Azerbaijan Journal of Mathematics
V. 2, No 1, 2012, January
ISSN 2218-6816

Essential spectrum a brief survey of concepts and applications

Mohammed Benharrat , Bekkai Messirdi

Abstract. We give a survey of results concerning various essential spectra of bounded linear operators in a Banach space. We define the generalized Kato essential spectrum of an operator, and we also give some relationships between this spectrum and other essential spectra found in Fredholm theory and the SVEP theory.

Key Words and Phrases: Semi-regular operator, Kato type operator, Generalized Kato spectrum, Essential spectrum

2010 Mathematics Subject Classifications: 47A10

1. Introduction

Let X be an infinite-dimensional complex Banach space. We denote by $\mathcal{L}(X)$ the set of all bonded linear operators from X into X. Let I denote the identity operator in X. For $T \in \mathcal{L}(X)$ we denote by T^* the adjoint of T, $R(T)$ its range, and $N(T)$ its null space. An operator $T \in \mathcal{L}(X)$ is said to be semi-regular if $R(T)$ is closed and $N(T^n) \subseteq R(T)$, for all $n \geq 0$. T admits a generalized Kato decomposition, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, where $T|M$ is semi-regular and $T|N$ is quasi-nilpotent. A bounded operator on X is said to be quasi-nilpotent if its spectrum $\sigma(T) = \{0\}$.

The Kato decomposition for bounded operator on Banach spaces arises from the classical treatment of perturbation theory of Kato [32], and its flourishing has greatly benefited from the work of many authors in the last ten years, in particular from the work of Mbekhta [41, 43, 44], Aiena [1] and Q. Jiang-H. Zhong [30]. The operators which satisfy this property form a class which includes the class of quasi-Fredholm operators, semi-regular, Kato type, semi-Fredholm and B-Fredholm operators. This concept leads in a natural way to the generalized Kato spectrum $\sigma_{gk}(T)$, an important subset of the ordinary spectrum which is defined as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ does not admit a generalized Kato decomposition. It was shown in [30, Corollary 2.3] that $\sigma_{gk}(T)$ is a compact subset of $\mathbb{C}$. Note that $\sigma_{gk}(T)$ is not necessarily non-empty. For example, a quasi-nilpotent operator T has empty generalized Kato spectrum.

The aim of this paper is to define the generalized Kato essential spectrum of an operator (the essential version of $\sigma_{gk}(T)$), and we also give some relationships between this spectrum and other essential spectra found in Fredholm theory and the SVEP theory. We present a survey of results for various essential spectra and we consider their stability under some perturbations.

2. The semi-regular spectrum and its essential version

The semi-regular spectrum was first introduced by Apostol [2] for operators on Hilbert spaces and successively studied by several authors Muller[47] and Rakocevic [54], Mbekhta and Ouahab [45] and Mbekhta [42] in the more general context of operators acting on Banach spaces. Trivial examples of semi-regular operators are surjective operators as well as injective operators with closed range, Fredholm operators and semi-Fredholm operators with jump equal zero (for more details see [1]). Some other examples of semi-regular operators may be found in Mbekhta [45] and Labrousse [37]. For an essential version of semi-regular operators we use the following notation, for subspaces $M, L \subset X$ we write $M \subset_e L$ if there exists a finite-dimensional subspace F of X for which $M \subset L + F$. Obviously

$$M \subset_e L \Leftrightarrow \dim \frac{M}{M \cap L} < \infty$$

An operator $T \in \mathcal{L}(X)$ is called essentially semi-regular if $R(T)$ is closed and $N(T^n) \subset_e R(T)$, for all integers $n \geq 0$.

The semi-regular spectrum of a bounded operator T on X is defined by

$$\sigma_{se}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}$$

and its essential version by

$$\sigma_{es}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not essentially semi-regular}\}$$

The sets $\sigma_{se}(T)$ and $\sigma_{es}(T)$ are always non-empty compact subsets of the complex plane, $\sigma_{se}(f(T)) = f(\sigma_{se}(T))$ and $\sigma_{es}(f(T)) = f(\sigma_{es}(T))$ for any analytic function f in a neighborhood of $\sigma(T)$(See [54]). Now we recall some results about $\sigma_{se}(T)$ and $\sigma_{es}(T)$

Theorem 1 ([54])**.** *Let $T \in \mathcal{L}(X)$.*

1. *$\sigma_{se}(T) = \sigma_{se}(T^*)$ and $\sigma_{es}(T) = \sigma_{es}(T^*)$.*

2. *$\partial\sigma(T) \subseteq \sigma_{se}(T)$; where $\partial\sigma(T)$ is the boundary of the spectrum of T.*

3. *$\lambda \in \sigma_{se}(T) \setminus \sigma_{es}(T)$ if and only if λ is an isolated point of $\sigma_{se}(T)$, $\sup_{n\in\mathbb{N}} \dim \dfrac{N(\lambda I - T) + N((\lambda I - T)^n)}{N(\lambda I - T)} < \infty$ and $R(T - \lambda I)$ is closed.*

Let $K(X)$ the closed two-sided ideal in $\mathcal{L}(X)$ of all compact operators and $F(X)$ denotes the set of all finite rank operators on X.

Theorem 2 ([54])**.** *Let $T \in \mathcal{L}(X)$. Then*

$$\begin{aligned}\sigma_{es}(T) &= \bigcap_{K\in K(X), KT=TK} \sigma_{se}(T+K)\\ &= \bigcap_{F\in F(X), FT=TF} \sigma_{se}(T+F)\end{aligned}$$

Let us mention that the mappings $T \to \sigma_{se}(T)$ and $T \to \sigma_{es}(T)$ are not upper semi-continuous at T in general [54, Remark 4.4].

Theorem 3 ([54])**.** *Let $T, T_n \in \mathcal{L}(X)$. and $TT_n = T_nT$ for each positive integer n. Then*

$$\limsup_{n\in\mathbb{N}} \sigma_{se}(T_n) \subset \sigma_{se}(T) \text{ and } \limsup_{n\in\mathbb{N}} \sigma_{es}(T_n) \subset \sigma_{es}(T)$$

3. Generalized Kato spectrum and its essential version

Now, we introduce an important class of bounded operators which involves the concept of semi-regularity.

Definition 1. *An operator $T \in \mathcal{L}(X)$, is said to admit a generalized Kato decomposition, if there exists a pair of closed subspaces (M, N) of X such that :*

1. *$X = M \oplus N$.*

2. *$T(M) \subset M$ and $T|M$ is semi-regular.*

3. *$T(N) \subset N$ and $T|N$ is quasi-nilpotent (i.e $\sigma(T|N) = \{0\}$.*

(M, N) is said a generalized Kato decomposition of T, abbreviated as GKD(M, N).

If we assume in the definition above that $T|N$ is nilpotent, then there exists $d \in \mathbb{N}$ for which $(T|N)^d = 0$. In this case T is said to be of Kato type of order d. Clearly, every semi-regular operator is Kato type with $M = X$ and $N = \{0\}$ and a quasi-nilpotent operator has a GKD with $M = \{0\}$ and $N = X$. Note that if T is essentialy semi-regular then N is finite-dimensional and $T|N$ is nilpotent, since every quasi-nilpotent operator on a finite-dimensional space is nilpotent. Discussions of operators which admit a generalized decomposition may be found in [43, 44].
For every operator $T \in \mathcal{L}(X)$, let us define the Kato type spectrum and the generalized Kato spectrum as follows respectively:

$$\sigma_k(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not of Kato type}\}$$

$$\sigma_{gk}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not admit a generalized Kato decomposition}\}$$

$\sigma_{gk}(T)$ is not necessarily non-empty. For example, each quasi-nilpotent operator T has empty generalized Kato spectrum.
The following results shows that the generalized Kato spectrum of a bounded operator is a closed subset of the spectra $\sigma(T)$ of T. The next theorem is due to Q. Jiang , H. Zhong ([30, Theorem 2.2]) :

Theorem 4. *Suppose that $T \in \mathcal{L}(X)$, admits a GKD(M, N). Then there exists an open disc $\mathbb{D}(0, \epsilon)$ for which $\lambda I - T$ is semi-regular for all $\lambda \in \mathbb{D}(0, \epsilon) \setminus \{0\}$*

Since $\sigma_{gk}(T) \subseteq \sigma_k(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{se}(T)$, as a straightforward consequence of Theorem 4, we easily obtain that these spectra differ from each other on at most countably many isolated points.

Proposition 1 ([1, 30])**.** *The sets $\sigma_{se}(T) \setminus \sigma_{gk}(T)$, $\sigma_{se}(T) \setminus \sigma_k(T)$, $\sigma_{es}(T) \setminus \sigma_k(T)$, $\sigma_{es}(T) \setminus \sigma_{gk}(T)$ and $\sigma_k(T) \setminus \sigma_{gk}(T)$ are at most countable.*

For $T \in \mathcal{L}(X)$, there are two linear subspaces of X defined in [43], the quasi-nilpotent part $H_0(T)$ of T:

$$H_0(T) = \left\{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\right\}$$

and the analytical core $K(T)$ of T:

$$K(T) = \{x \in X : \text{ there exist a sequence } (x_n) \text{ in } X \text{ and a constant } \delta > 0$$

such that

$$Tx_1 = x, Tx_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n \in \mathbb{N}\}$$

It easily follows, from the definitions, that $H_0(T)$ and $K(T)$ are generally not closed and $T(K(T)) = K(T)$. Observe that if Y is a closed subspace of X such that $T(Y) = Y$, then $Y \subset K(T)$ [60, Proposition 2]. Furthermore if T is quasi-nilpotent then $H_0(T) = X$ and $K(T) = \{0\}$.

Theorem 5 ([1]). *Suppose that (M, N) is a GKD for $T \in \mathcal{L}(X)$. Then we have:*

1. *$K(T) = K(T|M)$ and $K(T)$ is closed;*
2. *$K(T) \cap N(T) = N(T|M)$.*

Theorem 6 ([1]). *Assume that $T \in \mathcal{L}(X)$, admits a GKD (M, N). Then*

$$H_0(T) = H_0(T|M) \oplus H_0(T|N) = H_0(T|M) \oplus N \tag{1}$$

Theorem 7 ([20]). *Assume that $T \in \mathcal{L}(X)$, X a Banach space. The following assertions are equivalent:*

1. *0 is an isolated point in $\sigma(T)$;*
2. *$K(T)$ is closed and $X = K(T) \oplus H_0(T)$*
3. *$H_0(T)$ is closed and $X = K(T) \oplus H_0(T)$*
4. *there is a bounded projection P on X such that $R(P) = K(T)$ and $N(P) = H_0(T)$.*

Here $\oplus$ denotes the algebraic direct sum.

Definition 2. *Let $T \in \mathcal{L}(X)$. The generalized Kato essential spectrum is defined by*

$$\sigma_{eg}(T) = \{\lambda \in \mathbb{C}; \lambda I - T \textit{ is not admits a GKD and } R(T) \textit{ is not closed}\}$$

Note that by Theorem 4 the generalized Kato essential spectrum is a closed set of the spectrum $\sigma(T)$ of T and $\sigma_{eg}(T) \subseteq \sigma_{gk}(T)$. Moreover $\sigma_{gk}(T) \setminus \sigma_{eg}(T)$ is at most countable, this is a direct consequence of the following theorem:

Theorem 8 ([5]). *The symmetric difference $\sigma_{gk}(T)\Delta\sigma_{ec}(T)$ is at most countable, where $\sigma_{ec}(T) = \{\lambda \in \mathbb{C} \ ; \ R(\lambda I - T)$ is not closed$\}$.*

Theorem 9 ([4]). *The symmetric difference $\sigma_k(T)\Delta\sigma_{ec}(T)$ is at most countable.*

Proposition 2. *The sets $\sigma_{gk}(T)\setminus\sigma_{eg}(T)$, $\sigma_{ec}(T)\setminus\sigma_{eg}(T)$, $\sigma_k(T)\setminus\sigma_{eg}(T)$, $\sigma_{es}(T)\setminus \sigma_{eg}(T)$, and $\sigma_{se}(T) \setminus \sigma_{eg}(T)$ are at most countable.*

Proof. We have by definition of $\sigma_{eg}(T)$,

$$\sigma_{gk}(T) \setminus \sigma_{eg}(T) \subseteq \sigma_{gk}(T)\Delta\sigma_{ec}(T) \text{ and } \sigma_{ec}(T) \setminus \sigma_{eg}(T) \subseteq \sigma_{gk}(T)\Delta\sigma_{ec}(T)$$

Then by Theorem 8 we obtain the result.

Proposition 3. *If $\lambda \in \partial\sigma(T)$ is non-isolated point, then $\lambda \in \sigma_{eg}(T)$.*

Proof. Let $\lambda \in \partial\sigma(T)$ a non-isolated point. Since $\partial\sigma(T) \subseteq \sigma_{se}(T)$, then $\lambda \in \sigma_{se}(T)$ is non-isolated point, hence $\lambda \in \sigma_{eg}(T)$.

Now we want to study the influence of perturbations on the spectrum. Our hope is that at least some parts of the spectrum remain invariant.

We will consider for every $T \in \mathcal{L}(X)$ the following properties :

(P1) $\sigma_i(T) \neq \emptyset$.

(P2) $\sigma_i(T)$ is closed.

(P3) $\sigma_i(T+U) = \sigma_i(T)$ whenever $TU = UT$ and $\|U\| < \epsilon$ for some $\epsilon > 0$.

(P4) $\sigma_i(T+F) = \sigma_i(T)$ for every $F \in F(X)$ commuting with T.

(P5) $\sigma_i(T+K) = \sigma_i(T)$ for every $K \in K(X)$ commuting with T.

(P6) $\sigma_i(T+Q) = \sigma_i(T)$ for every quasi-nilpotent operator Q commuting with T.

(P7) $\sigma_i(T)$ verifies the spectral mapping theorem: $f(\sigma_i(T)) = \sigma_i(f(T))$ where f is an analytic function defined on a neighborhood of $\sigma(T)$.
The properties of σ_i $(i = se, es, k, gk, eg)$ are summarized in the following table:

Table 1:

	(P1) $\sigma_i \neq \emptyset$	(P2) σ_i closed	(P3) Small com. pert.	(P4) com. fin. rank pert.	(P5) com. comp. pert.	(P6) com. quasi nilp. pert.	(P7) sp. map. theorem
$\sigma_{se}(T)$	yes	yes	yes	no	no	yes	yes
$\sigma_{es}(T)$	yes	yes	yes	yes	yes	yes	yes
$\sigma_k(T)$	no	yes	no	?	no	no	?
$\sigma_{gk}(T)$	no	yes	no	?	no	yes	?
$\sigma_{eg}(T)$	no	yes	no	?	no	no	?

Comments.

1. It well-known that $\partial\sigma(T) \subseteq \sigma_{se}(T)$ and $\partial\sigma_{ef}(T) \subseteq \sigma_{se}(T)$, so both are non-empty (for infinite dimentional Banach spaces). Here σ_{ef} denotes the essential Fredholm spectrum (see the next section).

2. For property (P3) for σ_{se} and σ_{es} see [48].

3. For semi-regular and essentially semi-regular operators the property (P6) was proved in [35], for σ_{gk} is proved in [6].

4. The stability of essentially semi-regular spectrum under commuting compact perturbation was shown in [54], and under not necessary commuting finite rank perturbation in [34].

5. Consider the identity operator in a Hilbert space and let P be a $1-$ dimensional orthogonal projection. Then $I-P$ is not onto and (P4) and (P5) fail for semi-regular operators.

6. The boxes marked by "?" represent open problems.

4. Fredholm, Weyl, and Browder spectra

We introduce some important classes of operators in Fredholm theory. In the sequel, for every operator $T \in \mathcal{L}(X)$, we shall denote by $\alpha(T)$ the nullity of T, defined as $\alpha(T) := \dim N(T)$, whilst the deficiency $\beta(T)$ of T is defined by $\beta(T) := \operatorname{codim} R(T)$ and the number $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. is called the index of T. We recall (see, for example [26]) that for $T \in \mathcal{L}(X)$, the ascent $p(T)$ and the descent $q(T)$ of T are respectively defined by

$$p(T) = \min\{p \in \mathbb{N} \;:\; N(T^p) = N(T^{p+1})\}$$

and

$$q(T) = \min\{q \in \mathbb{N} \;:\; R(T^q) = R(T^{q+1})\}$$

The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X) := \{T \in \mathcal{L}(X) \text{such that } \alpha(T) < \infty \text{ and } R(T) \text{ is closed }\},$$

the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X) := \{T \in \mathcal{C}(X) : \beta(T) < \infty\},$$

the set of semi-Fredholm operators is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X),$$

the set of Fredholm operators is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X),$$

the set of upper semi-Weyl operators is defined by

$$\mathcal{W}_+(X) := \{T \in \Phi_+(X) : \operatorname{ind}(T) \leq 0\},$$

the set of lower semi-Weyl operators is defined by

$$\mathcal{W}_-(X) := \{T \in \Phi_-(X) : \operatorname{ind}(T) \geq 0\},$$

the set of Weyl operators is defined by

$$\mathcal{W}(X) := \mathcal{W}_+(X) \cap \mathcal{W}_-(X) = \{T \in \Phi(X) : \operatorname{ind}(T) = 0\},$$

the set of upper semi-Browder operators operators is defined by

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

the set of lower semi-Browder operators is defined by

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\},$$

the set of Browder operators is defined by

$$\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X) = \{T \in \Phi(X) : p(T), q(T) < \infty\},$$

Theses various classes of operators motivate the definition of several essential spectra:

- $\sigma_{uf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\}$,
- $\sigma_{lf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_-(X)\}$
- $\sigma_{sf}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(X)\} = \sigma_{uf}(T) \cap \sigma_{lf}(T)$,
- $\sigma_{ef}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\} = \sigma_{uf}(T) \cup \sigma_{lf}(T)$,
- $\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_+(X)\}$,
- $\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_-(X)\}$,
- $\sigma_{ew}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}(X)\} = \sigma_{uw}(T) \cup \sigma_{lw}(T)$,
- $\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(X)\}$.
- $\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\}$.
- $\sigma_{eb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(X)\} = \sigma_{ub}(T) \cup \sigma_{lb}(T)$.
- $\sigma_{ec}(T) := \{\lambda \in \mathbb{C} \ : \ R(\lambda I - T) \text{ is not closed}\}$

The subsets $\sigma_{uf}(.)$ and $\sigma_{lf}(.)$ are the Gustafson and Weidmann essential spectra [27]. $\sigma_{sf}(.)$ is defined by Kato [33]. $\sigma_{ef}(.)$ is the Wolf essential spectrum [63, 56, 62]. $\sigma_{ew}(.)$ is the Schechter essential spectrum [27, 56, 57], and $\sigma_{eb}(.)$ denotes the Browder essential spectrum [27, 31, 56], $\sigma_{ec}(T)$ is the Goldberg spectrum (see [25]). $\sigma_{uw}(.)$ is the essential approximate point spectrum [51, 52]. $\sigma_{lw}(.)$ is the essential defect spectrum [52]. $\sigma_{uw}(.)$ and $\sigma_{lw}(.)$ was introduced by Rakočević in [53]. Note that all these sets of essential spectra (except $\sigma_{ec}(T)$) are closed and in general satisfy the following inclusions

$$\sigma_{ec}(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{sf}(T) \subseteq \sigma_{ef}(T) \subseteq \sigma_{ew}(T) \subseteq \sigma_{eb}(T);$$

$$\sigma_{ec}(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{se}(T);$$

$$\sigma_{eg}(T) \subseteq \sigma_{gk}(T) \subseteq \sigma_{k}(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{sf}(T).$$

and

$$\sigma_{eg}(T) \subseteq \sigma_{gk}(T) \subseteq \sigma_{k}(T) \subseteq \sigma_{se}(T).$$

Remark 1. *If λ in the continuous spectrum $\sigma_c(T)$ of T then $R(\lambda - T)$ is not closed. Therefore $\lambda \in \sigma_i(T)$, $i \in \Lambda = \{ec, es, se, lf, uf, ef, ew, eb\}$. Consequently we have*

$$\sigma_c(A) \subset \bigcap_{i \in \Lambda} \sigma_i(A).$$

Proposition 4 ([1])**.** *The following properties hold:*

1. $\partial\sigma_{eb}(T) \subseteq \partial\sigma_{ew}(T) \subseteq \partial\sigma_{ef}(T) \subseteq \partial\sigma_{sf}(T)$.
2. *If $\lambda \in \partial\sigma_{ef}(T)$ is a non-isolated point of $\sigma_{ef}(T)$ then $\lambda \in \sigma_k(T)$. Moreover, similar statements hold if, instead of boundary points of $\sigma_{ef}(T)$, we consider boundary points of $\sigma_{lf}(T)$, $\sigma_{uf}(T)$ and $\sigma_{sf}(T)$.*

Theorem 10 ([59])**.** *Let $T \in \mathcal{L}(X)$. Then*

$$\begin{aligned}\sigma_{ew}(T) &= \bigcap_{K \in K(X)} \sigma(T+K) \\ &= \bigcap_{F \in F(X)} \sigma(T+F)\end{aligned}$$

Theorem 11 ([59])**.** *Let $T \in \mathcal{L}(X)$. Then*

$$\begin{aligned}\sigma_{eb}(T) &= \bigcap_{K \in K(X)} \sigma(T+K) \\ &= \bigcap_{F \in F(X)} \sigma(T+F)\end{aligned}$$

The approximate point spectrum is defined by

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$$

the surjectivity spectrum is defined by

$$\sigma_{su}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}$$

By the closed range theorem we easily know that the approximate point spectrum and the surjectivity spectrum are dual to each other, in the sense that $\sigma_{ap}(T) = \sigma_{su}(T^*)$ and $\sigma_{ap}(T^*) = \sigma_{su}(T)$.

Theorem 12 ([55]). *Let $T \in \mathcal{L}(X)$. Then*

$$\begin{aligned}\sigma_{ub}(T) &= \bigcap_{K\in K(X), KT=TK} \sigma_{ap}(T+K)\\ &= \bigcap_{F\in F(X), FT=TF} \sigma_{ap}(T+F)\end{aligned}$$

and

$$\begin{aligned}\sigma_{lb}(T) &= \bigcap_{K\in K(X), KT=TK} \sigma_{su}(T+K)\\ &= \bigcap_{F\in F(X), FT=TF} \sigma_{su}(T+F).\end{aligned}$$

The properties (P1)-(P7) for these sets of essential spectra defined above are summarized in the following table:

Table 2:

	(P1) $\sigma_i \neq \emptyset$	(P2) σ_i closed	(P3) Small com. pert.	(P4) com. fin. rank pert.	(P5) com. comp. pert.	(P6) com. quasi nilp. pert.	(P7) sp. map. theorem
$\sigma_{eb}(T)$	yes	yes	yes!	yes!	yes!	yes!	yes
$\sigma_{ew}(T)$	yes	yes	yes!	yes!	yes!	yes!	$\supseteq$
$\sigma_{ef}(T)$	yes	yes	yes!	yes!	yes!	yes!	yes
$\sigma_{sf}(T)$	yes	yes	yes!	yes!	yes!	yes!	$\subseteq$
$\sigma_{uf}(T)$	yes	yes	yes!	yes !	yes !	yes !	yes
$\sigma_{lf}(T)$	yes	yes	yes!	yes !	yes!	yes !	yes
$\sigma_{ub}(T)$	yes	yes	yes	yes	yes	yes	yes
$\sigma_{lb}(T)$	yes	yes	yes	yes	yes	yes	yes
$\sigma_{uw}(T)$	yes	yes	yes	yes	no	yes	$\supseteq$
$\sigma_{lw}(T)$	yes	yes	yes	yes	no	yes	$\supseteq$
$\sigma_{ec}(T)$	no	no	no	no	no	no	no

Comments.

1. The boxes marked by "yes!" means that the commutation is not necessary.

2. The properties (P1)-(P7) when are valid for σ_i, $i \in \{lb, ub, lw, uw, lf, uf, ef, eb, ew\}$ see [1], [59], [33] and [48].

3. The property (P7) is valid for σ_i, $i \in \{lb, ub, lf, uf, ef, eb\}$ by [24], [49], [55], [50]. The interested reader may find further results on the spectral mapping Theorem also in Schmoeger [61]. In the same paper Schmoeger has described the set of all $T \in \mathcal{L}(X)$ such that property (P7) holds for σ_i, $i \in \{lw, uw, sf, ew\}$.

4. The table 2 is valid for $\sigma_i(T)$, $i \in \{lb, ub, lw, uw, lf, uf, ef, eb, ew\}$ for all closed densely defined linear operators on X (see [59], [33] and [29]).

5. The closed-range spectrum (or Goldberg spectrum) $\sigma_{ec}(T)$ has not good properties:

   (a) $\sigma_{ec}(T)$ is not necessarily non-empty. For example, $T = 0$.

   (b) $\sigma_{ec}(T)$ may be not closed. There exists an operator A such that $R(A)$ is closed but $R(\lambda I - A)$ is not closed for all $\lambda \in \mathbb{D}(0,1) \setminus \{0\}$

   (c) It is possible that $R(A)$ is closed but $R(A^2)$ is not. let $A = \begin{pmatrix} V & I \\ 0 & 0 \end{pmatrix}$ be an operator defined on $\ell^2 \oplus \ell^2$, where V has the following properties that $V^2 = 0$ and $R(V)$ is not closed. Then $R(A)$ is closed, $R(A^2)$ is not closed, $A^3 = 0$.

   (d) Conversely, it is also possible that $R(A^2)$ is closed but $R(A)$ is not. Let A be defined on ℓ^2 by
   $$A(x_1, x_2, x_3, \dots) = (0, x_1, 0, \frac{1}{3}x_2, 0, \frac{1}{5}x_3, 0, \dots)$$
   The operator A is compact and $R(A)$ is not closed, $A^2 = 0$ and $R(A^2)$ is closed.

   (e) $\sigma_{ec}(T)$ is unstable under nilpotent perturbation. For example, $A = 0$ and N the nilpotent operator defined in (4.). Then $0 \in \sigma_{ec}(A + N)$ but $0 \notin \sigma_{ec}(A)$.

5. B-Fredholm, B-Browder, B-Weyl and quasi-Fredholm spectra

Given $n \in \mathbb{N}$, we denote by T_n the restriction of $T \in \mathcal{L}(X)$ on the subspace $R(T^n)$. According Berkani [7], T is said to be semi B-Fredholm (resp.

B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, viewed as a operator from the space $R(T^n)$ in to itself, is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). Analogously, $T \in \mathcal{L}(X)$ is said to be B-Browder (resp. upper semi B-Browder, lower semi B-Browder, B-Weyl, upper semi B-Weyl, lower semi B-Weyl), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is a Browder operator (resp. upper semi-Browder, lower semi-Browder, Weyl, upper semi-weyl, lower semi-Weyl). T is said to be quasi-Fredholm if there exists $d \in \mathbb{N}$ such that

1. $R(T^n) \cap N(T) = R(T^d) \cap N(T) \quad$ for all $n \geq d$.
2. $R(T^d) \cap N(T)$ and $R(T^d) + N(T)$ are closed in X.

This classes of operators motive the definition of several spectra.

The B-Ferdholm spectrum is defined by

$$\sigma_{bf}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not B-Ferdholm}\},$$

the semi B-Fredholm spectrum is defined by

$$\sigma_{sbf}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not semi B-Fredholm}\},$$

the upper semi B-Fredholm spectrum is defined by

$$\sigma_{ubf}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not upper semi B-Fredholm}\},$$

the lower semi B-Fredholm spectrum is defined by

$$\sigma_{lbf}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not lower semi B-Fredholm}\},$$

the B-Browder spectrum is defined by

$$\sigma_{bb}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not B-Browder}\},$$

the upper semi B-Browder spectrum is defined by

$$\sigma_{ubb}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not upper semi B-Browder}\},$$

the lower semi B-Browder spectrum is defined by

$$\sigma_{lbb}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not lower semi B-Browder}\},$$

the B-Weyl spectrum is defined by

$$\sigma_{bw}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not B-Weyl}\},$$

the upper semi B-Weyl spectrum is defined by

$$\sigma_{ubw}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not upper semi B-Weyl}\},$$

the lower semi B-Weyl spectrum is defined by

$$\sigma_{lbw}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ is not lower semi B-Weyl}\},$$

while the quasi-Fredholm spectrum is defined by

$$\sigma_{qf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not quasi-Fredholm}\}.$$

We have

$$\sigma_{bf}(T) = \sigma_{ubf}(T) \cup \sigma_{lbf}(T),$$
$$\sigma_{bw}(T) = \sigma_{ubw}(T) \cup \sigma_{lbw}(T),$$
$$\sigma_{bb}(T) = \sigma_{ubb}(T) \cup \sigma_{lbb}(T)$$

and

$$\sigma_{qf}(T) \subseteq \sigma_{bf}(T) \subseteq \sigma_{bw}(T) \subseteq \sigma_{bb}(T).$$

Note that all the B-spectra are compact subsets of $\mathbb{C}$ (see [7], [37]) , and may be empty. This is the case where the spectrum $\sigma(T)$ of T is a finite set of poles of the resolvent. Furthermore

$$\sigma_{eg}(T) \subseteq \sigma_{gk}(T) \subseteq \sigma_k(T) \subseteq \sigma_{bf}(T) \subseteq \sigma_{bb}(T) \subseteq \sigma_{bw}(T).$$

The properties (P1)-(P7) for these sets of essential spectra defined above are summarized in the following table:

Comments.

Table 3:

	(P1) $\sigma_i \neq \emptyset$	(P2) σ_i closed	(P3) Small com. pert.	(P4) com. fin. rank pert.	(P5) com. comp. pert.	(P6) com. quasi nilp. pert.	(P7) sp. map. theorem
$\sigma_{bb}(T)$	no	yes	no	yes!	no	?	yes
$\sigma_{bw}(T)$	no	yes	no	yes!	no	?	$\subseteq$
$\sigma_{bf}(T)$	no	yes	no	yes!	no	?	yes
$\sigma_{sbf}(T)$	no	yes	no	yes!	no	?	yes
$\sigma_{ubf}(T)$	no	yes	no	yes!	no	?	yes
$\sigma_{lbf}(T)$	no	yes	no	yes!	no	?	yes
$\sigma_{ubb}(T)$	no	yes	no	yes	no	?	yes
$\sigma_{lbb}(T)$	no	yes	no	yes	no	?	yes
$\sigma_{ubw}(T)$	no	yes	no	yes	no	?	$\supseteq$
$\sigma_{lbw}(T)$	no	yes	no	yes	no	?	$\supseteq$
$\sigma_{qf}(T)$	no	yes	no	yes	no	no	yes

1. Since every operators commutes with the zero operator, $\sigma_{qf}(T)$ cannot have properties (P1), (P3), (P5) and (P6).

2. All properties (P1)-(P7) for these sets of essential spectra are proved by Berkani in [7],[8], [9], [10], [11], [12], [14], [15], [16], [17].

3. If K is a compact operator such that $R(K^n)$ is not closed for every positive integer n, then K is not a B-Fredholm operator. So if F is a finite rank operator, then F is a B-Fredholm operator, but $K + F$ is not a B-Fredholm operator, otherwise $K = K+F-F$ would be a B-Fredholm operator. Hence the class of B-Fredholm operators is not stable under compact perturbation.

4. In the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of Kato type operators (see J.P Labrousse [37]). But in the case of Banach spaces the Kato type operator is also quasi-Fredholm, according to the remark following Theorem 3.2.2 in [37] the converse is true when $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are complemented in the Banach space X. The spectral mapping theorem for σ_{eq} in a Hilbert space was proved in [13]. For Banach spaces the theorem hold for every function f non constant on each component of its domain of definition (see[36]).

6. Essential spectrum and Drazin invertible operators

An operator $T \in \mathcal{L}(X)$ is said to be left Drazin invertible if $p = p(T) < \infty$ and $R(T^{p+1})$ is closed, and is said to be right Drazin invertible if $q = q(T) < \infty$ and $R(T^q)$ is closed, while $T \in \mathcal{L}(X)$ is said to be Drazin invertible if is both left and right Drazine invertible. The Drazin spectrum is defined by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} \ / \ \lambda I - T \text{ is not Drazin invertible}\}$$

The left Drazin spectrum and right Drazin spectrum are defined by

$$\sigma_{lD}(T) := \{\lambda \in \mathbb{C} \ / \ \lambda I - T \text{ is not left Drazin invertible}\}$$

and

$$\sigma_{rD}(T) := \{\lambda \in \mathbb{C} \ / \ \lambda I - T \text{ is not right Drazin invertible}\}$$

We have

$$\sigma_D(T) = \sigma_{lD}(T) \cup \sigma_{rD}(T)$$

It is well know that T is Drazin invertible if and only if T is finite ascent and descent, which is also equivalent to the fact that $T = R \oplus N$ where R is invertible and N is nilpotent (see [40] Corollary 2.2).

Corollary 1. *If $T \in \mathcal{L}(X)$ then $\sigma_{eg}(T) \subseteq \sigma_D(T)$*

Theorem 13 ([9])**.** *Let $T \in \mathcal{L}(X)$. Then*

$$\sigma_{bw}(T) = \bigcap_{F \in F(X)} \sigma_D(T + F)$$

Theorem 14 ([3])**.** *Let $T \in \mathcal{L}(X)$. If $N \in \mathcal{L}(X)$ is a nilpotent operator such that $TN = NT$. Then*

$$\sigma_{lD}(T) = \sigma_{lD}(T + N)$$

7. Essential spectrum and The SVEP theory

Let $T \in \mathcal{L}(X)$. We say that T has the single-valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighborhood $\mathcal{U}_{\lambda_0}$ of λ_0 the only analytic function $f : \mathcal{U}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0, \quad \text{for all } \lambda \in \mathcal{U}_{\lambda_0}$$

is the constant function $f \equiv 0$.
The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.

We collect some basic properties of the SVEP (see [1]):

1. Every operator T has the SVEP at an isolated point of the spectrum.
2. If $p(\lambda I - T) < \infty$, then T has the SVEP at λ.
3. If $q(\lambda I - T) < \infty$, then T^* has the SVEP at λ

For an arbitrary operator $T \in \mathcal{L}(X)$ let us consider the set

$$\Xi(T) = \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$$

The following theorems describe the relationships between an operator which admits a GKD(M, N) and the points where T, or its adjoint T^* have the SVEP.

Theorem 15 ([1]). *Suppose that $T \in \mathcal{L}(X)$ admits a GKD(M,N). Then the following assertions are equivalent:*

1. *T has the SVEP at 0;*
2. *$T|M$ has the SVEP at 0;*
3. *$T|M$ is injective;*
4. *$H_0(T) = N$;*
5. *$H_0(T)$ is closed;*
6. *$H_0(T) \cap K(T) = \{0\}$;*
7. *$H_0(T) \cap K(T)$ is closed.*

Theorem 16 ([1]). *Suppose that $T \in \mathcal{L}(X)$ admits a GKD(M,N). Then the following assertions are equivalent:*

1. *T^* has the SVEP at 0;*
2. *$T|M$ is surjective;*
3. *$K(T) = M$;*
4. *$X = H_0(T) + K(T)$;*
5. *$H_0(T) \cap K(T) = \{0\}$;*
6. *$X = H_0(T) + K(T)$ is norm dense in X.*

Theorem 17 ([1]). *Let $T \in \mathcal{L}(X)$. Then*

$$\sigma_{eb}(T) = \sigma_{ef}(T) \cup \Xi(T) \cup \Xi(T^*) \tag{2}$$

and

$$\sigma_{eb}(T) = \sigma_{ew}(T) \cup \Xi(T) = \sigma_{ew}(T) \cup \Xi(T^*) \tag{3}$$

Note that

$$\Xi(T) \subseteq \sigma_{ap}(T) \text{ and } \sigma(T) = \Xi(T) \cup \sigma_{su}(T)$$

In particular, if T (resp. T^*) has the SVEP then $\sigma(T) = \sigma_{su}(T)$ (resp. $\sigma(T) = \sigma_{ap}(T)$). In the next theorem we consider a situation which occurs in some concrete cases.

Theorem 18. *Let $T \in \mathcal{L}(X)$ an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then*

$$\sigma_{eg}(T) = \sigma_{gk}(T) = \sigma_k(T) = \sigma_{es}(T) = \sigma_{se}(T) = \sigma_{ap}(T).$$

Proof. Since $\lambda \in \partial\sigma(T)$ is non-isolated, according to Proposition 3, we obtain the result

Theorem 19. *Let $T \in \mathcal{L}(X)$ an operator for which $\sigma_{su}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then*

$$\sigma_{eg}(T) = \sigma_{ec}(T) = \sigma_{es}(T) = \sigma_{se}(T) = \sigma_{su}(T).$$

Proof. Since $\lambda \in \partial\sigma(T)$ is non-isolated, then $\sigma_{su}(T)$ cluster in λ. Observe that T^* has the SVEP at $\lambda \in \partial\sigma(T)$, then $\lambda I - T$ does not admit a generalized Kato decomposition and thus $\lambda \in \sigma_{eg}(T)$. So

$$\sigma_{su}(T) = \partial\sigma(T) \subseteq \sigma_{eg}(T) \subseteq \sigma_{gk}(T) \subseteq \sigma_k(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{se}(T) \subseteq \sigma_{su}(T).$$

Thus we obtain the result

Since the ascent implies the SVEP then we have

$$\Xi(T) \subseteq \sigma_{lD}(T) \text{ and } \Xi(T^*) \subseteq \sigma_{rD}(T)$$

The following theorem proves an equality up to $\Xi(T)$ between the left Drazin spectrum and the left B-Fredholm spectrum and by duality we find a similar result holds for the right Drazin spectrum and the right B-Fredholm spectrum.

Theorem 20 ([3])**.** *Let $T \in \mathcal{L}(X)$. Then*

$$\sigma_{lD}(T) = \sigma_{lbf}(T) \cup \Xi(T) \tag{4}$$

and

$$\sigma_{rD}(T) = \sigma_{rbf}(T) \cup \Xi(T^*) \tag{5}$$

Theorem 21 ([3])**.** *Let $T \in \mathcal{L}(X)$. Then*

$$\sigma_{ubb}(T) = \sigma_{qf}(T) \cup \Xi(T) = \sigma_{ubw}(T) \cup \Xi(T) \tag{6}$$

and

$$\sigma_{lbb}(T) = \sigma_{qf}(T) \cup \Xi(T^*) = \sigma_{lbw}(T) \cup \Xi(T^*) \tag{7}$$

Moreover,

$$\sigma_{bb}(T) = \sigma_{bw}(T) \cup \Xi(T) = \sigma_{bw}(T) \cup \Xi(T^*) \tag{8}$$

Corollary 2. *Let $T \in \mathcal{L}(X)$. Then we have*

1. $\sigma_{eb}(T) = \sigma_{ef}(T) \cup \sigma_{eg}(T)$, $\sigma_D(T) = \sigma_{bw}(T) \cup \sigma_{eg}(T)$ *and* $\sigma_{bb}(T) = \sigma_{qf}(T) \cup \sigma_{eg}(T)$.

2. If T has the SVEP then

$$\sigma_{qf}(T) = \sigma_{ubw}(T) = \sigma_{ubb}(T) \tag{9}$$

and

$$\sigma_{bw}(T) = \sigma_{bb}(T) = \sigma_{lbb}(T) = \sigma_{lbw}(T) \tag{10}$$

3. If T^* has the SVEP then

$$\sigma_{qf}(T) = \sigma_{lbw}(T) = \sigma_{lbb}(T) \tag{11}$$

and

$$\sigma_{bw}(T) = \sigma_{bb}(T) = \sigma_{ubb}(T) = \sigma_{ubw}(T) \tag{12}$$

4. If both T, T^* have SVEP then $\sigma_{eg}(T)$ is empty and

$$\sigma_{eb}(T) = \sigma_{ew}(T) = \sigma_{ef}(T) \tag{13}$$

$$\begin{aligned} \sigma_{qf}(T) = \sigma_D(T) = \sigma_{ubb}(T) = \\ = \sigma_{lbb}(T) = \sigma_{bb}(T) = \sigma_{lbw}(T) = \sigma_{ubw}(T) = \sigma_{bw}(T) \end{aligned} \tag{14}$$

From the definition of localized SVEP it is easily seen that $\Xi(T) \subseteq acc\sigma_{ap}(T)$; and dually $\Xi(T^*) \subseteq acc\sigma_{su}(T)$, where $accK$ denote the set off all accumulation points of $K \subseteq \mathbb{C}$.

Theorem 22 ([19])**.** *Let $T \in \mathcal{L}(X)$ an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then*

$$\sigma_{qf}(T) = \sigma_{ubb}(T) = \sigma_{ubw}(T) = \sigma_{ap}(T) = \sigma_{ub}(T) = \sigma_{uw}(T) = \sigma_{se}(T)$$

By duality we have

Theorem 23 ([19])**.** *Let $T \in \mathcal{L}(X)$ an operator for which $\sigma_{su}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then*

$$\sigma_{qf}(T) = \sigma_{lbb}(T) = \sigma_{lbw}(T) = \sigma_{su}(T) = \sigma_{lb}(T) = \sigma_{lw}(T) = \sigma_{se}(T).$$

By Theorem 18 and Theorem 22 we have

Corollary 3. *Let $T \in \mathcal{L}(X)$ an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then*

$$\begin{aligned} \sigma_{eg}(T) &= \sigma_{qf}(T) = \sigma_{ubb}(T) = \sigma_{ubw}(T) = \sigma_{ap}(T) = \sigma_{ub}(T) = \sigma_{uw}(T) = \sigma_{se}(T) \\ &= \sigma_k(T) = \sigma_{ec}(T) = \sigma_{es}(T) \end{aligned}$$

By duality we obtain by Theorem 19 and Theorem 23

Corollary 4. *Let $T \in \mathcal{L}(X)$ an operator for which $\sigma_{su}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then*

$$\begin{aligned}\sigma_{eg}(T) &= \sigma_{qf}(T) = \sigma_{lbb}(T) = \sigma_{lbw}(T) = \sigma_{su}(T) = \sigma_{lb}(T) = \sigma_{lw}(T) = \sigma_{se}(T) \\ &= \sigma_k(T) = \sigma_{ec}(T) = \sigma_{es}(T)\end{aligned}$$

Example 1. *We consider the Cesaro operator C_p on the classical Hardy space $H_p(\mathbb{D})$, where $\mathbb{D}$ is the open unit disc of $\mathbb{C}$ and $1 < p < \infty$. C_p is defined by*

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^{\lambda} \frac{f(\lambda)}{1-\lambda} d\mu, \ \textit{for all } f \in H_p(\mathbb{D}) \textit{ and } \lambda \in \mathbb{D}.$$

The spectrum of the operator C_p is the closed disc Γ_p centred at $\frac{p}{2}$ with radius $\frac{p}{2}$, see [1], and $\sigma_{ef}(C_p) \subseteq \sigma_{ap}(C_p) = \partial\Gamma_p$. From Corollary 3 we also have

$$\sigma_{eg}(C_p) = \sigma_{qf}(C_p) = \sigma_{lbb}(C_p) = \sigma_{lbw}(C_p) = \sigma_{ap}(C_p) = \sigma_{lb}(C_p) = \sigma_{lw}(C_p)$$

$$= \sigma_{se}(C_p) = \sigma_k(C_p) = \sigma_{ec}(C_p) = \sigma_{es}(C_p) = \sigma_{ef}(C_p = \partial\Gamma_p$$

8. Application of the quasi-nilpotent perturbations to transport Equations

In this section, we shall apply the results of the quasi-nilpotent perturbations to the one-dimensional transport equation on L_p-spaces, with $p \in [1,\ \infty)$. Let

$$X_p = L_p([-a,a] \times [-1,1], dx\, d\xi), a > 0 \text{ and } p \in [1,\infty).$$

We consider the boundary spaces :

$$X_p^o := L_p[\{-a\} \times [-1,0], |\xi| d\xi] \times L_p[\{a\} \times [0,1], |\xi| d\xi] := X_{1,p}^o \times X_{2,p}^o$$

and

$$X_p^i := L_p[\{-a\} \times [0,1], |\xi| d\xi] \times L_p[\{a\} \times [-1,0], |\xi| d\xi] := X_{1,p}^i \times X_{2,p}^i$$

respectively equipped with the norms

$$\|\psi^o\|_{X_p^o} = \left(\|\psi_1^o\|^p_{X_{1,p}^o} + \|\psi_2^o\|^p_{X_{2,p}^o}\right)^{\frac{1}{p}} =$$

$$\left[\int_{-1}^0 |\psi(-a,\xi)|^p |\xi|\, d\xi + \int_0^1 |\psi(a,\xi)|^p |\xi|\, d\xi\right]^{\frac{1}{p}}$$

and

$$\|\psi^i\|_{X_p^i} = \left(\|\psi_1^i\|^p_{X_{1,p}^i} + \|\psi_2^i\|^p_{X_{2,p}^i}\right)^{\frac{1}{p}} =$$

$$\left[\int_0^1 |\psi(-a,\xi)|^p|\xi|\,d\xi + \int_{-1}^0 |\psi(a,\xi)|^p|\xi|\,d\xi\right]^{\frac{1}{p}}.$$

Let $\mathcal{W}_p$ the space defined by:

$$\mathcal{W}_p = \left\{\psi \in X_p \;:\; \xi\frac{\partial\psi}{\partial x} \in X_p\right\}.$$

It is well-known that any function ψ in $\mathcal{W}_p$ has traces on $-a$ and a in X_p^o and X_p^i. They are denoted, respectively by ψ^o and ψ^i, and represent the outgoing and the incoming fluxes.
We define the operator T_H by:

$$\begin{cases} T_H : \mathcal{D}(T_H) \subseteq X_p \longrightarrow X_p \\ \qquad\qquad\qquad\qquad \psi \longrightarrow T_H\psi(x,\xi) = -\xi\dfrac{\partial\psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi) \\ \\ \mathcal{D}(T_H) = \left\{\psi \in \mathcal{W}_p \,\text{such that}\, \psi^i = H(\psi^o)\right\} \end{cases}$$

Where $\sigma(.) \in L^\infty(-1,1)$ and H is bounded from X_p^0 to X_p^i.

The function $\psi(x,\xi)$ represents the number density of gas particles having the position x and the direction cosine of propagation ξ. The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x-direction. The function $\sigma(.)$, is called the collision frequency.
The spectrum of the operator T_0 (i.e., $H = 0$) was analyzed in [46]. in particular we have

$$\sigma(T_0) = \sigma_c(T_0) = \{\lambda \in \mathbb{C} \;:\; Re\lambda \leq -\lambda^*\}, \tag{15}$$

where $\sigma_c(T_0)$ is the continuous spectrum of T_0 and $\lambda^* = -\lim \inf\limits_{|\xi|\to 0} \sigma(\xi)$, (for more detail see [46]).
Combining the inclusions in Remark 1 with Eq. (15) we obtain

$$\sigma_{ei}(T_0) = \{\lambda \in \mathbb{C} \;:\; Re\lambda \leq -\lambda^*\},$$

$$i \in \{ec, es, se, lf, uf, ef, ew, uw, lw, eb, ub, lb\} \tag{16}$$

Let us now consider the resolvent equation for T_H

$$(\lambda - T_H)\psi = \varphi \tag{17}$$

where φ is a given element of X_p and the unknown ψ must be found in $\mathcal{D}(T_H)$. For $\mathrm{Re}\lambda + \lambda^* > 0$, where $\lambda^* = -\lim \inf_{|\xi|\to 0} \sigma(\xi)$, the solution of (17) is formally given by :

$$\begin{cases} \psi(x,\xi) = \psi(-a,\xi)\, e^{-\frac{(\lambda+\sigma(\xi))|a+x|}{|\xi|}} + \frac{1}{|\xi|}\int_{-a}^{x} e^{\frac{-(\lambda+\sigma(\xi))|x-x^{'}|}{|\xi|}} \psi(x,\xi')\, dx', \\ \text{ if } 0<\xi<1, \\ \\ \psi(x,\xi) = \psi(a,\xi)\, e^{-\frac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}} + \dfrac{1}{|\xi|}\int_{x}^{a} e^{\frac{-(\lambda+\sigma(\xi))|x-x^{'}|}{|\xi|}} \psi(x',\xi)\, dx', \\ \text{ if } -1<\xi<0. \end{cases}$$

where

$$\begin{cases} \psi(a,\xi) = \psi(-a,\xi)\, e^{\frac{-2a(\lambda+\sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|}\int_{-a}^{a} e^{2a\frac{-(\lambda+\sigma(\xi))}{|\xi|}} \psi(x,\xi)\, dx', \\ \text{ if } 0<\xi<1 \\ \\ \psi(-a,\xi) = \psi(a,\xi)\, e^{-\frac{2a(\lambda+\sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|}\int_{x}^{a} e^{\frac{-(\lambda+\sigma(\xi))|a-x|}{|\xi|}} \psi(x,\xi)\, dx, \\ \text{ if } -1<\xi<0 \end{cases}$$

In the sequel we shall consider the following operators:

$$\begin{cases} M_\lambda : X_p^i \longrightarrow X_p^0, M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \quad \text{where} \\ M_\lambda^+ u = u(-a,\xi)\, e^{\frac{-2a}{|\xi|}(\lambda+\sigma(\xi))}, \quad \text{if } -1<\xi<0 \\ M_\lambda^- u = u(a,\xi)\, e^{\frac{-2a}{|\xi|}(\lambda+\sigma(\xi))} \quad \text{if } 0<\xi<1 \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \longrightarrow X_p; B_\lambda = \chi_{(-1,0)}(\xi) B_\lambda^+ u + \chi_{(0,1)(\xi)} B_\lambda^- u \\ (B_\lambda^- u)(x,\xi) = u(-a,\xi)\, e^{\frac{(\lambda+\sigma(\xi))}{|\xi|}|a-x|} \quad \text{if } 0<\xi<1 \\ B_\lambda^+ u(x,\xi) = u(-a,\xi)\, e^{\frac{(\lambda+\sigma(\xi))}{|\xi|}|a-x|} \text{ if } ; -1<\xi<0 \end{cases}$$

$$\begin{cases} G_\lambda : X_p^i \longrightarrow X_p, G_\lambda u := (G_\lambda^+\varphi, G_\lambda^-\varphi) \;\; \text{where} \\ \\ G_\lambda^+\varphi = \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|a-x|}\varphi(x,\xi)\,dx \;\; \text{if } 0<\xi<1 \\ \\ G_\lambda^-\varphi = \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|a+x|}\varphi(x,\xi)\,dx \;\; \text{if } -1<\xi<0 \end{cases}$$

$$\begin{cases} C_\lambda : X_p \longrightarrow X_p; C_\lambda\varphi = \chi_{(-1,0)}(\xi)C_\lambda^+\varphi + \chi_{(0,1)}(\xi)C_\lambda^-\varphi \quad \text{where} \\ \\ C_\lambda^-\varphi = \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{x} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|x'-x|}\varphi(x',\xi)\,dx' \;\; \text{if } 0<\xi<1 \\ \\ C_\lambda^+\varphi = \dfrac{1}{|\xi|}\displaystyle\int_{x}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|x'-x|}\varphi(x,\xi)\,dx \;\; \text{if } -1<\xi<0 \end{cases}$$

where $\chi_{(-1,0)}$ and $\chi_{(0,1)}$ denote, respectively the characteristic functions of the intervals $(-1,0)$ and $(0,1)$. The operators $M_\lambda,\ B_\lambda,\ G_\lambda$ and C_λ are bounded on their respective domains respectively, by $e^{-2a(Re\lambda+\lambda*)}$, $[p(\mathrm{Re}\lambda+\lambda^*)]^{\frac{-1}{p}}$, $[(\mathrm{Re}\lambda+\lambda^*)]^{\frac{-1}{q}}$ and $[(\mathrm{Re}\lambda+\lambda^*)]^{-1}$ where q denotes the conjugate of p. We define the real λ_0 by

$$\lambda_0 = \begin{cases} -\lambda^*, \quad \text{if} \quad ||H|| \le 1 \\ \\ \frac{1}{2a}\log||H|| - \lambda^* \quad \text{if} \quad ||H|| > 1 \end{cases}$$

It follows from the norm estimate of M_λ that, for $\mathrm{Re}\lambda > \lambda_0$, $||M_\lambda H|| < 1$ and consequently

$$\psi_0 = \sum_{n=0}^{+\infty}(M_\lambda H)^n G_\lambda\varphi \tag{18}$$

On the other hand, we have

$$\begin{aligned}\psi &= B_\lambda H\psi_0 + C_\lambda\varphi \\ &= (B_\lambda H\sum_{n=0}^{+\infty}(M_\lambda H)^n G_\lambda + C_\lambda)\varphi\end{aligned}$$

Hence, $\{\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \lambda_0\} \subset \rho(T_H)$ and for $\mathrm{Re}\lambda > \lambda_0$

$$(\lambda - T_H)^{-1} = B_\lambda H(I - M_\lambda H)^{-1}G_\lambda + C_\lambda \tag{19}$$

Theorem 24. *Suppose that the boundary operator H is quasi-nilpotent operator, then*

$$\sigma_i(T_H) = \sigma_i(T_0), \quad i = lf,\ uf,\ sf,\ ef, ew, eb.$$

Proof. If $\text{Re}\lambda > \lambda_0,$ then $\lambda \in \rho(T_H) \cap \rho(T_0)$ and

$$(\lambda - T_H)^{-1} - (\lambda - T_0)^{-1} = D_\lambda,$$

where

$$D_\lambda = B_\lambda H \sum_{n=0}^{+\infty} (M_\lambda H)^n G_\lambda$$

Since H is a quasi-nilpotent operator, then D_λ is quasi-nilpotent. This implies the statement of theorem.

Next we consider the transport operator

$$A_H = T_H + K$$

where K is the bounded operator given by

$$\begin{cases} K : X_p \longrightarrow X_p \\ \qquad \psi \longrightarrow \displaystyle\int_{-1}^{\xi} \kappa(x, \xi, \xi')\psi(x, \xi')\, d\xi' \end{cases}$$

and κ satisfies the following assumptions:

$(\mathbf{H_1}) \begin{cases} \kappa(.,.,.) \text{ is a measurable function form } [-a, a] \times [-1, 1] \times [-1, 1] \text{ to } \mathbb{R} \text{ and} \\ |\kappa(x, \xi, \xi')| \leq c < \infty. \end{cases}$

Lemma 1. *If κ satisfies $(\mathbf{H_1})$ then, for any integer $n \geq 1$*

$$\|K^n\| \leq \frac{2^{\frac{1}{q}+n+1}}{n!} c^n$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $\psi \in X_p$. Holder's inequalities implies that

$$\begin{aligned} |K\psi(x, \xi)| &= \left| \int_{-1}^{\xi} \kappa(x, \xi, \xi')\psi(x, \xi') d\xi' \right| \\ &\leq \left(\int_{-1}^{\xi} |\kappa(x, \xi, \xi')|^q \, d\xi' \right)^{\frac{1}{q}} \left(\int_{-1}^{1} |\psi(x, \xi')|^p \, d\xi' \right)^{\frac{1}{p}} \\ &\leq c(\xi + 1)^{\frac{1}{q}} \left(\int_{-1}^{1} |\psi(x, \xi')|^p \, d\xi' \right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned}\left|K^2\psi(x,\xi)\right| &= \left|\int_{-1}^{\xi}\kappa(x,\xi,\xi')K\psi(x,\xi')d\xi'\right| \\ &\leq c^2\int_{-1}^{\xi}(\xi'+1)^{\frac{1}{q}}d\xi'\left(\int_{-1}^{1}\left|\psi(x,\xi')\right|^p d\xi'\right)^{\frac{1}{p}} \\ &\leq c^2\frac{1}{\frac{1}{q}+1}(\xi+1)^{\frac{1}{q}+1}\left(\int_{-1}^{1}\left|\psi(x,\xi')\right|^p d\xi'\right)^{\frac{1}{p}}\end{aligned}$$

we proceed by induction to obtain

$$|K^n\psi(x,\xi)| \leq c^n\frac{1}{(\frac{1}{q}+1)(\frac{1}{q}+2)\dots(\frac{1}{q}+n)}(\xi+1)^{\frac{1}{q}+n}\left(\int_{-1}^{1}\left|\psi(x,\xi')\right|^p d\xi'\right)^{\frac{1}{p}}$$

then, by Fubini's theorem we deduce

$$\int_{-a}^{a}\int_{-1}^{1}\left|\int_{-1}^{\xi}\kappa(x,\xi,\xi')\psi(x,\xi')d\xi'\right|^p d\xi' dx \leq 2c^n\frac{1}{n!}(\xi+1)^{\frac{1}{q}+n}\|\psi\|_{X_p}^p$$

this shows the result.

Theorem 25. *Let $p \geq 1$ and suppose that the collision operator satisfies* $(\mathbf{H_1})$ *on X_p, then*

$$\sigma_i(A_H) = \sigma_i(T_H), \quad i = lf,\ uf,\ sf,\ ef, ew, eb.$$

Furthermore, if the boundary operator H is quasi-nilpotent operator then

$$\sigma_i(A_H) = \sigma_i(T_0) = \{\lambda \in \mathbb{C}:\ Re\lambda \leq -\lambda^*\}, \quad i = lf,\ uf,\ sf,\ ef, ew, eb.$$

Proof. Let $p \geq 1$. by virtue of Lemma 1 the operator K is quasi-nilpotent. Then (see Table 2., (P6) page 10)

$$\sigma_i(A_H) = \sigma_i(T_H + K) = \sigma_i(T_H), \quad i = lf,\ uf,\ sf,\ ef, ew, eb.$$

Furthermore, if the boundary operator H is quasi-nilpotent, the desired result follows from the relation (16) and Theorem 24.

Remark 2. *Since the differential operator T_H not commute with the collusion operator K, the Theorem 25 is not valid for σ_i when we need the commutation.*

References

[1] P. Aiena. Fredholm and local spectral theory, with applications to multipliers. Kluwer Academic Publishres, 2004.

[2] C. Apostol. The reduced minimum modulus. Michigan Math. J. 32 , 279-294, 1985.

[3] M. Amouch, H. Zguitti. B-Fredholm an Drazin invertible operators through localized SVEP. Mathematica Bohemica, vol. 136 (1), pp. 39-49, 2011.

[4] M. Benharrat, B. Messirdi. Relationship between the Kato essential spectrum and a variant of essential spectrum. To appear in General Mathematics Review 2011.

[5] M. Benharrat, B. Messirdi. On the generalized Kato spectrum. (Submitted).

[6] M. Benharrat, B. Messirdi. Perturbations of generalized Kato spectrum by quasi-nilpotent operators. (Submitted)

[7] M. Berkani. Restriction of an operator to the range of its powers. Studia Math., 140 , 163-175, 2000.

[8] M. Berkani. On a class of quasi-Fredholm operators. Integral Equations Operator Theory 34, 244-249, 1999.

[9] M. Berkani. Index of B-Fredholm operators and generalization of a Weyl theorem. Proc. Amer. Math. Soc. 130(6), 1717-1723, 2001.

[10] M. Berkani. B-Weyl spectrum and poles of the resolvent. J. Math. Anal. Applications, 272(2), 596-603, 2002.

[11] M. Berkani, M. Amouch. Preservation of property (gw) under perturbations. Acta. Sci. Math. (Szeged), 74 , 767-779, 2008.

[12] M. Berkani, A. Arroud. Generalized Weyl's theorem and hyponormal operators. J. Aust. Math. Soc. 76, 291-302, 2004.

[13] M. Berkani , A. Ouahab. Thorme de l'application spectrale pour le spectre essentiel quasi-Fredholm. Proc. Amer. Math. Soc. 125, 763-774, 1997.

[14] M. Berkani , J. J. Koliha. Weyl type theorems for bounded linear operators. Acta Sci. Math. (Szeged) 69, 359-376, 2003.

[15] M. Berkani, M. Sarih. On the semi B-Fredholm operators. Glasgow Math. J. 43, 457-465, 2001.

[16] M. Berkani, H. Zariouh. New extended Weyl type theorems. Matemat. Vesnik, 62(2), 145-154, 2010.

[17] M. Berkani, H. Zariouh. Extended Weyl type theorems and perturbations. Mathematical Proceedings of the Royal Irish Academy. 110, 73-82, 2010.

[18] M. Berkani, H. Zariouh. Perturbations results for Weyl type theorems. Acta. Math. Univer. Come., 2010.

[19] C.R. Carpinto and al. B-Browder spectra and localized SVEP. Rend. Circ. Math. Palermo, 57,239-254, 2008.

[20] A. Dajic and J. J. Koliha. The σg-Drazin inverse and the generalized Mbekhta decomposition. Integr. equ. oper. theory 99, 1-26, 2006.

[21] N. Dunford and J. T. Schwartz. Linears operators. Interscience Publishers Inc. New-York, Part 1, 1958.

[22] S. Grabiner. Ascent, descent, and compact perturbations. Proc. Amer. Math. Soc. 71, 79-80, 1978.

[23] S. Grabiner. Uniform ascent and descent of bounded operators. J. Math. Soc. Japan 34, 317-337, 1982.

[24] B. Gramsch, D.C. Lay. Spectral mapping theorems for essential spectra. Math. Ann. 192 , 17-32, 1971.

[25] S. Goldberg Unbounded Linear Operators. McGraw-Hill, New-York, 1966.

[26] K. Gustafson. On algebic multiplicty. Indiana. Univ. Math. J. 25, 6, 769-781, 1976.

[27] K. Gustafson and J. Weidmann, On the essential spectrum. J. Math. Anal. Appl. 25, 6, 121-127, 1969.

[28] A. Jeribi, Fredholm operators and essential spectra. Arch. Inequal. Appl., 2, no. 2-3, 123-140, 2004.

[29] A. Jeribi, N. Moalla. A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation. J. Math. Anal. Appl. 358, 434-444, 2009.

[30] Q. Jiang , H. Zhong Generalized Kato decomposition, single-valued extension property and approximate point spectrum,J. Math. Anal. Appl. 356, 322-327, 2009.

[31] M.A. Kaashoek and D.C. Lay, . Ascent, descent, and commuting perturbations, Trans. Amer. Math. Soc., 169, 35-47, 1972.

[32] T. Kato. Perturbation theory for nullity, deficiency and other quantities of linear operators. J. Anal. Math. 6, 261-322, 1958.

[33] T. Kato. Perturbation theory for linear operators. Springer-Verlag, New York, 1966.

[34] V. Kordula. The Apostol essential spectrum and finite-dimensional perturbations. Proc. Roy. Irish. Aca. Vol 96A No 1, 105-109, 1996.

[35] V. Kordula and V. Muller. The distance from the Apostol spectrum. Proc. Amer. Math. Soc. 124, 3055-3061, 1996..

[36] V. Kordula and V. Muller. On the axiomatic theory of the spectrum. Studia Math. 119, 109-128, 1996.

[37] J-P. Labrousse. Les opérateurs quasi-Fredholm une géniralisation des opérateurs semi-Fredholm. Rend. Circ. Math. Palermo (2) 29, 161-258, 1987.

[38] J-P. Labrousse. Relation entre deux définitions possible de spectre essentiel d'un opérateur. Rev. Roum. Math. Pures et Appl., Tome XXV, N9, 1391-1394, Bucarest, 1980.

[39] K. Latrach. Some remarks on the essential spectrum of transport operators with abstract boundary conditions. Comm. Pure Appl. Math., 7, 633-647, 1954.

[40] D. C. Lay. Spectral analysis using ascent, descent, nullity and defect. Math. Ann. 184,, 197-214, 1970.

[41] M. Mbekhta. Résolvant généralisé et théorie spectrale. J.Operator Th., 21, 69-105, 1989.

[42] M. Mbekhta. On the generalized resolvent in Banach spaces. J. Math. Anal. Appl., 189, 362-377, 1995.

[43] M. Mbekhta. Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux. Glasgow Math. J. 29, 159-175, 1987.

[44] M. Mbekhta. Sur l'unicité de la décomposition de kato généralisée. Acta Sci. Math. (Szeged) 54 (1990) 367-377.

[45] M. Mbekhta, A. Ouahab. Opérateur s-régulier dans un espace de Banach et théorie spectrale. Pub. Irma. Lille. Vol. 22, No XII, 1990.

[46] M. Mokhtar-Kharroubi, Time asymptotic behaviour and compactness in neutron transport theory. Euro. J. Mech., B Fluids, (**11**), 39-68, 1992.

[47] V. Muller. On the regular spectrum. Operator Theory 31, 363-380, 1994.

[48] V. Muller. spectral theory of linear operators and spectral systems in Banach algebra. Birkhauser, 2007.

[49] R.D. Nussbaum. Spectral mapping theorems and perturbation theorems for Browders essential spectrum. Trans. Amer. Math. Soc. 150, 445-455, 1970.

[50] K.K. Oberai. Spectral mapping theorem for essential spectra. Rev. Roum. Math. Pures et Appl., XXV 3, 365-373, 1980.

[51] V. Rakočević. On one subset of M. Schechter's essential spectrum. Math. Vesnik, 5, 389-391, 1981.

[52] V. Rakočević. Approximate point spectrum and commuting compact. Glasgow Math. J., 28, 193-198, 1986.

[53] V. Rakočević. Semi Browder operators and perturbations. Studia Math. 122 (2), 131-137, 1997.

[54] V.Rakočević. Generalized spectrum and commuting compact perturbations. Proc. Edinb. Math. Soc. 36, 197-209, 1993.

[55] V. Rakočević. Approximate point spectrum and commuting compact perturbations. Glasgow Math. J. 28,193-198, 1986.

[56] M. Schechter. On the essential spectrum of an arbitrary operator. J. Math. Anal. Appl., 13, 205-215, 1966.

[57] M. Schechter. Invariance of essential spectrum. Bull. Amer. Math. Soc., 71, 365-367, 1971.

[58] M. Schechter. Riesz operators and Fredholm perturbations. Bull. Amer. Math. Soc., 74, 1139-1144, 1968.

[59] M. Schechter. Principles of functional analysis. Academic Press, 1971.

[60] C. Schmoeger. On isolated points of the spectrum of a bounded operators. Proc. Ame. Math. Soc. 117, 715-719, 1993.

[61] C. Schmoeger. The spectral mapping theorem for the essential approximate point spectrum. Colloq. Math. 74, 2, 167-176, 1997.

[62] F. Wolf. On the essential spectrum of partial differential boundary problems. Comm. Pure Appl. Math., 12, 211-228, 1959.

[63] F. Wolf. On the invariance of the essential spectrum under a change of the boundary conditions of partial differential operators. Indag. Math., 21, 142-147, 1959.

Mohammed Benharrat
Department of Mathematics and Informatics. ENSET Oran, B.P. 1523 Oran El M'Naouar, Algeria
E-mail: mohammed.benharrat@gmail.com/ Benharrat2006@yahoo.fr

Bekkai Messirdi
Department of Mathematics. University of Oran, Algeria
E-mail: messirdi.bekkai@univ-oran.dz
Received: 15 March 2011
Published: 12 October 2011