Azerbaijan Journal of Mathematics
V. 2, No 1, 2012, January
ISSN 2218-6816

Embedding between variable exponent Lebesgue spaces with measures

R.A. Bandaliev

Abstract. In this paper a necessary and sufficient condition for validity of the continuously embedding between variable exponent Lebesgue space with the different measures is found.

Key Words and Phrases: Variable Lebesgue space; measure; embedding; boundedness

2010 Mathematics Subject Classifications: 46E30; 26D15

1. Introduction

It is well known that the variable exponent Lebesgue space appeared in the literature for the first time already in a 1931 by Orlicz [11]. In [11] the Hölder's inequality for variable exponent discrete Lebesgue space was proved. Orlicz also considered the variable exponent Lebesgue space on the real line, and proved the Hölder inequality in this setting.

However, after this paper, Orlicz abandoned the study of variable exponent Lebesgue spaces, to concentrate on the theory of the Orlicz spaces (see also [8]). Further development of this theory was connected with the theory of modular function spaces. The first systematic study of modular spaces is due to Nakano [9]. In the appendix, Nakano mentions explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considers. Somewhat later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see [8]).

The next step in the investigation of variable exponent spaces was given in the paper by Sharapudinov [14] and Kováčik and Rákosník [7]. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [1], [12], [15]). In the papers [6] and [7] was proved the criterion for Hardy inequality on weight functions.

The idea of considering such problems stems from the work [10], in connection with a boundedness of Hardy-Littlewood maximal function. Therefore in [10] was proved a embeddings theorem between discrete weighted Lebesgue spaces with variable exponent. We derive a continuous version of the result by A. Nekvinda in [10]. The approach is different than in the work [10]. Also, note that Theorem 3 of this paper isn't investigated in the [10].

2. Preliminaries

Let S be an arbitrary set and Σ be an $\sigma-$algebra of subsets of S. Let $(S,\,\Sigma,\,\mu)$ and $(S,\,\Sigma,\,\nu)$ be $\sigma-$ finite, complete measure spaces. By $\mathcal{P}\left(S\right)$ we define the set all μ-measurable functions such that $p:\ S\to[1,\infty)$. Also by $\mathcal{Q}\left(S\right)$ we define the set all ν-measurable functions $q:\ S\to[1,\infty)$. The functions $p\in\mathcal{P}\left(S\right)$ and $q\in\mathcal{Q}\left(S\right)$ are called exponents on S. Assume $\underline{p}=\text{ess}\inf\limits_{x\in S}p(x)$ and $\overline{p}=\text{ess}\sup\limits_{x\in S}p(x)$. Let $p'(x)$ be the conjugate exponent function defined by $\dfrac{1}{p(x)}+\dfrac{1}{p'(x)}=1,\ x\in S$. Obviously, $\sup\limits_{x\in S}p'(x)=\overline{p}'=\dfrac{\underline{p}}{\underline{p}-1}$ and $\inf\limits_{x\in S}p'(x)=\underline{p}'=\dfrac{\overline{p}}{\overline{p}-1}$. The Lebesgue measure of a set $A\subset S$ will be denoted by $|A|$.

Definition 1. *Let $p\in\mathcal{P}\left(S\right)$. By $L_{p(x),\,\mu}(S)$ we denote the space of μ-measurable functions f on S such that for some $\lambda_0>0$*

$$\int\limits_S\left(\frac{|f(x)|}{\lambda_0}\right)^{p(x)}\,d\mu(x)<\infty.$$

This set becomes a Banach function space(see [4]) when equipped with the norm

$$\|f\|_{L_{p(x),\,\mu}(S)}=\|f\|_{p(\cdot),\,\mu}=\inf\left\{\lambda>0:\ \int\limits_S\left|\frac{f(x)}{\lambda}\right|^{p(x)}\,d\mu(x)\le1\right\}.$$

For the absolutely continuous measures the space $L_{p(x),\,\mu}\left(S\right)$ coincides with the weighted variable Lebesgue spaces $L_{p(x),\,\omega}\left(S\right)$, where ω is a weight function on S.

The following theorem is characterizes the dual space of $L_{p(x),\,\mu}(S)$.

Theorem 1. *[4] Let $p\in\mathcal{P}\left(S\right)$ with $\overline{p}<\infty$. Then $\left(L_{p(x),\,\mu}(S)\right)^*\cong L_{p'(x),\,\mu}(S)$ and*

$$\|f\|_{L_{p'(\cdot),\,\mu}(S)}\le\|T_f\|_{\left(L_{p(x),\,\mu}(S)\right)^*}\le2\,\|f\|_{L_{p'(\cdot),\,\mu}(S)}\,,$$

where $f\in L_{p'(x),\,\mu}\left(S\right)$. Hereby, T_f is given by $<T_f,h>=\int_S f(x)h(x)\,d\mu(x)$ for $h\in L_{p(x),\,\mu}(S)$.

It is obvious that, the mapping $T_f : h \mapsto \int_S f(x)h(x)\,d\mu(x)$ is a continuous, linear functional on $L_{p(x),\,\mu}(S)$, i.e. $T_f \in \left(L_{p(x),\,\mu}(S)\right)^*$.

In [13] it was proved the following lemma.

Lemma 1. *Let* $1 \le \alpha(x) \le \beta(x) \le \overline{\beta} < \infty$, $x \in \Omega \subset R^n$. *Then*

$$\|f\|_\beta^{\overline{\alpha}} \le \|f^\alpha\|_{\beta/\alpha} \le \|f\|_\beta^{\underline{\alpha}}, \quad \textit{if } \|f\|_\beta \le 1,$$

$$\|f\|_\beta^{\underline{\alpha}} \le \|f^\alpha\|_{\beta/\alpha} \le \|f\|_\beta^{\overline{\alpha}}, \quad \textit{if } \|f\|_\beta \ge 1,$$

where $f^\beta = |f(x)|^{\beta(x)}$, $\overline{\alpha} = \sup\limits_{x\in\Omega} \alpha(x)$ *and* $\underline{\alpha} = \inf\limits_{x\in\Omega} \alpha(x)$. *If* $\alpha(x)$ *and* $\beta(x)$ *are continuous on* Ω, *there exists a point* $x_0 \in \Omega$ *depending on* f *such that* $\|f^\alpha\|_{\beta/\alpha} = \|f\|_\beta^{\alpha(x_0)}$.

Remark 1. *Let* α, $\beta \in \mathcal{P}(S)$ *and* $1 \le \alpha(x) \le \beta(x) \le \overline{\beta} < \infty$, *where* $x \in S$. *Then analog of Lemma 1 is proved in the same way.*

Hereby, we say that a function is simple if it is the finite linear combination of characteristic functions of μ-measurable sets with finite measure, i.e. g is finite if $g(x) = \sum_{i=1}^n a_i \chi_{A_i}(x)$ with $\mu(A_1), \dots, \mu(A_n) < \infty$, $a_1, \dots, a_n \in R$.

In [4] it was proved the following lemma.

Lemma 2. *Let* $p \in \mathcal{P}(S)$ *with* $\overline{p} < \infty$. *Then the set of all* μ*-measurable simple functions on* S *is dense in* $L_{p(x),\,\mu}(S)$.

Remark 2. *Note that in the case when* μ *is Lebesgue measure the Lemma 2 was proved in [5] (see also [13]).*

3. Main results

Assume that the embedding $L_{q(x),\,\nu}(S) \hookrightarrow L_{p(x),\,\mu}(S)$ holds.Then we show that μ is absolutely continuous with respect to ν. Indeed, if there is a set $E \subset \Sigma$ such that $\nu(E) = 0$ and $\mu(E) > 0$, then the function

$$f(x) = \begin{cases} +\infty, & \text{for } x \in E \\ 0, & \text{for } x \in S \setminus E \end{cases}$$

belongs to $L_{q(x),\,\nu}(S)$, but $f \notin L_{p(x),\,\mu}(S)$, i.e. $L_{q(x),\,\nu}(S) \not\hookrightarrow L_{p(x),\,\mu}(S)$. Therefore, it suffices to consider the case when μ is absolutely continuous with respect to ν.

Theorem 2. *Let* $p \in \mathcal{P}(S)$, $q \in \mathcal{Q}(S)$ *and* $1 \le p(x) \le q(x) \le \overline{q} < \infty$. *Then the following conditions are equivalent:*

(a) $L_{q(x),\,\nu}(S)\hookrightarrow L_{p(x),\,\mu}(S)$, *i.e.* $\|f\|_{p(\cdot),\,\mu}\le M\,\|f\|_{q(\cdot),\,\nu}$ *for some* $M>0$;

(b) $\dfrac{d\mu}{d\nu}\in L_{r(x),\nu}(S)$, *where* $r(x)=\frac{q(x)}{q(x)-p(x)}$.

Proof. $(a)\Rightarrow(b)$: Let $\|f\|_{p(\cdot),\,\mu}\le M\,\|f\|_{q(\cdot),\,\nu}$ for all $f\in L_{q(x),\,\nu}(S)$. By Remark 1 from $f\in L_{q(x),\,\nu}(S)$ implies $|f|^{p(x)}\in L_{\frac{q(x)}{p(x)},\,\nu}(S)$. Then the functional

$$\rho_{p(\cdot)}(f)=\int\limits_S |f(x)|^{p(x)}\,d\mu(x),$$

is defined on $L_{\frac{q(x)}{p(x)},\,\nu}(S)$ and is bounded. By Radon-Nikodym theorem, we have

$$\rho_{p(\cdot)}(f)=\int\limits_S |f(x)|^{p(x)}\,d\mu(x)=\int\limits_S |f(x)|^{p(x)}\,\frac{d\mu}{d\nu}(x)\,d\nu(x).$$

Since $|f|^{p(x)}\in L_{\frac{q(x)}{p(x)},\,\nu}(S)$, by Theorem 1 $\dfrac{d\mu}{d\nu}\in L_{r(x),\,\nu}(S)$.

This proves the first part.

$(b)\Rightarrow(a)$: By virtue of Young's inequality we have

$$ab\le\frac{a^{t(x)}}{t(x)}+\frac{b^{t'(x)}}{t'(x)},$$

where $a,b\ge0$, $t(x)\ge1$ and $t'(x)=\dfrac{t(x)}{t(x)-1}$. Taking

$$a=\left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{p(x)},b=\frac{d\mu}{d\nu}\Big/\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu},$$

$$t(x)=\frac{q(x)}{p(x)}\ \textit{and}\ t'(x)=\frac{q(x)}{q(x)-p(x)}=r(x)$$

and applying Young's inequality we get

$$\left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{p(x)}\frac{\frac{d\mu}{d\nu}}{\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}}\le\frac{1}{t(x)}\left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{q(x)}+\frac{1}{r(x)}\left(\frac{\frac{d\mu}{d\nu}}{\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}}\right)^{r(x)}\le$$

$$\left(\sup_{x\in S}\frac{1}{t(x)}\right)\left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{q(x)}+\left(\sup_{x\in S}\frac{1}{r(x)}\right)\left(\frac{\frac{d\mu}{d\nu}}{\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}}\right)^{r(x)}.$$

Obviously, $1 \le \sup\limits_{x\in S} \frac{1}{t(x)} + \sup\limits_{x\in S} \frac{1}{r(x)} \le 2$. Integrating by S, using the definition of the norm we get

$$\int\limits_S \left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{p(x)} \frac{\frac{d\mu}{d\nu}(x)}{\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}}\, d\nu(x) = \frac{1}{\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}} \int\limits_S \left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{p(x)} d\mu(x) \le$$

$$\left(\sup_{x\in S}\frac{1}{t(x)}\right) \int\limits_S \left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{q(x)} d\nu(x) + \left(\sup_{x\in S}\frac{1}{r(x)}\right) \int\limits_S \left(\frac{\frac{d\mu}{d\nu}}{\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}}\right)^{r(x)} d\nu(x) \le$$

$$\sup_{x\in S}\frac{1}{t(x)} + \sup_{x\in S}\frac{1}{r(x)} = A_1 + A_2.$$

Thus

$$\int\limits_S \left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{p(x)} d\mu(x) \le (A_1 + A_2) \left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}. \tag{1}$$

Let $(A_1 + A_2)\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu} \le 1$. Then by (1)

$$\int\limits_S \left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{p(x)} d\mu(x) \le 1.$$

Therefore $\|f\|_{p(\cdot),\,\mu} \le \|f\|_{q(\cdot),\,\nu}$.

Let $(A_1 + A_2)\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu} > 1$. Then by the convexity of $\rho_{p(\cdot)}$ and by (1), we get

$$\rho_{p(\cdot)}\left(\frac{f}{(A_1+A_2)\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu} \|f\|_{q(\cdot),\,\nu}}\right) =$$

$$= \int\limits_S \left(\frac{|f(x)|}{(A_1+A_2)\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu} \|f\|_{q(\cdot),\,\nu}}\right)^{p(x)} d\mu(x) \le$$

$$\frac{1}{(A_1+A_2)\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}} \int\limits_S \left(\frac{|f(x)|}{\|f\|_{q(\cdot),\,\nu}}\right)^{p(x)} d\mu(x) \le 1.$$

Thus, we conclude that

$$\|f\|_{p(\cdot),\,\mu} \le \max\left\{1, (A_1+A_2)\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}\right\}\|f\|_{q(\cdot),\,\nu}\,,$$

where $M = \max\left\{1, (A_1+A_2)\left\|\frac{d\mu}{d\nu}\right\|_{r(\cdot),\,\nu}\right\}$.

For $p(x) = q(x)$ suppose that $L_{r(x),\,\nu} = L_{\infty,\,\nu}$.

The proof of Theorem 2 is complete.◀

Remark 3. *Note that in the case $\mu = \nu$ under conditions $1 \in L_{r(x),\,\mu}(S)$ and atom-less of μ the Theorem 2 was proved in [4] (see also [3]). In the case $S = [0,1]$ and when μ and ν are Lebesgue measures Theorem 2 was proved in [14]. In the case $S = \Omega \subset R^n$ and when μ and ν are Lebesgue measures with $|\Omega| < \infty$ Theorem 2 was proved in [5] (see also [13]). In this paper was found positive response on P. Hästö on embedding theorem between discrete weighted Lebesgue spaces with variable exponent.*

Remark 4. *Let $p, q, \nu, \mu : \mathbb{Z} \mapsto (0,\infty)$. Then the embedding $\ell_{q_n}(\nu_n) \hookrightarrow \ell_{p_n}(\mu_n)$ was proved in [10].*

For weighted Lebesgue spaces with variable exponent the analog of Theorem 2 has the following form.

Corollary 1. *[2] Let $1 \le p(x) \le q(x)$, $\dfrac{1}{s(x)} = \dfrac{1}{p(x)} - \dfrac{1}{q(x)}$ a.e. $x \in \Omega$. Suppose that $v_1(x)$ and $v_2(x)$ are weights in $\Omega \subset R^n$ satisfying the condition*

$$\left\|\frac{v_2}{v_1}\right\|_{L_{s(\cdot),\,v_1}(\Omega)} < \infty.$$

Then

$$L_{q(x),\,v_1(x)}(\Omega) \hookrightarrow L_{p(x),\,v_2(x)}(\Omega),$$

i.e.

$$\|f\|_{L_{p(\cdot),\,v_2}(\Omega)} \le \left\{\max 1, d\left\|\frac{v_2}{v_1}\right\|_{L_{s(\cdot),\,v_1}(\Omega)}\right\}\|f\|_{L_{q(\cdot),\,v_1}(\Omega)},$$

and $d = \sup\limits_{x\in\Omega}\dfrac{p(x)}{s(x)} + \sup\limits_{x\in\Omega}\dfrac{p(x)}{q(x)}$.

Now we prove the following theorem.

Theorem 3. *Let* $p \in \mathcal{P}(S)$, $q \in \mathcal{Q}(S)$ *and* $1 \le q(x) \le p(x) \le \overline{p} < \infty$. *Then the following conditions are equivalent:*

(a) $L_{q(x),\nu}(S) \hookrightarrow L_{p(x),\mu}(S)$, *i.e.* $\|f\|_{p(\cdot),\mu} \le C\,\|f\|_{q(\cdot),\nu}$ *for some* $C > 0$;

(b) $\|\chi_E\|_{p(\cdot),\mu} \le M\,\|\chi_E\|_{q(\cdot),\nu}$ *for all* $E \in S$ *with* $\nu(E) < \infty$, *where* χ_E *is the characteristic function of a set* E *and constant* $M > 0$ *is independent of* E.

Proof. $(a) \Rightarrow (b)$: Let $L_{q(x),\nu}(S) \hookrightarrow L_{p(x),\mu}(S)$, $E \in \Sigma$ with $\chi_E \in L_{q(x),\nu}(S)$. Then there exists a constant $C > 0$ such that $\|\chi_E\|_{p(\cdot),\mu} \le C\,\|\chi_E\|_{q(\cdot),\nu}$, where a constant $C > 0$ is independent of E and $M = C$.

$(b) \Rightarrow (a)$: Let now $\|\chi_E\|_{p(\cdot),\mu} \le M\,\|\chi_E\|_{q(\cdot),\nu}$ for all $E \in S$ with $\nu(E) < \infty$. We prove that $\|f\|_{p(\cdot),\mu} \le C\,\|f\|_{q(\cdot),\nu}$, where f is ν-measurable, simple function and ν-integrable on S. Since μ is absolutely continuous with respect to ν, then f is μ-measurable, simple function and μ-integrable on S. Therefore f is equivalent to the some ν-measurable (also μ-measurable) simple function $g = \sum\limits_{i=1}^{n} a_i\,\chi_{E_i}$, where $\nu(E_i) < \infty$ and $i = 1, 2, \ldots, n$. Consequently, we have

$$\|g\|_{p(\cdot),\mu} \le \sum_{i=1}^{n} |a_i|\;\|\chi_{E_i}\|_{p(\cdot),\mu} \le M \sum_{i=1}^{n} |a_i|\;\|\chi_{E_i}\|_{q(\cdot),\nu} < \infty.$$

Therefore, by virtue of Lemma 2 the proof of Theorem 3 is completed.◂

Acknowledgement.

The authors is supported by the Science Development Foundation under the President of the Republic of Azerbaijan EIF-2010-1(1)-40/06. The author would like to thank the referees for valuable suggestions.

References

[1] E.Acerbi, G.Mingione. *Gradient estimates for a class of parabolic systems,* Duke Math.J., (136)(2007), 285-320.

[2] R.A.Bandaliev. *On an inequality in Lebesgue space with mixed norm and with variable summability exponent,* Mat. Zametki, **3**, (84)(2008), 323-333.(In Russian.) English translation: *Math. Notes,* **3**, (84)(2008), 303-313 (2008).

[3] L.Diening. *Maximal function on generalized Lebesgue spaces* $L^{p(\cdot)}$, Math. Inequal. Appl., **7**, (2)(2004), 245-253.

[4] L.Diening. *Lebesgue and Sobolev spaces with variable exponent,* Habilitation thesis. Germany, Freiburg, 2007.

[5] O.Kováčik, J.Rákosník. *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czech. Math. J., **41**, (1991), 592-618.

[6] F.I.Mamedov, A.Harman. On a weighted inequality of Hardy type in spaces $L_p^(\cdot)$; J. Math. Anal. Appl. (2) 353, 521-530(2009).

[7] F.I.Mamedov, A.Harman. On a Hardy type general weighted inequality in spaces $L_p^(\cdot)$; Integ. Equations and Oper. Theory, (4) 66, 565-592(2010).

[8] J.Musielak, W.Orlicz. *On modular spaces,* Studia Math.**18**, (1959), 49-65.

[9] H.Nakano. *Modulared semi-ordered linear spaces,* Maruzen. Co., Ltd., Tokyo, 1950.

[10] A.Nekvinda.*Embeddings between discrete weighted Lebesgue spaces with variable exponent,* Math. Ineq. Appl., **10**, (1)(2007), 255-265.

[11] W.Orlicz. *Über konjugierte exponentenfolgen,* Studia Math.,**3**, (1931), 200-212.

[12] M.Růžička. *Electrorheological fluids: Modeling and mathematical theory,* Lecture Notes in Math. 1748. Berlin: Springer, 2000.

[13] S.G.Samko. *Convolution type operators in $L^{p(x)}$*, Integ. Trans. and Special Funct., **7**, (1998), 123-144.

[14] I.I.Sharapudinov. *On the basis property of the Haar system in the space $L^{p(t)}([0,1])$ and the principle of localization in the mean,* Matem. Sbornik, **130**, (6)(1986), 275-283 (in Russian): English translation: *Sb. Math.,* **58**, (4)(1987), 279-287.

[15] V.V.Zhikov. Averaging of functionals of the calculus of variations and elasticity theory. Izv. Akad. Nauk Russian. (50)(1986), 675-710.(In Russian.)

Rovshan A. Bandaliev
Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan
E-mail: bandalievr@rambler.ru

Received 5 November 2011
Accepted 2 January 2012