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Inverse Scattering Problem for a Difference Sturm-Liouville Equation on the Line

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Abstract. We study the inverse scattering problem for a difference Sturm-Liouville equation on the real axis, where scattering occurs only in one direction. We derive a necessary and a sufficient condition on the scattering data so that the inverse problem is uniquely solvable.

Key Words and Phrases: The inverse spectral problem, scattering problem, Jost solution, Weyl function

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1. Introduction

Consider the following difference Sturm-Liouville equation on the real axis

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in Z, \tag{1}$$

where $a_n > 0$ and b_n are real coefficients. In the case when the coefficients $a_n,\ b_n$ guarantee scattering in both directions, the inverse scattering problem for equation (1) has been investigated in various contexts by many authors (see [1]-[15] and the bibliography therein).

However, in the case of scattering problem for equation (1) has been studied enough.

We will study the inverse scattering problem for equation (1), where the coefficients a_n and b_n satisfy the following assumption:

A. The usual condition for the existence of scattering holds on the negative half-line (see, e.g. [8], [15])

$$\sum_{n<0} |n| \left\{ |a_n - 1| + |b_n| \right\} < \infty \,. \tag{2}$$

B. The boundary problem

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \geq 0,$$

$$y_{-1} = 0,$$

generates a self-adjoint operator L_0 in $\ell_2\,[0,\,\infty)$ with a bounded pure discrete spectrum $\sigma(L_0)$, which has only a finite number of limit points.

Note that assumption B holds only if $a_n \to 0$ and $b_n \to 0$ as $n \to +\infty$.

In the case of unbounded coefficients the inverse problem was studied in [3], using the spectral matrix of equation (1). However, the technique developed in [3] is, in general, not applicable to the inverse problem in the class A,B. Also, the work [12] should be mentioned, in which the inverse problem for the equation (1) with bounded coefficients and continuous spectrum of multiplicity 2 is investigated, using right and left Weyl functions and the known coefficient a_{-1}.

In this paper we derive conditions on the scattering data which are necessary and sufficient for the existence of unique coefficients $a_n > 0$, b_n satisfying assumptions A and B. To solve the scattering problem we use the Weyl solution instead of the Jost solution referenced to $+\infty$. In addition, there is no Marchenko-type main equation at the right end. For these reasons some classical arguments needed to be modified considerably.

Note that in the case when scattering occurs only at the right end, the inverse problem for the Sturm-Liouville equation was studied in [13].

2. The direct scattering problem

The spaces $\ell_2\,[N,\,\infty)\,,\; \ell_2\,(-\infty,\,N)\,,\; \ell_2\,(-\infty,\infty)$ with the corresponding inner products $< \cdot\,, \cdot >$ are taken in the conventional way (see, for instance, [3]). We denote by Ω the set-of limit points of the spectrum $\sigma(L_0)$.

For the sake of simplicity, in what follows we assume that the set $\sigma(L_0)\bigcup\Omega$ lies in the interval $(-2,\,2)$.

It is known (see [3], [17]) the eigenvalues $\lambda_n, \quad n = 1, 2, \ldots,$ of the self –adjoint operator L_0 are simple. Consequently, the spectral function of the operator L_0, which we denote by $\rho(\lambda)$, is a step function concentrated at the points $\lambda_n, \quad n = 1, 2, \ldots.$

Denote by $P_n(\lambda)$, $Q_n(\lambda)$ the solutions of equation (1) satisfying the conditions

$$P_{-1}(\lambda) = \; Q_0(\lambda) = 0, \; P_0(\lambda) = 1, \; Q_1(\lambda) = a_0^{-1}.$$

Consider the spectral function

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \frac{1}{\alpha_n},$$

where

$$\alpha_n = \sum_{k=0}^{\infty} P_k^2(\lambda_n), \sum_{n=1}^{\infty} \frac{1}{\alpha_n} = 1.$$

Following [3], [17], [18], we introduce the Weyl function $m(\lambda) =< R_\lambda \delta,\, \delta >$ of the operator L_0, where R_λ is the resolvent of the operator L_0 and $\delta = (1, 0, 0, ...) \in \ell_2\,[0, \infty)$. The Weyl function is related to the spectral function (see [1],[3]) by the equality

$$m(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(\tau)}{\tau - \lambda},$$

which implies that

$$m(\lambda) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n (\lambda_n - \lambda)}. \tag{3}$$

Note that the Weyl function $m(\lambda)$ may be used to reconstruct the spectral function $\rho(\lambda)$. Indeed, by the Stieltjes-Perron formula (see [1], ch.3, §4), for $\lambda \neq \lambda_n, n = 1, 2, ...,$ we have

$$\rho(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_{-1}^{\lambda} Im\, m(\tau + i\varepsilon) d\tau. \tag{3$'$}$$

At the points $\lambda_n,\ n = 1, 2, ...,$ we define the function by left continuity, that is, by the formula

$$\rho(\lambda_n) = \lim_{\lambda \to \lambda_n - 0} \rho(\lambda). \tag{3$''$}$$

We also introduce the Weyl solution

$$\psi_n(\lambda) = Q_n(\lambda) + m(\lambda) P_n(\lambda),\ n \in Z, \tag{4}$$

of equation (1). By (3) and (4), the Weyl solution is analytic on the whole complex λ–plane except for the simple poles $\lambda_k, \quad k = 1, 2, \ldots,$ (the point $\lambda = 0$ is a nonisolated singularity of the Weyl solution). In addition, it is known (see, for instance, [3] or [18]) that for the equality $\psi_n(\lambda) = (R_\lambda \delta)_n$ is valid. Consequently, for every $N > -\infty$ the Weyl solution belongs to $\ell_2\,[N, \infty)$ with respect to the variable n. By virtue of the well-known relation $\frac{d}{d\lambda}R_\lambda = R_\lambda^2$ this last property is also shared by $\frac{d}{d\lambda}\psi_n(\lambda)$.

Now consider equation (1) with parameter $\lambda = z + z^{-1}$, where $|z| \leq 1$. Throughout what follows we assume that $\lambda_n = z_n + z_n^{-1}$, where $|z_n| = 1, Im z_n > 0, z_n \to i$ as $n \to \infty$. We introduce the notation

$$M(z) = m\left(z + z^{-1}\right), \tag{5}$$

$$\Psi_n(z) = \psi_n\left(z + z^{-1}\right). \tag{6}$$

Now suppose that condition (2) is satisfied. Then equation (1) with parameter $\lambda = z + \frac{1}{z}$ possesses a Jost solution $f_n(z)$ (see, for example, [8] and [15]), which can be represented in the form

$$f_n(z) = \sum_{m \leq n} K(n,m) z^{-m}, n \in Z, \tag{7}$$

where the kernel $K(n,m)$ satisfies the relations

$$K(n,n) > 0, \;\; K(n,n) \to 1 \quad as \quad n \to -\infty,$$

$$K(n,m) = O\Big(\sum_{r < [\frac{n+m}{2}]} \{|a_r - 1| + |b_r|\}\Big), \tag{8}$$

$$as \quad n + m \to -\infty,$$

here, $[\cdot]$ denotes the integer part of a number. Consider the Jost solution $f_n(z)$. According to (7), (8), for any n the solution $f_n(z)$ is an analytic function of z in the disc $|z| < 1$ it is continuous up to its boundary. For all such that $|z| = 1, z^2 \neq 1$ the pairs $f_n(z)$, $\overline{f_n(z)} = f_n\left(\frac{1}{z}\right)$ form the fundamental systems of solutions of equation (1) because their Wronskian $W\left[f_n(z), \overline{f_n(z)}\right] = a_n\left(f_n(z)\overline{f_{n+1}(z)} - f_{n+1}(z)\overline{f_n(z)}\right)$ is equal to $z - z^{-1}$ (see [8], [15]). Hence, the solution $\Psi_n(z)$ may be represented as a linear combination of them. On the other hand, by virtue of (4), (6), the solution takes real values for $|z| = 1$, $z \overline{\in} \Omega$. Therefore, for $|z| = 1, z^2 \neq 1$, $z \overline{\in} \Omega$, the following expansions are valid

$$\Psi_n(z) = a(z)\overline{f_n(z)} + \overline{a(z)} f_n(z). \tag{9}$$

Setting $n = -1$ and $n = 0$ in (9) we find that

$$a(z) = \frac{f_0(z) + a_{-1} M(z) f_{-1}(z)}{z - z^{-1}}. \tag{10}$$

Obviously, $a(z)$ is continuous for $|z| = 1$, $z\overline{\in}\ \Omega$, may be except at the points $z = \pm 1$.

Consider the function $M(z)$. By virtue (3),(5) we have

$$M(z) = z\sum_{n=1}^{\infty}\frac{z_n}{\alpha_n(z - z_n)(1 - zz_n)}, \tag{11}$$

which means that the function $M(z)$ is analytic in the disc $|z| < 1$. Taking into account that $\sum\limits_{n=1}^{\infty}\frac{1}{\alpha_n} = 1$, we find from (11) that

$$M(z)z^{-1} \to -1 \quad as \quad z \to 0. \tag{12}$$

By virtue of the properties of the functions $f_n(z)$ and $M(z)$ established above, the function $a(z)$ may be extended by analytic continuation to the disc $|z| < 1$.

Lemma 1. *The function $a(z)$ can only have a finite number of zeros in the disc $|z| < 1$. All these zeros are real and simple.*

Proof. Note that the zeros of the function $a(z)$ are eigenvalues of the operator defined on $\ell_2(-\infty, \infty)$ by the equation (1). By virtue of assumption A,B this operator is self-adjoint. Consequently, the zeros of the function $a(z)$ are real .

Denote the greatest lower bound of the distances between the neighbouring zeros of the function $a(z)$ by r. We shall show that $r > 0$. Assuming the contrary we see that there exist subsequences $\hat{æ}_k, æ_k$ of zeros such that

$$\lim_{k\to\infty}\hat{æ}_k = \lim_{k\to\infty}æ_k = æ_0,$$

where $-1 \leq æ_0 \leq 1$. According to (10) the function $a(z)$ may be extended by analytic continuation to the disc $|z| < 1$. Therefore, if $-1 \leq æ_0 \leq 1$ then by the identity theorem for analytic function we obtain $a(z) \equiv 0$, which is impossible . Now suppose, for instance, that $æ_0 = 1$. Without loss of generality we can assume that $\hat{æ}_k > æ_k > \frac{1}{2}$, $k = 1, 2, ...$ Using (7) –(8) we choose a number $N > 0$ such that for all $n \leq -N$ the inequality $f_n(z) > \frac{1}{2}z^{-n}$ be valid. Then we have

$$\sum_{n\leq -N} f_n(æ_k)f_n(\hat{æ}_k) > \frac{1}{4}\sum_{n\leq -N}æ_k^{-2n} = \frac{1}{4}\frac{æ_k^{2N}}{1 - æ_k^2} > \frac{1}{4^{N+1}}.$$

As the eigenfunctions in the discrete spectrum are orthogonal, using the equalities

$$f_n(æ_k) = c(æ_k)\Psi_n(æ_k), f_n(\hat{æ}_k) = c(\hat{æ}_k)\Psi_n(\hat{æ}_k),$$

which are obvious, implies that

$$0=\sum_{n\in Z} f_n(æ_k)f_n(\hat{æ}_k)=\sum_{n\leq -N} f_n(æ_k)f_n(\hat{æ}_k)+$$
$$+\sum_{n=1-N}^{-1} f_n(æ_k)f_n(\hat{æ}_k)+c(æ_k)c(\hat{æ}_k)\sum_{n\geq 0}\Psi_n(æ_k)\Psi_n(\hat{æ}_k) \quad . \tag{13}$$

Since

$$\lim_{k\to\infty} f_n(\hat{æ}_k)=\lim_{k\to\infty} f_n(æ_k)=f_n(1),$$

we have

$$\lim_{k\to\infty} c(æ_k)c(\hat{æ}_k)=f_{-1}^2(1)\Psi_{-1}^2(1)\geq 0,$$

$$\lim_{k\to\infty}\sum_{n=1-N}^{-1} f_n(æ_k)f_n(\hat{æ}_k)=\sum_{n=1-N}^{-1} f_n^2(1)>0,$$

$$\lim_{k\to\infty}\sum_{n\geq 0}\Psi_n(æ_k)\Psi_n(\hat{æ}_k)=\|R_\lambda\delta\|^2\,|_{\lambda=1}>0.$$

Hence, passing to the limit as $k\to\infty$ on both sides of (13), we arrive at $0>4^{-N-1}$, which is false. Thus, the assumption made above ($r=0$) is wrong and the function $a(z)$ can only have a finite number of zeros $æ_1, æ_2, ..., æ_p$.

Now we will show that these zeros are simple. Let

$$f_n(æ_k)=c_k\Psi_n(æ_k),\ k=1,...,p, \tag{14}$$

Consider the relations

$$\left(1-z^{-2}\right)u_n\vartheta_n=W\left[\dot{u}_{n-1},\vartheta_{n-1}\right]-W\left[\dot{u}_n,\vartheta_n\right]=$$
$$=W\left[\dot{\vartheta}_{n-1},u_{n-1}\right]-W\left[\dot{\vartheta}_n,u_n\right] \ ,$$

where $W\left[u_n,\vartheta_n\right]=a_n\left[u_n\vartheta_{n+1}-u_{n+1}\vartheta_n\right]$ and the dot denotes differentiation with respect to z. Summing these equalities for $u_n=\Psi_n(z)$, $\vartheta_n=f_n(z)$, $z=æ_k$ and taking (14) into account we obtain

$$W\left[\dot{\Psi}_n(æ_k),f_n(æ_k)\right]=\left(1-æ_k^{-2}\right)c_k^{-1}\sum_{m=n+1}^{\infty}\left|f_m(æ_k)\right|^2,$$
$$W\left[\Psi_n(æ_k),\dot{f}_n(æ_k)\right]=\left(1-æ_k^{-2}\right)c_k^{-1}\sum_{m=-\infty}^{n}\left|f_m(æ_k)\right|^2.$$

Consequently

$$\left.\frac{dW\left[\Psi_n(z),f_n(z)\right]}{dz}\right|_{z=æ_k}=\left(1-æ_k^{-2}\right)c_k^{-1}m_k^{-2}.$$

On the other hand, it follows from (9) that

$$a(z)=\frac{W\left[\Psi_n(z),f_n(z)\right]}{z^{-1}-z}\ .$$

Since $a(æ_k)=0$, the last two equalities imply

$$æ_k\frac{da(æ_k)}{dz}=-c_k^{-1}m_k^{-2}\ ,\tag{15}$$

and this means the zeros of the function $a(z)$ are simple.

The proof of lemma is complete.◀

It follows from the lemma that the function $t(z)=a^{-1}(z)$ can only have a finite number of simple real poles $æ_k,\ k=1,...,p$ in the disc $|z|<1$.

On the other hand, if we consider the boundary value problem

$$\begin{aligned}&a_{n-1}^{(N)}y_{n-1}+b_n^{(N)}y_n+a_n^{(N)}y_{n+1}=\lambda y_n,\quad n\le N,\\&y_{N+1}=0,\end{aligned}$$

where

$$a_n^{(N)}=\begin{cases}a_n & for\quad |n|\le N,\\ 1 & for\quad n<-N,\end{cases}\qquad b_n^{(N)}=\begin{cases}b_n & for\quad |n|\le N,\\ 1 & for\quad n<-N,\end{cases}$$

and, following the notation introduced above, set

$$m^{(N)}(\lambda)=\sum_{k=1}^{N+1}\frac{1}{\left(\lambda_k^{(N)}-\lambda\right)\sum\limits_{j=1}^{N}p_j^2\left(\lambda_k^{(N)}\right)}\ ,$$

where $p_{N+1}\left(\lambda_k^{(N)}\right)=0,\ k=1,...,N+1$, and

$$M^{(N)}(z)=m^{(N)}\left(z+z^{-1}\right),a^{(N)}(z)=\frac{f_0^{(N)}(z)+a_{-1}M^{(N)}(z)f_{-1}^{(N)}(z)}{z-z^{-1}}\ ,$$

then we find that $M^{(N)}(z)\to M(z)$ (see [3]) and $f_n^{(N)}(z)\to f_n(z)$ (see [8]) as $N\to\infty$. This means that $a^{(N)}(z)\to a(z)$ as $N\to\infty$. It is easily seen that the function $\frac{1}{a^{(N)}(z)}$ is bounded near the points $z=\pm1$. Now passing to the limit we establish that the function $t(z) = a^{-1}(z)$ is also bounded near the points $z = \pm 1$ and, hence, the function $za^{-1}(z) \prod_{k=1}^{p} \frac{z-æ_k}{1-æ_k z}$ is bounded in the disc $|z| < 1$.

The function $t(z) = a^{-1}(z)$ is referred to as the transmission coefficient. Setting $n = -1$ in (9) we obtain the relation

$$-\frac{1}{a_{-1}} = a(z)\overline{f_{-1}(z)} + \overline{a(z)}f_{-1}(z),$$

according to which $a(z) \neq 0$ for $|z| = 1$, $z \overline{\in} \Omega$. Taking (10) into account this means that the transmission coefficient $t(z)$ is a continuous function for $|z| = 1$, $z \overline{\in} \Omega$, except possibly at the points $z = \pm 1$.

We shall demonstrate that at the points of the set $\bar{\Omega} \backslash \Omega$. Suppose $æ \in \Omega$. It is known (see [12]) that between two poles of the Weyl function $m(\lambda)$ there is exactly one zero of this function. Consequently, there exists a sequence $\tilde{z}_n$such that $|\tilde{z}_n| = 1$, $\tilde{z}_n \to æ$ as $n \to \infty$ and $t(\tilde{z}_n) \to \frac{æ - æ^{-1}}{f_0(æ)}$ as $n \to \infty$. Here, $f_0(æ) \neq 0$. Indeed, $f_n(æ)$ and $\overline{f_n(æ)}$ form a fundamental system of solutions for the equation (1) with $\lambda = æ + \frac{1}{æ}$; hence $P_n(æ)$ can be written as a nontrivial linear combination of them. As a result, by virtue of the condition $P_0(æ) = 1$ we obtain $f_0(æ) \neq 0$.

On the other hand, $t(z_n) \to 0$ as $n \to \infty$. Thus, it has been established that there no exists $\lim_{z \to æ} t(z)$.

Consider formula (10). By virtue of properties of the function $a(z)$ established above, the function $t(z)$ is analytic at the points of the disc $|z| < 1$ at which $a(z)$ is nonzero.

Moreover, by virtue of (7), (10), (12), we have the asymptotic formula

$$zt(z) = K^{-1}(0,0) \ \ as \ \ z \to 0. \tag{16}$$

Let us virtue out some of the properties proved above in the form of a condition that will be needed in what follows.

Condition 1. *The function $t(z) = a^{-1}(z)$ is continuous for $|z| = 1$, may be except at the points $z = \pm 1$ and at the points of the finite set Ω. At the points $z \in \Omega$ it has no limit. The following relations hold*

$$\overline{t(z)} = t\left(\frac{1}{z}\right), \ t(z_n) = 0, \ n = 1, 2, ...$$

The function $t(z)$ may be extended by analytic continuation to the disc $|z| < 1$ except possibly for a simple pole $z = 0$ and, perhaps, a finite number of simple real poles $æ_1, æ_2, ..., æ_p$ in this case, $zt(z) \to d > 0$ as $z \to 0$ and the function $zt(z) \prod_{k=1}^{p} \frac{z-æ_k}{1-æ_k z}$ is bounded in the disc $|z| < 1$.

The collection of quantities

$$\left\{t(z) = a^{-1}(z)\ , |z| = 1;\ æ_k,\ 0 < æ_k^2 < 1;\ m_k > 0,\ \ k = 1, ..., p\right\},$$

is called the scattering data for equation (1). The inverse scattering problem for equation (1) is to recover the coefficients $a_n,\ b_n$ of this equation from the scattering data.

For our problem, we introduce equations relating the scattering data to the Kernel $K\,(n,\,m)$:

$$F(n) = \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{\overline{a(z)}}{a(z)} z^{-n-1} dz + \sum_{k=1}^{p} m_k^2 æ_k^{-n}\,. \tag{17}$$

Theorem 1. *The following relations hold*

$$F(n+m) + \frac{K(n,m)}{K(n,n)} + \sum_{r<n} \frac{K(n,r)}{K(n,n)} F(r+m) = 0, \qquad m < n \leq 0\,, \tag{18}$$

$$K^{-2}(n,n) = 1 + F(2n) + \sum_{r<n} \frac{K(n,r)}{K(n,n)} F(r+n), \quad n \leq 0\,. \tag{19}$$

Proof. Consider identity (9) for $n \leq 0$. We substitute the representation (7) into it, multiply it by $\frac{1}{2\pi i}\frac{1}{a(z)} z^{-(m+1)},\ \ m \leq n$, and integrate along the circle $|z| = 1$. Then, using (4)-(6), (11) taking due account of the fact that

$$\Psi_n(z) z^{-(n+1)} \to h_n \quad as \quad z \to 0,$$

where

$$h_n = \begin{cases} -1 \quad for \quad n = 0, \\ -\frac{1}{a_n \dots a_{-1}} \quad for \quad n \leq -1, \end{cases}$$

and applying the reside theorem we obtain relations (18) and (19).◄

Relation (18) is called Marchenko- type main equation. Applying research techniques available for Marchenko-type equation (see [8]), we can obtain the following necessary condition on the scattering data.

Condition 2. *The following estimate holds*

$$\sum_{n \leq -1} |n|\,|F(n+2) - F(n)| < \infty\,.$$

Now we find the relation between the transmission coefficient and the spectral function of L_0. *Specifically, applying* (5), (10) *gives another necessary condition on the scattering data.*

Condition 3. *The function* $m(\lambda)$ *defined by formulas* (5) *and* (10) *is analytic on the whole complex plane, except at the simple poles* $\lambda_n = z_n + z_n^{-1}$ *takes real values on the real axis. Moreover, the function* $\rho(\lambda)$*, defined by (3'), (3") is the spectral function of some self – adjoint difference Stur Liouville operator* L_0 *on the positive half-axis with the boundary condition* $y_{-1} = 0$.

3. The inverse scattering problem

In the previous sections, we obtained the necessary Conditions 1-3 on the scattering data. Let us prove that these conditions are also sufficient for uniquely recovering the coefficients from the scattering data.

Theorem 2. Suppose that the conditions 1 and 2 are satisfied. Then for any n, with $n \leq 0$ equation (18) has a unique solution in $\ell_p\,(-\infty,\, n-1]\,,\, p = 1, 2$.

Proof. First of all, we note that by virtue of Condition 2, equation (18) is generated by a completely continuous operator on $\ell_p\,(-\infty,\, n-1]\,,\, p = 1, 2$. Therefore, in accordance with the Fredholm alternative, equation (18) has a unique solution $\ell_p\,(-\infty,\, n-1]$ in if the homogeneous equation has no nontrivial solutions in $\ell_p\,(-\infty,\, n-1]$.

Consider the homogeneous equation

$$h_m + \sum_{r<n} F(m+r)h_r = 0, \qquad m < n \leq 0. \tag{20}$$

Since $F(n)$ is real –valued, it may be assumed that all the h_m are real. Moreover, since each solution of equation (20) from $\ell_1\,(-\infty,\, n-1]$ also belongs to $\ell_2\,(-\infty,\, n-1]$ it suffices to show that (20) only has the zero solution in $\ell_2\,(-\infty,\, n-1]$. Let h_m be a solution of (2.1) from $\ell_2\,(-\infty,\, n-1]$. We introduce the function

$$g(z) = \sum_{r<n} h_r z^{-r}, \qquad |z| < 1. \tag{21}$$

The function $g(z)$ obviously belongs to the class H_2 (see [16], ch.2, p.175).

Now multiplying both sides of equation (20) by h_m and summing over m from $-\infty$ to $n-1$, in similar way to [8] and [15], we establish that the function

$$g_0(z) = ig(z), \tag{22}$$

satisfies the relations

$$\frac{\overline{g_0(z)}}{\overline{a(z)}} = \frac{g_0(z)}{a(z)} \quad for \quad almost \quad all \quad z, \quad |z| = 1, \tag{23}$$

$$g_0(æ_k) = 0,\ k = 1, ..., p.$$

By the generalized reflection principle (see, f.i., [11] ch.3, §2) the function

$$g_1(z) = \frac{g_0(z)}{a(z)}, \tag{24}$$

is analytic in the whole complex plane and

$$g_1(z) = \overline{g_1\left(\frac{1}{z}\right)} \quad for \quad |z| > 1. \tag{25}$$

Then, taking account of Condition 1, it follows from (21)-(25) that

$$g_1(z) \to -idh_{n-1}\delta_{on} \quad as \quad z \to \infty,$$

where $d > 0$ and δ_{on} is the Kronecker delta. By Liouville's theorem we have the identity

$$g_1(z) \equiv -idh_{n-1}\delta_{on}\ ,$$

according to which $g_1(z) \equiv 0$ for $n < 0$. Moreover, if $n = 0$ and $h_{-1} \neq 0$,then we have $g_1(z) \equiv -idh_{n-1}$. The last identity contradicts (23). Therefore, $h_{-1} = 0$. Thus, $g_1(z) \equiv 0$ and consequently, $h_m \equiv 0$ for $m < n$.

The theorem is proved.◄

Note that the fact that $F(m)$ is real-valued and equation (18) is uniquely solvable suggest that $K(n, m)$ are real. In addition, it may be shown similarly to [2], [6] and [7] that the function $f_n(z)$ defined by (7), (18), (19) satisfies an equation of the form (1) for $n < 0$ with parameter $\lambda = z + \frac{1}{z}$ and coefficients

$$\begin{aligned} a_n &= \tfrac{K(n,n)}{K(n+1,n+1)}, \\ b_n &= \tfrac{K(n,n-1)}{K(n,n)} - \tfrac{K(n+1,n)}{K(n+1,n+1)}, \end{aligned} \tag{26}$$

Moreover, by virtue of Condition 2 the assumption A holds true (see, for example, [8],[15]).

Theorem 3. *For a set of quantities*

$$\left\{t(z) = a^{-1}(z)\ , |z| = 1; æ_k,\ \ 0 < æ_k^2 < 0;\ m_k > 0, \quad k = 1, ..., p\right\}$$

to be the scattering data of some equation of the form (1) *with coefficients satisfying assumptions A and B, it is necessary and sufficient that Conditions 1-3 be fulfilled.*

Proof. That Conditions 1-3 are necessary was established in 1. Now let Conditions 1-3 be satisfied. Then, for each $n \leq 0$, equation (18) has a unique solution $K(n,m)$. With the help of $K(n,m)$ we use formulas (26) to find $a_n$and b_n for $n \leq -1$. We construct $f_0(z)$and $f_{-1}(z)$according to (7), (18) and (19) and use them to define the function $m(\lambda)$ by formulas (5) an (10). As a third necessary condition, we assume that $\rho(\lambda)$ given by () and () be the spectral function of a difference Sturm-Liuville operator L_0 on the half-axis. Then the coefficients $a_n$and b_n with $n \geq 0$ are recovered from the spectral measure $d\rho(\lambda)$, as in [3]. By Condition 3, $d\rho(\lambda)$, is supported by a bounded countable set. Therefore, assumption B is also satisfied.

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