, T)$ *is a retro Banach frame of type* $P$.

*(b) Let* $\{x_n\} \subset E$ *be a sequence defined as*

$$\left.\begin{array}{l} x_1 = e_1, \\ x_n = e_{n-1}, \quad n = 2, 3, \dots \end{array}\right\}.$$

*Then, there exists a reconstruction operator* $T_0 : (E^*)_{d_0} = \{\{f(x_n)\} : f \in E^*\} \to E^*$ *such that* $(\{x_n\}, T_0)$ *is a retro Banach frame for* $E^*$ *which is not of type* $P$.

The following theorem provides a necessary and sufficient conditions for a retro Banach frame to be of type $P$.

**Theorem 2.** *Let* $(\{x_n\}, T)$ *be a retro Banach frame for* $E^*$ *with respect* $(E^*)_d$. *Then the following conditions are equivalent:*

(a) $(\{x_n\}, T)$ *is of type* $P$;

(b) $(\{x_n\}, T)$ *is exact and there exists no reconstruction operator* $V$ *such that* $(\{x_n - x_{n+1}\}, V)$ *is a retro Banach frame for* $E^*$.

*Proof.* (a)$\Rightarrow$(b): Let $V$ be the reconstruction operator such that $(\{x_n - x_{n+1}\}, V)$ is a retro Banach frame for $E^*$ with respect to some associated Banach space $(E^*)_{d_0}$. Then there exist positive constants $A$, $B$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n - x_{n+1})\}\|_{(E^*)_{d_0}} \leq B\|f\|_{E^*}, \text{ for all } \ f \in E^*.$$

Now, $\Psi(x_n - x_{n+1}) = 0$, for all $n \in \mathbb{N}$, where $\Psi$ is the associated functional to $(\{x_n\}, T)$. So, retro frame inequality for $(\{x_n - x_{n+1}\}, V)$ gives $\Psi = 0$, a contradiction.

(b)$\Rightarrow$(a): If there exists no reconstruction operator $V$ such that $(\{x_n - x_{n+1}\}, V)$ is a retro Banach frame for $E^*$, then by Lemma 1, there exists a non-zero $f_0 \in E^*$ such that $f_0(x_n - x_{n+1}) = 0$, for all $n \in \mathbb{N}$.

This gives $f_0(x_1) = f_0(x_2) = \ldots = \omega$ (say).

If $\omega = 0$, then $f_0(x_n) = 0$, for all $n \in \mathbb{N}$. By retro frame inequality of $(\{x_n\}, T)$, we obtain $f_0 = 0$, a contradiction. Thus, $\omega \neq 0$. Put $\Psi = \frac{1}{\omega} f_0$. Then, $\Psi \in E^*$ is such that $\Psi(x_n) = 1$, for all $n \in \mathbb{N}$. Hence, $(\{x_n\}, T)$ is of type $P$.◀

Let $(\{x_n\}, T)$ be an exact retro Banach frame for $E^*$ and $z_0$ a non-zero vector in $E$. If there exists a reconstruction operator $T_0$ such that $(\{x_n + z_0\}, T_0)$ is a retro Banach frame for $E^*$, then in general, $(\{x_n\}, T)$ is not of type $P$ (see Example 2). In this direction necessary and sufficient conditions under which an exact retro Banach frame turns out to be of type $P$ is given in the following theorem.

**Theorem 3.** *Let* $(\{x_n\}, T)$ *(where* $T : (E^*)_d \to E^*$*) be an exact retro Banach frame for* $E^*$*. Then, the following conditions are equivalent:*

(a) $(\{x_n\}, T)$ *is of type* $P$*;*

(b) *There exists a non zero vector* $z_0$ *in* $E$ *for which there is no reconstruction operator* $T_0$ *such that* $(\{x_n + z_0\}, T_0)$ *is a retro Banach frame for* $E^*$.

*Proof.* (a)$\Rightarrow$(b) Assume that $(\{x_n\}, T)$ is of type $P$ with associated functional $\Psi \in E^*$. Then there is an element $x \in E$ such that $\Psi(x) \neq 0$. Let $z_0 = -\frac{1}{\Psi(x)} x$. Then, $z_0$ is a non-zero vector in $E$ for which there is no reconstruction operator $T_0$ such that $(\{x_n + z_0\}, T_0)$ is a retro Banach frame for $E^*$.

Indeed, let $0 < A_0 \leq B_0 < \infty$ be frame bounds for $(\{x_n + z_0\}, T_0)$.

Then

$$A_0\|h\|_{E^*} \leq \|\{h(x_n + z_0)\}\|_{(E^*)_{d_0}} \leq B_0\|h\|_{E^*}, \text{ for all } \ h \in E^*,$$

where $(E^*)_{d_0}$ is the associated Banach space of scalar-valued sequences.

But $\Psi(x_n+z_0)=0$, for all $n\in\mathbb{N}$. So, retro frame inequality for $(\{x_n+z_0\},T_0)$ gives $\Psi=0$, a contradiction.

(b)⇒(a): If there is no reconstruction operator $T_0$ such that $(\{x_n+z_0\},T_0)$ is a retro Banach frame for $E^*$, then by Lemma 1, there is a non-zero functional $g\in E^*$ such that $g(x_n+z_0)=0$, for all $n\in\mathbb{N}$. By retro frame inequality for $(\{x_n\},T)$ we conclude that $g(z_0)\neq 0$. Put $\Psi=-\dfrac{1}{g(z_0)}g$. Then, $\Psi$ is a functional in $E^*$ such that $\Psi(x_n)=1$, for all $n\in\mathbb{N}$. Hence $(\{x_n\},T)$ is of type $P$.◄

The following example gives an application of Theorems 2 and Theorem 3.

**Example 2.** *Let* $E=l^2$.

*(a) Let* $\{x_n\}\subset E$ *be a sequence defined by*

$$x_n=e_n,\quad \textit{for all } n\in\mathbb{N},$$

*where* $\{e_n\}\subset E$ *is the sequence of canonical unit vectors. Then, by Lemma 1* $(\{x_n\},T)$ *(where* $T:(E^*)_d=\{\{f(x_n)\}:f\in E^*\}\to E^*$*) is an exact retro Banach frame for* $E^*$ *with respect to* $(E^*)_d$ *and with bounds* $A=B=1$*. Also there exists a reconstruction operator* $V$ *such that* $(\{x_n-x_{n+1}\},V)$ *is a retro Banach frame for* $E^*$ *with respect to* $(E^*)_{d_0}=\{\{f(x_n-x_{n+1})\}:f\in E^*\}$*. Thus, by Theorem 2* $(\{x_n\},T)$ *is not of type* $P$.

*(b) Let* $\{x_n\}\subset E$ *be a sequence defined by*

$$\left.\begin{array}{l} x_1=2e_1,\\ x_n=e_n,\quad n=2,3,\dots \end{array}\right\}.$$

*Then, there exists a reconstruction operator* $W:(E^*)_d=\{\{f(x_n)\}:f\in E^*\}\to E^*$ *such that* $(\{x_n\},W)$ *is an exact retro Banach frame for* $E^*$ *with respect to* $(E^*)_d$.
*Put* $z_0=-e_1$*. Then, there eixsts a reconstruction operator* $T_0:(E^*)_{d_0}=\{\{f(x_n+z_0)\}:f\in E^*\}\to E^*$ *such that* $(\{x_n+z_0\},T_0)$ *is a retro Banach frame for* $E^*$ *with respect to* $(E^*)_{d_0}$*. Hence by Theorem 2* $(\{x_n\},W)$ *is not of type* $P$.

The following theorem give characterization of retro Banach frames obtained by translation of retro Banach frames of type $P$ by a non-zero vector.

**Theorem 4.** *Let* $(\{x_n\},T)$ *be a retro Banach frame of type* $P$ *for* $E^*$ *with associated functional* $\Psi$*. Then, there exists a reconstruction operator* $U$ *such that* $(\{x_n+z_0\},U)$ *is a normalized tight retro Banach frame for* $E^*$ *if and only if* $\Psi(z_0)\neq -1$*, where* $z_0$ *is a given non-zero vector in* $E$.

*Proof.* Assume that $\Psi(z_0) \neq -1$. Let $L = I + z_0 \otimes \Psi$ be a bounded linear operator on $E$, where $I$ is the identity operator on $E$.
Then, $Lx = 0$ gives $x + \Psi(x)z_0 = 0$.

If $\Psi(x) \neq 0$, then $\Psi(z_0) = -1$, a contradiction. Therefore, $x = 0$. Thus, $L$ is one-one. By Theorem 1, $L$ is invertible. Hence there exists a reconstruction operator $U : (E^*)_d = \{\{f(x_n + z_0)\} : f \in E^*\} \to E^*$ such that $(\{x_n + z_0\}, U)$ is a retro Banach frame for $E^*$ with respect to $(E^*)_d$ and with bounds $A = B = 1$.

Conversely, if $\Psi(z_0) = -1$, then, by using retro frame inequality for $(\{x_n + z_0\}, U)$, we obtain $\Psi = 0$, a contradiction.◄

An application of Theorem 4 is given in the following example.

**Example 3.** *Let $E = l^1$. Define $\{x_n\} \subset E$ by*

$$\left.\begin{array}{ll} x_1 = 2e_1, & \\ x_n = e_n, & n = 2, 3, \ldots \end{array}\right\}.$$

*Then, there exists a reconstruction operator $T : (E^*)_{d_0} = \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that $(\{x_n\}, T)$ is a retro Banach frame of type $P$ for $E^*$ with respect to $(E^*)_{d_0}$ and with associated functional $\Psi = (1/2, 1, 1, 1, \ldots) \in E^*$.*

*Let $z_0 = -e_1$. Then, $z_0 \in E$ is a non-zero vector such that $\Psi(z_0) \neq -1$. Hence by Theorem 4, there exists a reconstruction operator $U$ such that $(\{x_n + z_0\}, U)$ is a normalized tight retro Banach frame for $E^*$.*

To conclude the section, we prove that if two Banach spaces $E^*$ and $F^*$ both have retro Banach frames of type $P$, then the conjugate space space $(E \times F)^*$ also has a retro Banach frame of type $P$.

**Theorem 5.** *Let $(\{x_n\}, T)$ ($\{x_n\} \subset E, T : (E^*)_d \to E^*$) and $(\{y_n\}, V)$ ($\{y_n\} \subset F$, $V : (F^*)_d \to F^*$) be retro Banach frames of type $P$ for Banach spaces $E^*$ and $F^*$, respectively. Then there exist a sequence $\{z_n\} \subset (E \times F)$, and a reconstruction operator $U : (E \times F)^*_d \to (E \times F)^*$ such that $(\{z_n\}, U)$ is a retro Banach frame of type $P$ for $(E \times F)^*$.*

*Proof.* Since $(\{x_n\}, T)$ and $(\{y_n\}, V)$ are of type $P$, there exists $\{f_n\} \subset E^*$ and $\{g_n\} \subset E^*$ such that $f_n(x_m) = \delta_{n,m} = g_n(y_m)$, for all $n, m \in \mathbb{N}$.

Let $\{z_n\} \subset E \times F$ and $\{h_n\} \subset (E \times F)^*$ be sequences defined by

$$\begin{cases} z_{2n} = (0, y_n), \\ z_{2n-1} = (x_n, 0), \end{cases} \quad n \in \mathbb{N};$$

$$\begin{cases} h_{2n}(x, y) = g_n(y), \\ h_{2n-1}(x, y) = f_n(x), \end{cases} \quad n \in \mathbb{N}.$$

Then, by Lemma 1, there exists a reconstruction operator $U : (E \times F)^*_d = \{\{\psi(z_n)\} : \psi \in (E \times F)^*\} \to (E \times F)^*$ such that $(\{z_n\}, U)$ is a retro Banach frame for $(E \times F)^*$ with respect to $(E \times F)^*_d$ and with bounds $A = B = 1$.

Now, by nature of construction of $\{z_n\} \subset E \times F$, $h_n(z_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$, that is $(\{z_n\}, U)$, is exact. Further, there exists no reconstruction operator $W$ such that $(\{z_n - z_{n+1}\}, W)$ is a retro Banach frame for $(E \times F)^*$. Therefore, by Theorem 2, $(\{z_n\}, U)$ is of type $P$.◄

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