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On a Class of Nonlinear Elliptic Systems Involving (p,q)-Laplacian

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Abstract. In this paper we study the existence of positive solutions for the system

$$\begin{cases} -\Delta_p u = \lambda f(u,v), & x \in \Omega, \\ -\Delta_q v = \lambda g(u,v), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where Δ_p is the so-called p-Laplacian operator i.e. $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$, p and q are real numbers satisfying $1 < p, q < N$, λ is a real positive parameter, Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\partial\Omega$, and f, g are C^1 functions satisfying $\lim_{s\to\infty} f(s,t) = \infty = \lim_{t\to\infty} g(s,t)$, where each limit is uniform with respect to the other variable, $\lim_{|(s,t)|\to\infty} \frac{f(s,t)}{s^{p-1}} = \sigma$, and $\lim_{|(s,t)|\to\infty} \frac{g(s,t)}{t^{q-1}} = \delta$. In particular we do not assume any sign conditions on $f(0,0)$ or $g(0,0)$. For λ large we prove the existence of a large positive solution. Our approach is based on the method of sub-super.

Key Words and Phrases: Elliptic system; Positive solutions
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1. Introduction

The aim of this article is to study the existence of positive solutions for some nonlinear elliptic systems of the form

$$\begin{cases} -\Delta_p u = \lambda f(u,v), & x \in \Omega, \\ -\Delta_q v = \lambda g(u,v), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1}$$

here Δ_p is the so-called p-Laplacian operator i.e. $\Delta_p z = div\,(|\nabla z|^{p-2}\nabla z)$, p and q are real numbers satisfying $1 < p, q < N$, λ is a real positive parameter, Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\partial\Omega$, and $f, g : [0,\infty) \times [0,\infty) \to R$ are C^1 functions satisfying the following assumptions:

(A1) $\lim_{s\to\infty} f(s,t) = \infty = \lim_{t\to\infty} g(s,t)$, where each limit is uniform with respect to the other variable;

(A2) $\lim_{|(s,t)|\to\infty} \frac{f(s,t)}{s^{p-1}} = \sigma$, and $\lim_{|(s,t)|\to\infty} \frac{g(s,t)}{t^{q-1}} = \delta$;

(A3) $s \mapsto f(s,t)$ and $s \mapsto g(s,t)$ are nondecreasing for every $t > 0$;

(A4) $t \mapsto f(s,t)$ and $t \mapsto g(s,t)$ are nondecreasing for every $s > 0$.

Let $\zeta_1(x),\ \zeta_2(x)$ be the positive solutions, respectively, of the problems

$$\begin{cases} -\Delta_p \zeta_1 = 1, & x \in \Omega, \\ \zeta_1 = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta_q \zeta_2 = 1, & x \in \Omega, \\ \zeta_2 = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is as before. Let $l_1 = ||\zeta_1||_\infty,\ l_2 = ||\zeta_2||_\infty$, and we assume that

$$\sigma < \frac{1}{l_1^{p-1}},\ \ \delta < \frac{1}{l_2^{q-1}}, \tag{2}$$

where σ and δ is in $(A2)$.

The problem (1) arises in the theory of quasiregular and quasiconformal mappings or in the study of non-Newtonian fluids [2]. In the later case the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluid and those with $p < 2$ are called pseudoplastics.

The solvability of system (1) has been studied by various methods, fibering [4], bifurcation [9], via the mountain pass theorem [3]. See [1, 5] where the authors discussed the system (1) when $p = q = 2$, $f(u,v) = \tilde{f}(u)$, $g(u,v) = \tilde{g}(u)$, $\tilde{f}, \tilde{g}$ are increasing and $\tilde{f}, \tilde{g} \geq 0$. In [11], the authors extended the study of [5], to the case when no sign conditions on $f(0)$ or $g(0)$ were required and in [12] they extend this study to the case when $p = q > 1$. Here we focus on further extending the study in [12] for the quasilinear elliptic systems with much stronger coupling. Due to this strong coupling conditions, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions, see [8]. We refer to [6, 7, 10, 13] for additional results on elliptic problems.

2. Existence results

To prove our existence results we use the method of sub-super solutions. To do so, we now define sub and super solutions of (1). Let $W_0^{1,s} = W_0^{1,s}(\Omega), s > 1$, denote the usual Sobolev space.

Definition 1. *A pair of nonnegative functions* (ψ_1, ψ_2), (z_1, z_2) *in* $W_0^{1,p} \times W_0^{1,q}$ *are called a weak subsolution and supersolution of* (1) *if they satisfy* $\psi_i(x) \le z_i(x)$ *in* Ω *for* $i = 1, 2,$ *and*

$$\int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \nabla w_1 dx \le \lambda \int_\Omega f(\psi_1, \psi_2)\, w_1 dx, \tag{3}$$

$$\int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 \nabla w_2 dx \le \lambda \int_\Omega g(\psi_1, \psi_1)\, w_2 dx, \tag{4}$$

$$\int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \nabla w_1 dx \ge \lambda \int_\Omega f(z_1, z_2)\, w_1 dx, \tag{5}$$

and

$$\int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \nabla w_2 dx \ge \lambda \int_\Omega g(z_1, z_2)\, w_2 dx, \tag{6}$$

for all $w_1(x) \in W_0^{1,p}$, $w_2(x) \in W_0^{1,q}$, *with* $w_1, w_2, \ge 0$.

We shall obtain the existence of positive solution to system (1) by constructing a positive subsolution (ψ_1, ψ_2) and supersolution (z_1, z_2).

Our main result is formulate in the following theorem.

Theorem 1. *Assume that hypotheses* $(A1)-(A4)$ *and* (2) *hold. Then there exists a positive number* λ_0 *such that* (1) *has a large positive solution* (u, v) *for* $\lambda > \lambda_0$.

Proof. Let λ_1 and λ_2 be the first eigenvalue of the problems, respectively,

$$-\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1}, \quad x \in \Omega, \quad \phi_1 = 0, \quad x \in \partial\Omega,$$
$$-\Delta_q \phi_2 = \lambda_2 \phi_2^{q-1}, \quad x \in \Omega, \quad \phi_2 = 0, \quad x \in \partial\Omega,$$

where ϕ_1 and ϕ_2 denote the corresponding positive eigenfunctions, respectively, satisfying $||\phi_i||_\infty = 1$ for $i = 1, 2,$. By $(A1)$ we can take $a_1, a_1 > 0$ such that

$$f(s,t) > -a_1, \;\; g(s,t) > -a_2,$$

for all $s, t \geq 0$. Let $b_1, b_2 > 0$ be such that

$$\lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -b_1, \qquad \lambda_2 \phi_2^q - |\nabla \phi_2|^q \leq -b_2, \qquad x \in \bar{\Omega}_\eta,$$

where $\bar{\Omega}_\eta = \{x \in \Omega \mid dist\,(x, \partial\Omega) \leq \eta\}$. (This is possible since $|\nabla \phi_i|^r \neq 0$ on $\partial\Omega$ while $\phi_i = 0$ on $\partial\Omega$ for $r = p, q$ and $i = 1, 2$). We shall verify that

$$(\psi_1, \psi_2) = ((\frac{\lambda\, a_1}{b_1})^{\frac{1}{p-1}} (\frac{p-1}{p})\, \phi_1^{\frac{p}{p-1}}, (\frac{\lambda\, a_2}{b_2})^{\frac{1}{q-1}} (\frac{q-1}{q})\, \phi_2^{\frac{q}{q-1}}),$$

is a subsolution of (1) for λ large. Let the test function $w_1(x) \in W_0^{1,p}$, with $w_1 \geq 0$. Then it follows from (3) that

$$\begin{aligned}
\int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 . \nabla w_1 &= \frac{\lambda\, a_1}{b_1} \int_\Omega \phi_1 |\nabla \phi_1|^{p-2} \nabla \phi_1 \nabla w_1 dx dx \\
&= \frac{\lambda\, a_1}{b_1} \{ \int_\Omega |\nabla \phi_1|^{p-2} \nabla \phi_1 \nabla (\phi_1 w_1) dx - \int_\Omega |\nabla \phi_1|^p w_1 dx \} \\
&= \frac{\lambda\, a_1}{b_1} \int_\Omega (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w_1 dx.
\end{aligned}$$

Since on $\bar{\Omega}_\eta$ we have $\lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -b_1$, which implies that

$$\frac{a_1}{b_1} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq f(\psi_1, \psi_2).$$

It is well known that $\frac{\partial \phi_i}{\partial n} < 0$ on $\partial\Omega$ where n is the unit outward normal for $i = 1, 2$. Hence there exists $\beta > 0$ such that $\phi_1 \geq \beta > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$. Therefore, from $(A1)$ for λ large, we have

$$\frac{a_1}{b_1} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq \frac{a_1}{b_1} \lambda_1 \leq f(\psi_1, \psi_2).$$

Hence

$$\int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \nabla w_1 dx \leq \lambda \int_\Omega f(\psi_1, \psi_2)\, w_1 dx.$$

Similarly, we have

$$\int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 . \nabla w_2 dx \leq \lambda \int_\Omega g(\psi_1, \psi_1)\, w_2 dx,$$

for all $w_2(x) \in W_0^{1,q}$, with $w_2 \geq 0$. Thus, (ψ, ψ) is a subsolution of (1).

Next, we construct a supersolution (z_1, z_2) of (1). We denote

$$(z_1, z_2) = (\lambda^{\frac{1}{p-1}}\, C_1\, \zeta_1,\ \lambda^{\frac{1}{q-1}}\, C_2\, \zeta_2),$$

where ζ_1, ζ_2 are as before, and $C_1, C_2 > 0$ are large enough such that

$$\lambda\, f(\lambda^{\frac{1}{p-1}}\, C_1\, l_1,\ \lambda^{\frac{1}{q-1}}\, C_2\, l_2) \leq (C_1\, \lambda^{1/p-1})^{p-1}, \tag{7}$$

and

$$\lambda\, g(\lambda^{\frac{1}{p-1}}\, C_1\, l_1,\ \lambda^{\frac{1}{q-1}}\, C_2\, l_2) \leq (C_2\, \lambda^{1/q-1})^{q-1}. \tag{8}$$

Here (7) and (8) are possible by $(A2)$, and (2). We shall verify that (z_1, z_2) is a supersolution of (1). To this end, let $w_1(x) \in W_0^{1,p}$, $w_2(x) \in W_0^{1,q}$, with $w_1, w_2, \geq 0$. Then we obtain from (7), (8), $(A3)$ and $(A4)$, that

$$\begin{aligned}
\int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \nabla w_1 dx &= \lambda\, C_1^{p-1} \int_\Omega |\nabla \zeta_1|^{p-2} \nabla \zeta_1 \nabla w_1 dx \\
&= \lambda\, C_1^{p-1} \int_\Omega w_1 dx \\
&\geq \lambda \int_\Omega f(\lambda^{\frac{1}{p-1}}\, C_1\, l_1,\ \lambda^{\frac{1}{q-1}}\, C_2\, l_2) w_1 dx \\
&\geq \lambda \int_\Omega f(\lambda^{\frac{1}{p-1}}\, C_1\, \zeta_1,\ \lambda^{\frac{1}{q-1}}\, C_2\, \zeta_2) w_1 dx \\
&= \lambda \int_\Omega f(z_1, z_2) w_1 dx.
\end{aligned}$$

Similarly we have

$$\int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \nabla w_2 dx \geq \lambda \int_\Omega g(z_1, z_2) w_2 dx.$$

Thus, (z_1, z_2) is a supersolution of (1) with $z_i \geq \psi_i$ in Ω for large C_1, C_2, $i = 1, 2$. Thus, by the comparison principle, there exists a solution $(u, v,)$ of (1) with $\psi_1 \leq u \leq z_1,\ \psi_2 \leq v \leq z_2$. This completes the proof of Theorem 1.◄

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