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On boundedness of sublinear operators in weighted Morrey spaces

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Abstract. In this paper we provide conditions on functions ω_1 and ω_2 which ensures the boundedness of sublinear operators from one weighted Morrey space $\mathcal{M}_{p,w_1}(v)$ to another $\mathcal{M}_{p,w_2}(v)$, where weight function v belongs to Muckenhoupt classes A_p.

Key Words and Phrases: Weighted Morrey spaces, sublinear operators, A_p weights

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1. Introduction

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C. Morrey in 1938 [12], in relation to the study of partial differential equations, were widely investigated during last decades, including the study of classical operators of Harmonic Analysis - maximal, singular and potential operators - in generalizations of these spaces ("so-called" Morrey-type spaces). It is well known that many classical operators in Harmonic Analysis are bounded on the weighted space $L_{p,v}$, $1 < p < \infty$, provided a weight v satisfies the A_p condition. Meanwhile the boundedness of these operators on weighted Morrey spaces was not studied good enough. We can refer to few works in this direction (see [7], [10], [11], for instance).

For this purpose we first recall the definition of weighted Morrey space $\mathcal{M}_{p,\omega}(v)$.

Definition 1 (see [10], for instance)**.** *(Weighted Morrey spaces) Let $1 \leq p < \infty$ and $\omega(x,r)$ be a positive continuous function on $\mathbb{R}^n \times (0,\infty)$. Let v be a weight function on $\mathbb{R}^n$. We denote by $\mathcal{M}_{p,\omega}(v) = \mathcal{M}_{p,\omega}(\mathbb{R}^n, v)$ a weighted Morrey space, the space of all functions $f \in L^{\mathrm{loc}}_{p,v}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{\mathcal{M}_{p,\omega}(v)} = \sup_{x\in\mathbb{R}^n,\, r>0} \omega(x,r)^{-\frac{1}{p}} \|f\|_{L_{p,v}(B(x,r))},$$

where $B(x,r)$ denotes the open ball centered at x of radius r.

In case $v \equiv 1$, $\mathcal{M}_{p,\omega}(v)$ is the generalized Morrey space $\mathcal{M}_{p,\omega}$, which are introduced by Nakai in [13]. Moreover, if $\omega(x,r) \equiv 1$, then $\mathcal{M}_{p,\omega}(v) = L_p$; if $\omega(x,r) = r^{\lambda}$, $0 < \lambda < n$, then $\mathcal{M}_{p,\omega}(v)$ is just the standart Morrey space $\mathcal{M}_{p,\lambda}$; and if $\omega(x,r)$ is independent of x, then $\mathcal{M}_{p,\omega}(v)$ is the generalized Morrey space introduced by Mizuhara in [9].

Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$:

$$|Tf(x)| \le c_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \tag{1}$$

where c_0 is independent of f and x.

We point out that the condition (1) was first introduced by Soria and Weiss in [14]. The condition (1) are satisfied by many interesting operators in Harmonic Analysis, such as the Calderón-Zygmund singular integral operators, Carleson's maximal operators, Hardy-Littlewood maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on (see also [4], [8], [14]-[17] for details).

In the present paper we study the boundedness of sublinear operators on weighted Morrey spaces $\mathcal{M}_{p,w}(v)$. Method of investigation is based on obtaining weighted L_p-estimates over balls for these operators and applying the boundedness of appropriate Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions, which is the extension to the weighted case of one used for investigation of the boundedness of maximal, singular and potential operators in Morrey-type spaces (see, for instance, [1], [2], [3]).

2. Definitions and Preliminary Results

Now we make some conventions. Throughout the paper we always denote by c or C a positive constant, which is independent of main parameters, but it may vary from line to line. By $A \lesssim B$ we mean that $A \le cB$ with some positive constant c independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Constant, with subscript such as c_1, does not change in different occurrences. For a measurable set E, χ_E denotes the characteristic function of E. We define the Lebesgue measure of E by $|E|$. Given $\lambda > 0$ and a cube Q, λQ denotes the cube with the same center as Q and whose side is λ times that of Q. For a fixed p with $p \in [1, \infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$. For any measurable set E and any integrable function f on E, we denote by f_Q the mean value of f over E, that is, $f_Q = (1/|Q|) \int_E f(x)dx$.

A weight is a locally integrable function on $\mathbb{R}^n$, which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E we define $w(E) =$

$\int_E w(x)dx$. The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by $L_{p,w}$ with $0 < p < \infty$. Given a weight w, we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any cube Q we have $w(2Q) \le Dw(Q)$. When w satisfies this condition, we denote $w \in \mathcal{D}$, for short.

Denote

$$L^{\rm loc}_{p,v}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} \text{ measurable} \mid f \in L_{p,v}(K) \text{ for all compact subsets } K \text{ of } \mathbb{R}^n\}.$$

When $v \equiv 1$, we will omit v in the notation of $L^{\rm loc}_{p,v}(\mathbb{R}^n)$ and denote it by $L^{\rm loc}_{p}(\mathbb{R}^n)$.

Let $f \in L^{\rm loc}_1(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)|dy.$$

For the sake of completeness we recall the definition of spaces, we are going to use, and some properties of them.

We consider weights in the Muckenhoupt classes A_p, $1 \le p \le \infty$, which are defined as follows. Let w be a weight function and $1 \le p < \infty$. We say that $w \in A_p$ if there exists a constant $c_p > 0$ such that for every ball $B \subset \mathbb{R}^n$:

$$\left(\int_B w(x)dx\right)\left(\int_B w(x)^{1-p'}dx\right)^{p-1} \le c_p|B|^p,$$

when $1 < p < \infty$, and for $p = 1$:

$$\int_B w(y)dy \le c_1|B|w(x), \quad \text{for a.e. } x \in B,$$

which can be equivalently written as $Mw(x) \le c_1 w(x)$ for a.e. $x \in \mathbb{R}^n$. The smallest possible c_p here is denoted by $[w]_{A_p}$. Finally, we set $A_\infty = \bigcup_{p\ge1} A_p$. It is well known that the Muckenhoupt classes characterize the boundedness of the Hardy-Littlewood maximal function M on weighted Lebesgue spaces. Namely, M is bounded on $L_{p,w}(\mathbb{R}^n)$ if and only if $w \in A_p$, $1 < p < \infty$.

Lemma 1 ([5])**.** (1) *If $w \in A_p$ for some $1 \le p < \infty$, then $w \in \mathcal{D}$. More precisely, for all $\lambda > 1$ we have*

$$w(\lambda Q) \le \lambda^{np}[w]_{A_p} w(Q).$$

(2) *If $w \in A_p$ for some $1 \le p < \infty$, then there exist $c > 0$ and $\delta > 0$ such that for any cube Q and a measurable set $S \subset Q$:*

$$\frac{w(S)}{w(Q)} \le c\left(\frac{|S|}{|Q|}\right)^\delta.$$

Definition 2 ([10]). *Let $p \in (0,\infty)$ and v be a weight on $\mathbb{R}^n$. We say that the pair (ω_1,ω_2) of positive measurable functions defined on $\mathbb{R}^n \times (0,\infty)$ belongs to the class $\mathcal{Z}_{p,n}(v)$, if there is a constant $c>0$ such that for any $x \in \mathbb{R}^n$ and for any $t>0$:*

$$\left(\int_t^\infty \left(\frac{\operatorname{ess\,inf}_{r<s<\infty} \omega_1(x,s)}{\|v\|_{L_1(B(x,r))}}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \le c\frac{\omega_2(x,t)}{\|v\|_{L_1(B(x,t))}}. \tag{2}$$

We need the following statement on the boundedness of the Copson operator

$$(H^*g)(t) := \int_t^\infty g(r)dr,\ 0<t<\infty.$$

Theorem 1. *Let v_1, v_2 and w be a weight functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_1(t) \int_t^\infty g(s)w(s)ds \le c \operatorname*{ess\,sup}_{t>0} v_2(t)g(t), \tag{3}$$

holds for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$A := \operatorname*{ess\,sup}_{t>0} v_1(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<x<\infty} v_2(x)} < \infty, \tag{4}$$

and $c \approx A$.

Proof. Sufficiency. Assume that (4) holds. Whenever F, G are non-negative functions on $(0,\infty)$ and F is non-decreasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} G(s), \quad t\in(0,\infty). \tag{5}$$

By (5) we have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t>0} v_1(t) \int_t^\infty g(s)w(s)ds \\
&\qquad = \operatorname*{ess\,sup}_{t>0} v_1(t) \int_t^\infty g(s)w(s) \frac{\operatorname{ess\,sup}_{s<r<\infty} v_2(r)}{\operatorname{ess\,sup}_{s<r<\infty} v_2(r)} ds \\
&\qquad \le \operatorname*{ess\,sup}_{t>0} v_1(t) \int_t^\infty \frac{w(s)}{\operatorname{ess\,sup}_{s<r<\infty} v_2(r)} ds \cdot \sup_{t>0} g(t) \operatorname*{ess\,sup}_{t<r<\infty} v_2(r) \\
&\qquad = \operatorname*{ess\,sup}_{t>0} v_1(t) \int_t^\infty \frac{w(s)}{\operatorname{ess\,sup}_{s<r<\infty} v_2(r)} ds \cdot \sup_{t>0} g(t)v_2(t) \\
&\qquad \le A \sup_{t>0} g(t)v_2(t).
\end{aligned}$$

Necessity. Assume that the inequality (3) holds. The function

$$g(t) = (\operatorname*{ess\,sup}_{t<r<\infty} v_2(r))^{-1},$$

is nonnegative and non-decreasing on $(0,\infty)$. Thus

$$\sup_{t>0} v_1(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_2(\tau)} \le c \sup_{t>0} v_2(t) (\operatorname*{ess\,sup}_{t<r<\infty} v_2(r))^{-1} \le c. \blacktriangleleft$$

3. Local $L_{p,v}$-estimates of sublinear operators

In this section we present estimates of $L_{p,v}$-norms of sublinear operator over balls.

Lemma 2. *Let $1 < p < \infty$, $v \in A_p$, T be a sublinear operator satisfying condition (1) and bounded on $L_{p,v}(\mathbb{R}^n)$. Then for any ball $B = B(x_0, r)$ in $\mathbb{R}^n$ the following inequality holds*

$$\|Tf\|_{L_{p,v}(B)} \le c\,(v(B))^{\frac{1}{p}} \int_r^\infty (v(B(x_0,t)))^{-\frac{1}{p}} \|f\|_{L_{p,v}(B(x_0,t))} \frac{dt}{t}, \tag{6}$$

where constant $c > 0$ does not depend on B and f.

Proof. Let $B = B(x_0, r)$ be any ball in $\mathbb{R}^n$. We write $f(y) = f_1(y) + f_2(y)$ with $f_1(y) = f(y)\chi_{2B}(y)$ and $f_2(y) = \sum_{k=1}^\infty f(y)\chi_{2^{k+1}B\setminus 2^k B}(y)$. Taking into account the sublinearity of T, we have

$$\|Tf\|_{L_{p,v}(B)} \le \|Tf_1\|_{L_{p,v}(B)} + \|Tf_2\|_{L_{p,v}(B)}. \tag{7}$$

Since $f_1 \in L_{p,v}(\mathbb{R}^n)$, the boundedness of T in $L_{p,v}(\mathbb{R}^n)$ implies that

$$\|Tf_1\|_{L_{p,v}(B)} \le \|Tf_1\|_{L_{p,v}(\mathbb{R}^n)} \le c\|f_1\|_{L_{p,v}(\mathbb{R}^n)} = c\|f\|_{L_{p,v}(2B)}, \tag{8}$$

where c is independent of B and f. Note that

$$|B| \approx \|v\|_{L_1(B)}^{\frac{1}{p}} \|v^{-\frac{1}{p}}\|_{L_{p'}(B)}. \tag{9}$$

Using (9) and

$$\|f\|_{L_{p,v}(2B)} \approx r^n \|f\|_{L_{p,v}(2B)} \int_{2r}^\infty \frac{dt}{t^{n+1}} \lesssim r^n \int_{2r}^\infty \|f\|_{L_{p,v}(B(x_0,t))} \frac{dt}{t^{n+1}},$$

we get

$$\begin{aligned}\|f\|_{L_{p,v}(2B)} &\lesssim \|v\|_{L_1(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,v}(B(x_0,t))} \|v^{-\frac{1}{p}}\|_{L_{p'}(B(x_0,t)} \frac{dt}{t^{n+1}} \\ &\approx \|v\|_{L_1(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,v}(B(x_0,t))} \|v\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.\end{aligned} \tag{10}$$

By (8) and (10) we obtain

$$\|Tf_1\|_{L_{p,v}(B)} \lesssim \|v\|_{L_1(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,v}(B(x_0,t))} \|v\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}. \tag{11}$$

By inequality (1) we have for $x \in B$

$$|Tf_2(x)| \le \sum_{k=1}^{\infty} |Tf_k(x)| \le c_0 \sum_{k=1}^{\infty} \int_{2^{k+1}\backslash 2^k B} \frac{|f(y)|}{|x-y|^n} dy = c_0 \int_{\mathbb{R}^n \backslash (2B)} \frac{|f(y)|}{|x-y|^n} dy.$$

It's clear that $x \in B$, $y \in \mathbb{R}^n \backslash (2B)$ implies $1/2|x_0-y| \le |x-y| \le 3/2|x_0-y|$. Therefore we get

$$\|Tf_2\|_{L_{p,v}(B)} \lesssim \|v\|_{L_1(B)}^{\frac{1}{p}} \int_{\mathbb{R}^n \backslash (2B)} \frac{|f(y)|}{|x_0-y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^n \backslash (2B)} \frac{|f(y)|}{|x_0-y|^n} dy &\approx \int_{\mathbb{R}^n \backslash (2B)} |f(y)| \left(\int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} \right) dy \\ &\approx \int_{2r}^{\infty} \left(\int_{2r \le |x_0-y| \le t} |f(y)| dy \right) \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \left(\int_{B(x_0,t)} |f(y)| dy \right) \frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality, in view of (9), we get

$$\begin{aligned}\int_{\mathbb{R}^n \backslash (2B)} \frac{|f(y)|}{|x_0-y|^n} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,v}(B(x_0,t))} \|v^{-\frac{1}{p}}\|_{L_{p'}(B(x_0,t)} \frac{dt}{t^{n+1}} \\ &\approx \int_{2r}^{\infty} \|f\|_{L_{p,v}(B(x_0,t))} \|v\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.\end{aligned}$$

Thus,

$$\|Tf_2\|_{L_{p,v}(B)} \lesssim \|v\|_{L_1(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,v}(B(x_0,t))} \|v\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}. \tag{12}$$

Finally, by (7), (11) and (12), we arrive at (6).◄

Remark 1. *Note that Lemma 2 is new even for $v = 1$.*

4. Boundedness of sublinear operators in weighted Morrey spaces

The boundedness of sublinear operators on generalized Morrey spaces $\mathcal{M}_{p,\omega}$ was established in [4].

Theorem 2 ([4], Theorem 1). *Assume that there is a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and any $r > 0$:*

$$r \leq t \leq 2r \Rightarrow C^{-1} \leq \omega(x,t)/\omega(x,r) \leq C, \tag{13}$$

$$\int_r^\infty \frac{\omega(x,t)}{t^{n+1}} dt \leq C \frac{\omega(x,r)}{r^n}. \tag{14}$$

Let $p \in (1,\infty)$. If a sublinear operator T, satisfying (1), *is bounded on $L_p(\mathbb{R}^n)$, then T is also bounded on $\mathcal{M}_{p,\omega}$.*

Note that in [4] the statement of the Theorem 2 was proved for more general sublinear operators, namely, for sublinear operators with rough kernel.

Now we prove the main result of the present paper. In the following theorem we present sufficient conditions on functions ω_1 and ω_2 which ensures the boundedness of sublinear operators from $\mathcal{M}_{p,w_1}(v)$ to $\mathcal{M}_{p,w_2}(v)$, where weight function v belongs to Muckenhoupt classes A_p. The conditions for the boundedness of sublinear operators are given in terms of Zygmund-type integral inequalities on (ω_1,ω_2), which do not assume any assumption on monotonicity of ω_1, ω_2 in r.

Theorem 3. *Let $1 < p < \infty$, $v \in A_p$, $(\omega_1,\omega_2) \in \mathcal{Z}_{p,n}(v)$, T be a sublinear operator satisfying condition* (1) *and bounded on $L_{p,v}(\mathbb{R}^n)$. Then operator T is bounded from $\mathcal{M}_{p,\omega_1}(v)$ to $\mathcal{M}_{p,\omega_2}(v)$ and*

$$\|Tf\|_{\mathcal{M}_{p,\omega_2}(v)} \leq c\|f\|_{\mathcal{M}_{p,\omega_1}(v)}, \tag{15}$$

constant $c > 0$ does not depend on f.

Proof. By Lemma 2 and Theorem 1 we have

$$\begin{aligned}\|Tf\|_{\mathcal{M}_{p,\omega_2}(v)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-\frac{1}{p}} \|v\|_{L_1(B(x,r))}^{\frac{1}{p}} \int_r^\infty \|f\|_{L_{p,v}(B(x,t))} \|v\|_{L_1(B(x,t))}^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega_1(x,r)^{-\frac{1}{p}} \|f\|_{L_{p,v}(B(x,r))} = \|f\|_{\mathcal{M}_{p,\omega_1}(v)}. \blacktriangleleft\end{aligned}$$

For further argumentation we need the following lemma.

Lemma 3 (see [13], for instance). *Suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. If there is a constant $c > 0$ such that for any $t > 0$:*

$$\int_t^\infty \varphi(r)\frac{dr}{r} \le c\varphi(t),$$

then there are constants $\varepsilon > 0$ and $c > 0$ such that for any $t > 0$:

$$\int_t^\infty r^\varepsilon \varphi(r)\frac{dr}{r} \le c\, t^\varepsilon \varphi(t). \tag{16}$$

Remark 2. *Note that if the condition* (14) *holds, then $(\omega, \omega) \in \mathcal{Z}_{p,n}(v)$.*

Indeed, assume that the condition (14) holds. By Lemma 3, there are constants $\varepsilon > 0$ and $C > 0$ such that for any $x \in \mathbb{R}^n$ and $t > 0$

$$\int_t^\infty r^\varepsilon \frac{\omega(x,r)}{r^n}\frac{dr}{r} \le C\, t^\varepsilon \frac{\omega(x,t)}{t^n}. \tag{17}$$

Applying Hölder's inequality, we get

$$\left(\int_t^\infty \left(\frac{\operatorname{ess\,inf}_{r<s<\infty} \omega(x,s)}{r^n}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \le \left(\int_t^\infty \left(\frac{\omega(x,r)}{r^n}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p =$$

$$= \left(\int_t^\infty r^{-\frac{\varepsilon}{p}} r^{\frac{\varepsilon}{p}} \left(\frac{\omega(x,r)}{r^n}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p =$$

$$= \left(\int_t^\infty r^{-\varepsilon\frac{p'}{p}} \frac{dr}{r}\right)^{\frac{p}{p'}} \int_t^\infty r^\varepsilon \frac{\omega(x,r)}{r^n}\frac{dr}{r} \approx$$

$$\approx t^{-\varepsilon} \int_t^\infty r^\varepsilon \frac{\omega(x,r)}{r^n}\frac{dr}{r}. \tag{18}$$

In view of inequalities (18) and (17), we obtain that $(\omega, \omega) \in \mathcal{Z}_{p,n}(v)$.

In the recent work by Komori and Shirai the boundedness of the Hardy-Littlewood maximal function and a Calderón-Zygmund singular integral operator on weighted Morrey spaces was proved.

Theorem 4. *[7, Theorem 3.2 and Theorem 3.3] If $1 < p < \infty$, $0 < \kappa < 1$, and $w \in A_p$, then the maximal operator M and a Calderón-Zygmund singular integral operator T are bounded on $L^{p,\kappa}$, where weighted Morrey space $L^{p,\kappa}$ is defined by*

$$L^{p,\kappa} := \left\{ f \in L^{loc}_{p,w} : \|f\|_{L^{p,\kappa}(w)} := \sup_Q \left(\frac{1}{w(Q)^\kappa}\int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty \right\},$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Remark 3. *Theorem 3 contains Theorem 4 as a special case. To show this let $1 < p < \infty$, $0 < \kappa < 1$, $v \in A_p$ and let $\omega_1(x,t) = \omega_2(x,t) = v(B(x,t))^\kappa$.*

By Lemma 1 there exists $c > 0$ such that for all $x \in \mathbb{R}^n$ and $r > t$:

$$v(B(x,r)) \geq c\left(\frac{r}{t}\right)^{n\delta} v(B(x,t)),$$

thus

$$\begin{aligned}
&\left(\int_t^\infty \left(\frac{\inf_{r<s<\infty}\omega_1(x,s)}{v(B(x,r))}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \\
&\qquad = \left(\int_t^\infty (v(B(x,r)))^{\frac{\kappa-1}{p}} \frac{dr}{r}\right)^p \\
&\qquad \lesssim \left(\int_t^\infty \left(\left(\frac{r}{t}\right)^{n\delta} v(B(x,t))\right)^{\frac{\kappa-1}{p}} \frac{dr}{r}\right)^p \\
&\qquad \approx v(B(x,t))^{\kappa-1}\left(\int_t^\infty \left(\frac{r}{t}\right)^{n\delta\frac{\kappa-1}{p}} \frac{dr}{r}\right)^p \\
&\qquad \approx v(B(x,t))^{\kappa-1} \approx (v(B(x,t)))^{-1}\omega_2(x,t),
\end{aligned}$$

that is $(\omega_1, \omega_2) \in \mathcal{Z}_{p,n}(v)$.

Remark 4. *Let us consider weights of the form $v(x) = |x|^\alpha$, $-n < \alpha < n(p-1)$. We divide all balls $B(x_0, R)$ in $\mathbb{R}^n$ into two categories: balls of type I that satisfy $|x_0| \geq 3R$ and type II that satisfy $|x_0| < 3R$ (see, [6, p. 285], for instance).*

For balls of type I we observe that

$$\nu_n R^n \begin{cases} (|x_0| - R)^\alpha & \text{when } \alpha \geq 0, \\ (|x_0| + R)^\alpha & \text{when } \alpha < 0, \end{cases} \leq$$

$$\int\limits_{B(x_0,R)} |x|^\alpha dx \leq \nu_n R^n \begin{cases} (|x_0| + R)^\alpha & \text{when } \alpha \geq 0, \\ (|x_0| - R)^\alpha & \text{when } \alpha < 0. \end{cases}$$

If $|x_0| \geq 3R$, we have $|x_0| + R \leq 4(|x_0| - R)$ and $|x_0| - R \geq 1/4(|x_0| + R)$, from which by previous inequalities we get

$$\int\limits_{B(x_0,R)} |x|^\alpha dx \approx R^n \begin{cases} (|x_0| + R)^\alpha & \text{when } \alpha \geq 0, \\ (|x_0| - R)^\alpha & \text{when } \alpha < 0. \end{cases} \tag{19}$$

For balls of type II we have

$$\int\limits_{B(x_0,R)} |x|^\alpha dx \approx R^{n+\alpha}. \tag{20}$$

The following theorem follows from Theorem 3.

Theorem 5. *Let* $1 < p < \infty$*,* $0 < \lambda < n$*,* $\lambda - n < \alpha < n(p-1)$*,* T *be a sublinear operator, satisfying condition* (1) *and bounded on* $L_{p,v}(\mathbb{R}^n)$*. Then operator* T *is bounded on* $\mathcal{M}_{p,\lambda}(|\cdot|^\alpha)$ *and*

$$\|Tf\|_{\mathcal{M}_{p,\lambda}(|\cdot|^\alpha)} \leq c\|f\|_{\mathcal{M}_{p,\lambda}(|\cdot|^\alpha)}, \tag{21}$$

with constant $c > 0$ *independent of* f*.*

Proof. Let us check that $(\omega_1, \omega_2) \in \mathcal{Z}_{p,n}(v)$, where $\omega_1(x,t) = \omega_2(x,t) = t^\lambda$, $0 < \lambda < n$ and $v(x) = |x|^\alpha$, $\lambda - n < \alpha < n(p-1)$. Then the statement follows from Theorem 3.

Consider positive and negative values of α separately.

a) Let $0 < \alpha < n(p-1)$. If $|x| < 3t$, then, by inequalities (20), we get for the left hand side of (2)

$$\left(\int_t^\infty \left(\frac{\inf_{r<s<\infty} \omega_1(x,s)}{\|v\|_{L_1(B(x,r))}}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \approx \left(\int_t^\infty r^{\frac{\lambda-n-\alpha}{p}} \frac{dr}{r}\right)^p.$$

We have

$$\int_t^\infty r^{\frac{\lambda-n-\alpha}{p}} \frac{dr}{r} \approx t^{\frac{\lambda-n-\alpha}{p}},$$

since $\lambda - n - \alpha < 0$. Therefore

$$\left(\int_t^\infty \left(\frac{\inf_{r<s<\infty} \omega_1(x,s)}{\|v\|_{L_1(B(x,r))}}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \lesssim t^{\lambda-n-\alpha}. \tag{22}$$

On the other hand, for the right hand side of the inequality (2), we have

$$\frac{\omega_2(x,t)}{\|v\|_{L_1(B(x,t))}} \approx t^{\lambda-n-\alpha}. \tag{23}$$

We have used inequalities (20), since $|x| < 3t$.

Inequalities (22) and (23) imply that the inequality (2) holds, when $|x| < 3t$.

If $|x| \geq 3t$, then, by inequalities (19) and (20), we get for the left hand side of (2)

$$\left(\int_t^\infty \left(\frac{\inf_{r<s<\infty} \omega_1(x,s)}{\|v\|_{L_1(B(x,r))}}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \approx \left(\int_t^{\frac{|x|}{3}} \left(\frac{r^{\lambda-n}}{(|x|+r)^\alpha}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p$$

$$+ \left(\int_{\frac{|x|}{3}}^\infty r^{\frac{\lambda-n-\alpha}{p}} \frac{dr}{r}\right)^p.$$

Taking into account that $(|x|+r)^{-1} \approx |x|^{-1}$, when $t < r < \frac{|x|}{3}$, for both integrals on the right hand side of previous relation, we obtain

$$\left(\int_t^{\frac{|x|}{3}} \left(\frac{r^{\lambda-n}}{(|x|+r)^\alpha}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \approx |x|^{-\alpha} \left(\int_t^{\frac{|x|}{3}} r^{\frac{\lambda-n}{p}} \frac{dr}{r}\right)^p \lesssim$$

$$|x|^{-\alpha} \left(\int_t^{\infty} r^{\frac{\lambda-n}{p}} \frac{dr}{r}\right)^p \approx |x|^{-\alpha} t^{\lambda-n}$$

and

$$\left(\int_{\frac{|x|}{3}}^{\infty} r^{\frac{\lambda-n-\alpha}{p}} \frac{dr}{r}\right)^p \lesssim |x|^{\lambda-n-\alpha}. \tag{24}$$

Thus

$$\left(\int_t^{\infty} \left(\frac{\inf_{r<s<\infty} \omega_1(x,s)}{\|v\|_{L_1(B(x,r))}}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \lesssim \frac{t^{\lambda-n} + |x|^{\lambda-n}}{|x|^\alpha}. \tag{25}$$

On the other hand, by (19), for the right hand side of the inequality (2), we have

$$\frac{\omega_2(x,t)}{\|v\|_{L_1(B(x,t))}} \approx \frac{t^{\lambda-n}}{(|x|+t)^\alpha}, \tag{26}$$

since $|x| \ge 3t$.

Obviously

$$\frac{t^{\lambda-n} + |x|^{\lambda-n}}{|x|^\alpha} \lesssim \frac{t^{\lambda-n}}{(|x|+t)^\alpha}, \quad \text{when} \quad |x| \ge 3t. \tag{27}$$

In view of inequalities (25), (26) and (27), the inequality (2) holds, when $|x| \ge 3t$. Thus, $(\omega_1, \omega_2) \in \mathcal{Z}_{p,n}(v)$.

b) Let $\lambda - n < \alpha < 0$. If $|x| < 3t$, then the inequality (2) holds, since, by (20), $\|v\|_{L_1(B(x,t))} \approx t^{n+\alpha}$ (see item (a)).

If $|x| \ge 3t$, then, by inequalities (19) and (20), we get for the left hand side of (2)

$$\left(\int_t^{\infty} \left(\frac{\inf_{r<s<\infty} \omega_1(x,s)}{\|v\|_{L_1(B(x,r))}}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \approx \left(\int_t^{\frac{|x|}{3}} \left(\frac{r^{\lambda-n}}{(|x|-r)^\alpha}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p + $$
$$+ \left(\int_{\frac{|x|}{3}}^{\infty} r^{\frac{\lambda-n-\alpha}{p}} \frac{dr}{r}\right)^p.$$

Taking into account that $2/3|x| \leq (|x|-r) \leq |x|$, when $t < r < \frac{|x|}{3}$, for the first integral on the right hand side of previous relation, we obtain

$$\left(\int_t^{\frac{|x|}{3}} \left(\frac{r^{\lambda-n}}{(|x|-r)^\alpha}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \lesssim |x|^{-\alpha} \left(\int_t^{\frac{|x|}{3}} r^{\frac{\lambda-n}{p}} \frac{dr}{r}\right)^p \lesssim$$

$$|x|^{-\alpha} \left(\int_t^{\infty} r^{\frac{\lambda-n}{p}} \frac{dr}{r}\right)^p \approx |x|^{-\alpha} t^{\lambda-n}.$$

In view of (24) we arrive at (25).

On the other hand, by (19), for the right hand side of the inequality (2), we have

$$\frac{\omega_2(x,t)}{\|v\|_{L_1(B(x,t))}} \approx \frac{t^{\lambda-n}}{(|x|-t)^\alpha}, \tag{28}$$

since $|x| \geq 3t$.

Obviously

$$\frac{t^{\lambda-n} + |x|^{\lambda-n}}{|x|^\alpha} \lesssim \frac{t^{\lambda-n}}{(|x|-t)^\alpha}, \quad \text{when} \quad |x| \geq 3t. \tag{29}$$

In view of inequalities (25), (28) and (29) the inequality (2) holds, when $|x| \geq 3t$. Thus, $(\omega_1, \omega_2) \in \mathcal{Z}_{p,n}(v)$.◀

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