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A martingale central limit theorem with random indices

Yilun Shang*

Abstract. Let $\{X_i\}_{i\geq 1}$ be a martingale difference sequence. Under some regularity conditions, we show that $N_n^{-1/2}(X_1+\cdots+X_{N_n})$ is asymptotically normal, where $\{N_i\}_{i\geq 1}$ is a sequence of positive integer-valued random variables tending to infinity.

Key Words and Phrases: Central limit theorem, Martingale

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1. Introduction

Martingale central limit theorems have very useful unifying properties in the sense that many specific central limit theorems follow as special cases of martingale's. The classical form of the martingale central limit theorem, as presented by Brown [6] and amplified by Dvoretzky [7], Scott [19] and McLeish [15], closely resembles the theorem of Lindeberg and Feller [3].

Let $n\geq 1$ be a fixed integer. Consider a finite sequence $\{X_1,\cdots,X_n\}$ of martingale difference random variables (i.e., X_i is $\mathcal{F}_i$-measurable and $E(X_i|\mathcal{F}_{i-1})=0$ a.s., where $\{\mathcal{F}_i\}_{0\leq i\leq n}$ is an increasing filtration and $\mathcal{F}_0$ is the trivial σ-algebra). Let $S_n=X_1+X_2+\cdots+X_n$ and $v_n^2=\sum_{i=1}^n E(X_i^2)$. The central limit theorem established by Brown [6] and Dvoretzky [7] states that under some Lindeberg-type condition

$$\triangle_n=\sup_{x\in\mathbb{R}}|P(S_n/v_n<x)-\Phi(x)|\to 0, \tag{1}$$

as $n\to\infty$, where $\Phi(x)=(1/\sqrt{2\pi})\int_{-\infty}^x e^{-u^2/2}\mathrm{d}u$ is the standard normal distribution function. More recent studies on martingale cental limit theorems and their convergence rates can be found in e.g. [12, 13, 14, 16, 21]. We refer the reader to books [5] and [9] for more about martingale central limit theorems.

A classical martingale central limit theorem with non-random norming is the following. See [9] for a proof.

*Corresponding author.

Theorem 1. *Let $\{S_i\}_{i\geq 0}$ be a zero-mean martingale sequence relative to $\{\mathcal{F}_i\}_{i\geq 0}$, and let $s_i^2 = E(X_i^2|\mathcal{F}_{i-1})$, where $X_i = S_i - S_{i-1}$ and $S_0 = 0$. Given t, let $\tau_t = \inf\{n : \sum_{i=1}^{n} s_i^2 \geq t\}$. Suppose*

(i) $|X_i| < \alpha < \infty$ for all i,

(ii) $P(\sum_{i=1}^{\infty} s_i^2 = \infty) = 1$,

(iii) $\tau_t/t \xrightarrow{P} 1$ (in probability).

Then

$$S_n/\sqrt{n} \xrightarrow{L} N(0,1), \tag{2}$$

as $n \to \infty$, where "$\xrightarrow{L} N(0,1)$" denotes convergence in distribution to standard normal distribution.

In this paper, we want to generalize Theorem 1 in another direction, that is, consider the central limit theorem for martingale with random indices. In other words, we investigate the sum of a random number of martingale difference sequence $\{X_i\}$. This question is important not only in probability theory itself but in sequential analysis, random walk problems, Monte Carlo methods, etc. Central limit problems for the sum of a random number of independent random variables have been addressed in the pioneer work of Anscombe [2], Rényi [18] and Blum et. al. [4]. More recent study can be found in e.g. [8, 10, 11, 17, 20], most of which, nevertheless, deals with independent cases.

The rest of the paper is organized as follows. In Section 2, we present our martingale cental limit theorem and in Section 3, we give the proof.

2. The Result

Theorem 2. *Let $\{S_i\}_{i\geq 0}$ be a zero-mean martingale sequence relative to $\{\mathcal{F}_i\}_{i\geq 0}$, and let $s_i^2 = E(X_i^2|\mathcal{F}_{i-1})$, where $X_i = S_i - S_{i-1}$ and $S_0 = 0$. Given t, let $\tau_t = \inf\{n : \sum_{i=1}^{n} s_i^2 \geq t\}$. Let $\{N_n\}_{n\geq 1}$ denote a sequence of positive integer-valued random variables such that*

$$\frac{N_n}{\omega_n} \xrightarrow{P} \omega, \tag{3}$$

as $n \to \infty$, where $\{\omega_n\}_{n\geq 1}$ is an arbitrary positive sequence tending to $+\infty$ and ω is a positive constant. Suppose

(i) $|X_i| < \alpha < \infty$ for all i,

(ii) $P(\sum_{i=1}^{\infty} s_i^2 = \infty) = 1$,

(iii) $\tau_t/t \xrightarrow{P} 1$ (in probability),

(iv) there exists some $k_0 \geq 0$ and $c > 0$ such that, for any $\lambda > 0$ and $n > k_0$, we have

$$P\big(\max_{k_0<k_1\leq k_2\leq n} |S_{k_2} - S_{k_1}| \geq \lambda\big) \leq \frac{c \cdot E(S_n - S_{k_0})^2}{\lambda^2}, \tag{4}$$

(v) $Cov(X_i, X_j) \geq 0$ for all i and j.

Then

$$S_{N_n}/\sqrt{N_n} \xrightarrow{L} N(0,1), \tag{5}$$

as $n \to \infty$.

We give some remarks here. Firstly, note that the assumption (iv) is for sufficiently large index of martingale difference sequence X_i, i.e., $\{X_i\}_{i>k_0}$. Secondly, if $\{X_i\}_{i\geq 1}$ is independent, then (iv) automatically holds for $k_0 = 0$ and $c = 1$ by using the Kolmogorov inequality or Doob martingale inequality (see e.g. [3]). Therefore, the assumption (iv) may be regarded as a "relaxed" Kolmogrov inequality. Thirdly, the assumption (v) says that each pair X_s, X_t of $\{X_i\}_{i\geq 1}$ is positively correlated. In view of the independent case [4], it seems likely that the assertion of Theorem 2 still holds when ω is a positive random variable.

3. Proof of Theorem 2

Let $0 < \varepsilon < 1/2$. From (3) it follows that there exists some n_0, for any $n \geq n_0$,

$$P(|N_n - \omega\omega_n| > \varepsilon\omega\omega_n) \leq \varepsilon. \tag{6}$$

For any $x \in \mathbb{R}$, we have

$$P\left(\frac{S_{N_n}}{\sqrt{N_n}} < x\right) = \sum_{n=1}^{\infty} P\left(\frac{S_n}{\sqrt{n}} < x, N_n = n\right). \tag{7}$$

By (6) and (7), we have for $n \geq n_0$,

$$\left| P\left(\frac{S_{N_n}}{\sqrt{N_n}} < x\right) - \sum_{|n-\omega\omega_n|\leq\varepsilon\omega\omega_n} P\left(\frac{S_n}{\sqrt{n}} < x, N_n = n\right)\right| \leq \varepsilon. \tag{8}$$

Let $n_1 = [\omega(1-\varepsilon)\omega_n]$ and $n_2 = [\omega(1+\varepsilon)\omega_n]$. Since ω_n tends to infinity, we have $n_1 \geq k_0$ for large enough n. Note that $S_{n_1} + \sum_{n_1<k\leq n} X_k = S_n$. Then we have for $|n - \omega\omega_n| \leq \varepsilon\omega\omega_n$,

$$P\left(\frac{S_n}{\sqrt{n}} < x, N_n = n\right) \leq P(S_{n_1} < \sqrt{n_2}x + Y, N_n = n), \tag{9}$$

where

$$Y = \max_{n_1<n\leq n_2}\left|\sum_{n_1<k\leq n} X_k\right|. \tag{10}$$

Likewise, we get

$$P\left(\frac{S_n}{\sqrt{n}} < x, N_n = n\right) \geq P(S_{n_1} < \sqrt{n_1}x - Y, N_n = n). \tag{11}$$

Involving the assumption (iv) and (10), we obtain

$$P(Y \geq \varepsilon^{\frac{1}{3}}\sqrt{n_1}) \leq \frac{c(n_2-n_1)\alpha^2}{\varepsilon^{\frac{2}{3}}n_1} \leq 4c\alpha^2\varepsilon^{\frac{1}{3}}, \tag{12}$$

the right-hand side of which is less than 1 when ε is small enough.

Denote by E the event that $Y < \varepsilon^{1/3}\sqrt{n_1}$. By virtue of (8), (9) and (12), we get

$$\begin{aligned} P\left(\frac{S_{N_n}}{\sqrt{N_n}} < x\right) &\leq P\left(\frac{S_{n_1}}{\sqrt{n_1}} < \sqrt{\frac{n_2}{n_1}}x + \varepsilon^{\frac{1}{3}}, E\right) + 4c\alpha^2\varepsilon^{\frac{1}{3}} + \varepsilon \leq \\ &\leq P\left(\frac{S_{n_1}}{\sqrt{n_1}} < \sqrt{\frac{1+2\varepsilon}{1-2\varepsilon}}x + \varepsilon^{\frac{1}{3}}\right) + (4c\alpha^2+1)\varepsilon^{\frac{1}{3}}. \end{aligned} \tag{13}$$

Similarly, from (8), (11) and (12) it follows that

$$P\left(\frac{S_{N_n}}{\sqrt{N_n}} < x\right) \geq P\left(\frac{S_{n_1}}{\sqrt{n_1}} < x - \varepsilon^{\frac{1}{3}}, E\right) - \varepsilon. \tag{14}$$

Using (14), (12) and the assumption (v), we may derive

$$\begin{aligned} P\left(\frac{S_{N_n}}{\sqrt{N_n}} < x\right) &\geq P\left(\frac{S_{n_1}}{\sqrt{n_1}} < x - \varepsilon^{\frac{1}{3}}\right) P(E) - \varepsilon \geq \\ &\geq \left(1 - 4c\alpha^2\varepsilon^{\frac{1}{3}}\right) P\left(\frac{S_{n_1}}{\sqrt{n_1}} < x - \varepsilon^{\frac{1}{3}}\right) - \varepsilon, \end{aligned} \tag{15}$$

where the first inequality is due to application of the FKG inequality (see e.g. [1]).

Now by Theorem 1 we obtain

$$\lim_{n_1\to\infty} P\left(\frac{S_{n_1}}{\sqrt{n_1}} < x\right) = \Phi(x), \tag{16}$$

where $\Phi(x)$ is the standard normal distribution function as defined above. Combining (13), (15) and (16), we then concludes the proof of Theorem 2.

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Yilun Shang
Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China
E-mail: shylmath@hotmail.com

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