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A Hilbert Integral Type Inequality with Non-homogeneous Kernel

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Abstract. It is shown that the Hilbert integral type inequality with non-homogeneous kernel function can be established by introducing two parameters and a proper logarithm function. And the constant factor expressed by product of the gamma function and the Riemann Zeta function is proved to be the best possible. As applications, some equivalent forms are considered.

Key Words and Phrases: weight function, non-homogeneous kernel function, Hurwitz Zeta function, Riemann Zeta function, Bernoulli number

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1. Introduction and Lemmas

Let $f(x),\ g(x) \in L^2(0, +\infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right) f(x)\, g(y)}{x - y} dxdy \leq \pi^2 \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}}. \tag{1.1}$$

And

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x + y} dxdy \leq \pi \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}}. \tag{1.2}$$

They are the famous Hilbert integral inequalities, where the constant factor π^2 and π are the best possible. And the equalities in (1.1) and (1.2) hold if and only if $f(x) = 0$, or $g(x) = 0$.These results can be found in papers [2] and [7]. Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various refinements, extensions and generalizations of (1.2) appear in a great deal of the articles. (such as [1], [3], [4], [5], [10], [11], [12], [13] etc.). However, the research articles of (1.1) are few.

Let $\alpha, \lambda > 0$. Define a non-homogeneous kernel function by

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$$k(x,y)=\begin{cases}\frac{|\ln xy|^{\alpha}}{|1-(xy)^{\lambda}|}, & xy\neq 1\\ 0, & xy=1\end{cases},\quad (x,y)\in(0,+\infty)\times(0,+\infty). \tag{1.3}$$

Throughout the paper we will use frequently the kernel function.

The aim of the present paper is to establish some extensions of (1.1) and to build an inequality of the form

$$\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy\leq\ C_\lambda(\alpha)\left\{\int_0^\infty\omega(x)f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty\omega(x)g^2(x)\,dx\right\}^{\frac{1}{2}}. \tag{1.4}$$

And the constant factor $C_\lambda(\alpha)$and the weight function $\omega(x)$will be given, and some important and especial results will be enumerated, and then some equivalent forms will be considered.

In order to prove our main results, we need to introduce the Hurwitz Zeta functions and some lemmas.

Let $Rez>1$ and $0<q<1$. Then the Hurwitz Zeta function is defined by

$$\zeta(z,q)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{t^{z-1}e^{-qt}}{1-e^{-t}}dt, \tag{1.5}$$

where $\Gamma(z)$is the gamma function.

The Hurwitz Zeta function can be expressed by series as follows:

$$\zeta(z,q)=\sum_{k=0}^{\infty}\frac{1}{(k+q)^z},\quad (Rez>1,\ \ q\neq 0,-1,-2,\ \cdots). \tag{1.6}$$

In case $q=0$, we obtain the famous Riemann Zeta function:

$$\zeta(z)=\sum_{k=1}^{\infty}\frac{1}{k^z},\qquad (Rez>1) \tag{1.7}$$

These results can be found in the papers [6] and [9].

Lemma 1.2. *Let α and λ be positive numbers. Then*

$$\int_0^\infty\frac{s^{\alpha}e^{-\frac{\lambda s}{2}}}{1-e^{-\lambda s}}ds=\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}\zeta(\alpha+1,\tfrac{1}{2}), \tag{1.8}$$

where $\Gamma(z)$is the gamma function and $\zeta(z,q)$is the Hurwitz zeta function, and that $Rez>1$ and $0<q<1$.

Proof. Let $t=\lambda s$. Then $\int\limits_0^\infty \frac{s^\alpha e^{-\frac{\lambda s}{2}}}{1-e^{-\lambda s}}ds = \frac{1}{\lambda^{\alpha+1}}\int\limits_0^\infty \frac{t^\alpha e^{-\frac{1}{2}t}}{1-e^{-t}}dt$, It follows from (1.5) that the equality (1.8) holds.◂

Lemma 1.3. *Let α be a positive number. Then*

$$\zeta\left(\alpha+1,\tfrac{1}{2}\right) = \left(2^{\alpha+1}-1\right)\zeta\left(\alpha+1\right), \tag{1.9}$$

where $\zeta(z)=\sum\limits_{k=1}^{\infty}\frac{1}{k^z}$, *(*$Re z>1$*) is the Riemann Zeta function.*

Proof. When $\alpha>0$, it is known from (1.6) that

$$\zeta\left(\alpha+1,\ \tfrac{1}{2}\right) = \sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{2}\right)^{\alpha+1}} = \sum_{k=0}^{\infty}\frac{2^{\alpha+1}}{\left(2k+1\right)^{\alpha}+1}.$$

Based on (1.7), we have

$\zeta(\alpha+1)=\sum\limits_{k=0}^{\infty}\frac{1}{(k+1)^{\alpha+1}}$and $\frac{1}{2^{\alpha+1}}\zeta\left(\alpha+1\right)=\sum\limits_{k=1}^{\infty}\frac{1}{(2k)^{\alpha+1}}$.

It is obvious that

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\alpha+1}} = \zeta\left(\alpha+1\right) - \frac{1}{2^{\alpha+1}}\zeta\left(\alpha+1\right) = \frac{2^{\alpha+1}-1}{2^{\alpha+1}}\zeta\left(\alpha+1\right).$$

It follows that the equality (1.9) holds.◂

Lemma 1.4. *With the assumptions as Lemma1.1, and define a function by*

$$C_\lambda\left(\alpha\right) = \frac{2\left(2^{\alpha+1}-1\right)}{\lambda^{\alpha+1}}\Gamma\left(\alpha+1\right)\zeta(\alpha+1), \tag{1.10}$$

where $\Gamma\left(z\right)$*is the gamma function and* $\zeta\left(z\right)$ *is the Riemann Zeta function. Then*

$$\int\limits_0^\infty \frac{|\ln u|^\alpha}{|1-u^\lambda|}u^{\frac{\lambda}{2}-1}du = C_\lambda\left(\alpha\right). \tag{1.11}$$

Proof. It is easy to duce that

$$\int\limits_0^\infty \frac{|\ln u|^\alpha}{|1-u^\lambda|}u^{\frac{\lambda}{2}-1}du = \int\limits_0^1 \frac{\left(\ln\frac{1}{u}\right)^\alpha}{1-u^\lambda}u^{\frac{\lambda}{2}-1}du + \int\limits_1^\infty \frac{(\ln u)^\alpha}{u^\lambda-1}u^{\frac{\lambda}{2}-1}du =$$

$$= \int\limits_0^1 \frac{\left(\ln\frac{1}{u}\right)^\alpha}{1-u^\lambda}u^{\frac{\lambda}{2}-1}du + \int\limits_0^1 \frac{\left(\ln\frac{1}{v}\right)^\alpha}{1-v^\lambda}v^{\frac{\lambda}{2}-1}dv =$$

$$= 2\int\limits_0^1 \frac{\left(\ln\frac{1}{u}\right)^\alpha}{1-u^\lambda}u^{\frac{\lambda}{2}-1}du = 2\int\limits_0^\infty \frac{s^\alpha e^{-\frac{\lambda s}{2}}}{1-e^{-\lambda s}}ds.$$

It follows from (1.8) and (1.9) that the equality (1.11) holds.◄

Lemma 1.5. *With the assumptions as Lemma1.1, define a function by*

$$\omega(\alpha,\lambda,x)=\int_0^\infty k\left(x,y\right)\left(\tfrac{x}{y}\right)^{1-\frac{\lambda}{2}}dy.$$

Then

$$\omega\left(\alpha,\lambda,x\right)=C_\lambda\left(\alpha\right)x^{1-\lambda}, \tag{1.12}$$

where $C_\lambda\left(\alpha\right)$is defined by (1.10).

Proof. It is easy to deduce that

$$\omega(\alpha,\lambda,x)=\int_0^\infty k\left(x,y\right)\left(\tfrac{x}{y}\right)^{1-\frac{\lambda}{2}}dy=\int_0^\infty \tfrac{|\ln(xy)|^\alpha}{\left|1-(xy)^\lambda\right|}\left(\tfrac{x}{y}\right)^{1-\frac{\lambda}{2}}dy=x^{1-\lambda}\int_0^\infty \tfrac{|\ln u|^\alpha}{\left|1-u^\lambda\right|}u^{\frac{\lambda}{2}-1}du.$$

It follows from (1.11) that the equality (1.12) holds.◄

Lemma 1.6. *With the assumptions as Lemma1.1, define a function by*

$$\tilde{\omega}(\alpha,\lambda,x)=\int_{\substack{0\\x\neq y}}^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{\left|x^\lambda-y^\lambda\right|}\left(\tfrac{x}{y}\right)^{1-\frac{\lambda}{2}}dy.$$

Then

$$\tilde{\omega}\left(\alpha,\lambda,x\right)=C_\lambda(\alpha)x^{1-\lambda}, \tag{1.13}$$

where $C_\lambda\left(\alpha\right)$is defined by (1.10). *Proof.* It is easy to deduce that

$$\tilde{\omega}(\alpha,\lambda,x)=\int_{\substack{0\\x\neq y}}^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{\left|x^\lambda-y^\lambda\right|}\left(\tfrac{x}{y}\right)^{1-\frac{\lambda}{2}}dy=\int_{\substack{0\\x\neq y}}^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{x^\lambda\left|1-\frac{y^\lambda}{x^\lambda}\right|}\left(\tfrac{x}{y}\right)^{1-\frac{\lambda}{2}}dy=$$

$$=x^{1-\lambda}\int_0^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha}{\left|1-u^\lambda\right|}u^{\frac{\lambda}{2}-1}du=x^{1-\lambda}\int_0^\infty \tfrac{|\ln u|^\alpha}{\left|1-u^\lambda\right|}u^{\frac{\lambda}{2}-1}du.$$

It follows from (1.11) that the equality (1.13) holds.◄

2. Main Results

In this section, we will prove our assertions.

Theorem 2.1. *Let $k(x,y)$ be a function by (1.3), f and g be two real functions. If $\int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\lambda} g^2(x)dx < +\infty$, then*

$$\int_0^\infty \int_0^\infty k(x,y) f(x) g(y)\, dxdy \le C_\lambda(\alpha) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty x^{1-\lambda} g^2(x)\, dx \right\}^{\frac{1}{2}}, \tag{2.1}$$

where $C_\lambda(\alpha)$ is defined by (1.10), and $C_\lambda(\alpha)$ in (2.1) is the best possible. And the equality in (2.1) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We may apply the Cauchy-Schwartz inequality to estimate the left-hand side of (2.1) as follows

$$\int_0^\infty \int_0^\infty k(x,y) f(x) g(y)\, dxdy =$$

$$= \int_0^\infty \int_0^\infty \left\{ (k(x,y))^{\frac{1}{2}} \left(\tfrac{x}{y}\right)^{\frac{2-\lambda}{4}} f(x) \right\} \left\{ (k(x,y))^{\frac{1}{2}} \left(\tfrac{y}{x}\right)^{\frac{2-\lambda}{4}} g(y) \right\} dxdy \le$$

$$\le \left(\int_0^\infty \int_0^\infty k(x,y) \left(\tfrac{x}{y}\right)^{\frac{2-\lambda}{2}} f^2(x) dxdy \right)^{\frac{1}{2}} \left(\int_0^\infty \int_0^\infty k(x,y) \left(\tfrac{y}{x}\right)^{\frac{2-\lambda}{2}} g^2(y) dxdy \right)^{\frac{1}{2}} = \tag{2.2}$$

$$= \left(\int_0^\infty \omega(\alpha,\lambda,x) f^2(x) dx \right)^{\frac{1}{2}} \left(\int_0^\infty \omega(\alpha,\lambda,x) g^2(x) dx \right)^{\frac{1}{2}}, \tag{2.3}$$

where $\omega(\alpha,\lambda,x) = \int_0^\infty k(x,y) \left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}} dy$.

It follows from (1.11) and (2.3) that the inequality (2.1) is valid.

If $f(x) = 0$, or $g(x) = 0$, then the equality in (2.1) is obviously valid. If $f(x) \ne 0$ and $g(x) \ne 0$, then $0 < \int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$ and $0 < \int_0^\infty x^{1-\lambda} g^2(x)dx < +\infty$. Let's consider (2.2). If (2.2) takes the form of the equality, then it is known from the paper [7] (pp.5.) that there exist a pair of non-zero constants c_1 and c_2 such that

$c_1 k(x,y) f^2(x) \left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}} = c_2 k(x,y) g^2(y) \left(\frac{y}{x}\right)^{1-\frac{\lambda}{2}}$ a.e. on $(0,+\infty) \times (0,+\infty.)$

Then we have

$c_1 x^{2-\lambda} f^2(x) = c_2 y^{2-\lambda} g^2(y) = C_0$.(constant) a.e. on $(0,+\infty) \times (0,+\infty)$.

Without losing the generality, we suppose that $c_1 \neq 0$, then

$$\int_0^\infty x^{1-\lambda} f^2(x)\,dx = \tfrac{C_0}{c_1}\int_0^\infty x^{-1}dx.$$

This contradicts that$0 < \int_0^\infty x^{1-\lambda} f^2(x)\,dx < +\infty$. Hence it is impossible to take the equality in (2.2). It shows that it is also impossible to take the equality in (2.1). In other words, the equality in (2.1) holds if and only if$f(x) = 0$, or $g(x) = 0$.

It remains to need only to show that$C_\lambda(\alpha)$ in (2.1) is the best possible.

$\forall\ n \in N$,define two functions by

$$f_n(x) = \begin{cases} x^{\frac{\lambda}{2}} - 1 + \frac{1}{2n}, & x \in (0,1) \\ 0, & x \in [1,\ \infty) \end{cases} \text{ and } g_n(x) = \begin{cases} 0, & x \in (0,1] \\ x^{\frac{\lambda}{2}} - 1 - \frac{1}{2n}, & x \in (1,\ \infty). \end{cases}$$

Then we have

$$\left(\int_0^1 x^{1-\lambda} f_n^2(x)dx\right)^{\frac{1}{2}} = \left(\int_1^\infty x^{1-\lambda} g_n^2(x)dx\right)^{\frac{1}{2}} = \sqrt{n}. \tag{2.4}$$

When $xy = 1$, It is known from (1.3) that the inequality (2.1) is obviously valid.

Consider the case $xy \neq 1$. Let $0 < A \leq C_\lambda(\alpha)$such that the inequality (2.1) is still valid, when $C_\lambda(\alpha)$in (2.1) is replaced by A. By using (2.4), we have

$$\frac{1}{n}\int_0^\infty\int_0^\infty k(x,y)\, f_n(x)\, g_n(y)\, dxdy = \frac{1}{n}\int_0^\infty\int_0^\infty \frac{|\ln xy|^{\alpha} f_n(x) g_n(y)}{\left|1-(xy)^\lambda\right|}dxdy \leq$$

$$\leq\ A\left(\frac{1}{n}\right)\left\{\int_0^\infty x^{1-\lambda} f_n^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty x^{1-\lambda} g_n^2(x)\,dx\right\}^{\frac{1}{2}} = A. \tag{2.5}$$

Let $k(1,xy) = \frac{|\ln xy|^\alpha}{\left|1-(xy)^\lambda\right|}$. Based on (2.5) and then by using Fubini's theorem, we have

$$A \geq \frac{1}{n}\int_0^\infty\int_0^\infty k(1,xy) f_n(x) g_n(y) dxdy =$$

$$= \frac{1}{n}\int_1^\infty y^{\frac{\lambda}{2}} - 1 - \frac{1}{2n}\left(\int_0^1 k(1,xy)x^{\frac{\lambda}{2}} - 1 + \frac{1}{2n}dx\right)dy$$

$$= \frac{1}{n}\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_0^y k(1,u)u^{\frac{\lambda}{2}} - 1 + \frac{1}{2n}du\right)dy =$$

$$= \frac{1}{n}\left\{\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_0^1 k(1,u)u^{\frac{\lambda}{2}} - 1 + \frac{1}{2n}du\right)dy + \right.$$

$$+\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_1^y k(1,u)u^{\frac{\lambda}{2}-1+\frac{1}{2n}}du\right)dy\Bigg\} =$$

$$= \frac{1}{n}\left\{\int_1^\infty n\left(\int_0^1 k(1,u)u^{\frac{\lambda}{2}-1+\frac{1}{2n}}du\right)+\int_1^\infty k(1,u)u^{\frac{\lambda}{2}-1+\frac{1}{2n}}\left(\int_u^\infty y^{-1-\frac{1}{n}}dy\right)du\right\} =$$

$$= \int_0^1 k(1,u)u^{\frac{\lambda}{2}-1+\frac{1}{2n}}du+\int_1^\infty k(1,u)u^{\frac{\lambda}{2}-1-\frac{1}{2n}}du.$$

By Fatou's lemma and (1.11), we have

$$A \geq \varliminf_{n\to\infty}\int_0^1 k(1,u)u^{\frac{\lambda}{2}-1+\frac{1}{2n}}du+\varliminf_{n\to\infty}\int_1^\infty k(1,u)u^{\frac{\lambda}{2}-1-\frac{1}{2n}}du \geq$$

$$\geq \int_0^1 \varliminf_{n\to\infty} k(1,u)u^{\frac{\lambda}{2}-1+\frac{1}{2n}}du+\int_1^\infty \varliminf_{n\to\infty} k(1,u)u^{\frac{\lambda}{2}-1-\frac{1}{2n}}du =$$

$$= \int_0^1 k(1,u)u^{\frac{\lambda}{2}-1}du+\int_1^\infty k(1,u)u^{\frac{\lambda}{2}-1}du =$$

$$= \int_0^\infty k(1,u)u^{\frac{\lambda}{2}-1}du = \int_0^\infty \frac{|\ln u|^\alpha}{|1-u^\lambda|}u^{\frac{\lambda}{2}-1}du = C_\lambda(\alpha).$$

It follows that $A = C_\lambda(\alpha)$in (2.1) is the best possible. Thus the proof of Theorem is Completed.◄

Based on Theorem 2.1, we have the following result when α is odd.

Theorem 2.2. *Let n be a positive integer and $\lambda > 0$. If $\int_0^\infty x^{1-\lambda}f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\lambda}g^2(x)dx < +\infty$, then*

$$\int_0^\infty\int_0^\infty \frac{|\ln xy|^{2n-1}f(x)g(y)}{|1-(xy)^\lambda|}dxdy \quad\leq\quad C_\lambda(2n-1)\left\{\int_0^\infty x^{1-\lambda}f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty x^{1-\lambda}g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{2.6}$$

where $C_\lambda(2n-1) = \frac{2^{2n-1}\left(2^{2n}-1\right)}{n}\left(\frac{\pi}{\lambda}\right)^{2n}B_n$, and the $B_{n's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, etc., and the constant factor

$C_\lambda(2n-1)$*is the best possible. And the equality in (2.6) holds if and only if* $f(x)=0$, *or* $g(x)=0$.

Proof. We need only to verify the constant factor $C_\lambda(2n-1)$in (2.6). When $\alpha=2n-1$, it is known from(1.10) that

$$C_\lambda(2n-1)=\frac{2\left(2^{2n}-1\right)}{\lambda^{2n}}\Gamma(2n)\,\zeta(2n)=$$

$$=\frac{2\left(2^{2n}-1\right)}{\lambda^{2n}}(2n-1)!\sum_{k=1}^{\infty}\frac{1}{k^{2n}}.$$

It is known from the paper [8] that

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n,$$

where the$B_{n's}$ are the Bernoulli numbers, viz. $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}$, *etc.*.

It follows that the constant factor $C_\lambda(2n-1)$in (2.6) is the best possible.◀

Theorem 2.3. *With the assumptions as Theorem 2.1, then*

$$\int_0^\infty\int_0^\infty\frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha f(x)\,g(y)}{|x^\lambda-y^\lambda|}dxdy\ \le\ C_\lambda(\alpha)\left\{\int_0^\infty x^{1-\lambda}f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty x^{1-\lambda}g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{2.7}$$

where $C_\lambda(\alpha)$*is defined by (1.10), and* $C_\lambda(\alpha)$*in (2.7) is the best possible. And the equality in (2.7) holds if and only if* $f(x)=0$, *or* $g(x)=0$.

Proof. We may apply the Cauchy-Schwartz inequality to estimate the left-hand side of (2.7) as follows

$$\int_0^\infty\int_0^\infty\frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha f(x)g(y)}{|x^\lambda-y^\lambda|}dxdy=\int_0^\infty\int_0^\infty\left(\frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{|x^\lambda-y^\lambda|}\right)^{\frac{1}{2}}\left(\frac{x}{y}\right)^{\frac{2-\lambda}{4}}f(x)\left(\frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{|x^\lambda-y^\lambda|}\right)^{\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{2-\lambda}{4}}\times$$

$$\times g(y)dxdy\le\left(\int_0^\infty\int_0^\infty\frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{|x^\lambda-y^\lambda|}\left(\frac{x}{y}\right)^{\frac{2-\lambda}{2}}f^2(x)dxdy\right)^{\frac{1}{2}}\left(\int_0^\infty\int_0^\infty\frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{|x^\lambda-y^\lambda|}\left(\frac{y}{x}\right)^{\frac{2-\lambda}{2}}g^2(y)dxdy\right)^{\frac{1}{2}}$$

$$=\left(\int_0^\infty\tilde{\omega}(\alpha,\lambda,x)f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty\tilde{\omega}(\alpha,\lambda,x)g^2(x)dx\right)^{\frac{1}{2}}, \tag{2.8}$$

where $\tilde{\omega}(\alpha,\lambda,x)=\int\limits_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{\left|x^\lambda-y^\lambda\right|}\left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}}dy.$

It follows from (1.13), (1.10) and (2.8) that the inequality (2.7) is valid.◄

The rest is similar to the proof of Theorem 2.1, it is omitted.

In particular, when α is odd, based on (2.7), an extension of (1.1) can be obtained.

Theorem 2.4. *Let n be a positive integer and $\lambda>0$. If $\int\limits_0^\infty x^{1-\lambda}f^2(x)dx<+\infty$ and $\int\limits_0^\infty x^{1-\lambda}g^2(x)dx<+\infty$, then*

$$\int\limits_0^\infty\int\limits_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n-1}f(x)\,g(y)}{x^\lambda-y^\lambda}dxdy\ \le\ C_\lambda(2n-1)\left\{\int\limits_0^\infty x^{1-\lambda}f^2(x)\,dx\right\}^{\frac{1}{2}}\times$$

$$\times\left\{\int\limits_0^\infty x^{1-\lambda}g^2(x)\,dx\right\}^{\frac{1}{2}},\tag{2.9}$$

where $C_\lambda(2n-1)=\frac{2^{2n-1}\left(2^{2n}-1\right)}{n}\left(\frac{\pi}{\lambda}\right)^{2n}B_n$, and the$B_{n's}$ are the Bernoulli numbers, viz. $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}$, $B_5=\frac{5}{66}$, etc., and the constant factor $C_\lambda(2n-1)$is the best possible. And the equality in (2.9) holds if and only if $f(x)=0$, or $g(x)=0$.

3. Some Applications

As applications, we will build some new inequalities.

Theorem 3.1. *Let α and λ be two positive numbers, $f$and g be two real functions. If $\int_0^\infty x^{1-\lambda}f^2(x)dx<+\infty$, then*

$$\int\limits_0^\infty y^{\lambda-1}\left\{\int\limits_0^\infty k(x,y)\,f(x)\,dx\right\}^2dy\le\ (C_\lambda(\alpha))^2\int\limits_0^\infty x^{1-\lambda}f^2(x)\,dx,\tag{3.1}$$

where $C_\lambda(\alpha)$is defined by (1.10), and $C_\lambda(\alpha)$in (3.1)is the best possible. And the equality in (3.1) holds if and only if $f(x)=0$. And the inequality (3.1) is equivalent to (2.1).

Proof. First, we assume that the inequality (2.1) is valid. Setting a real function $g(y)$as

$$g(y)=y^{\lambda-1}\int\limits_0^\infty k(x,y)\,f(x)\,dx,\quad y\in(0,\,+\infty).$$

By using (2.1), we have

$$\int\limits_0^\infty y^{\lambda-1}\left\{\int\limits_0^\infty k(x,y)\,f(x)\,dx\right\}^2dy=\int\limits_0^\infty\int\limits_0^\infty k(x,y)\,f(x)\,g(y)dxdy\le$$

$$\leq C_\lambda(\alpha)\left\{\int_0^\infty x^{1-\lambda}f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)dy\right\}^{\frac{1}{2}} =$$

$$= C_\lambda(\alpha)\left\{\int_0^\infty x^{1-\lambda}f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{\lambda-1}\left(\int_0^\infty k(x,y)\,f(x)dx\right)^2 dy\right\}^{\frac{1}{2}} \tag{3.2}$$

It follows from (3.2) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid, by applying in turn Cauchy-Schwartz's inequality and (3.1), we have

$$\int_0^\infty\int_0^\infty k(x,y)\,f(x)\,g(y)\,dxdy = \int_0^\infty y^{\frac{\lambda-1}{2}}\left\{\int_0^\infty k(x,y)\,f(x)\,dx\right\}y^{\frac{1-\lambda}{2}}g(y)\,dy \leq$$

$$\leq \left\{\int_0^\infty y^{\lambda-1}\left(\int_0^\infty k(x,y)\,f(x)\,dx\right)^2 dy\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)\,dy\right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{(C_\lambda(\alpha))^2\int_0^\infty x^{1-\lambda}f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)\,dy\right\}^{\frac{1}{2}} =$$

$$= (C_\lambda(\alpha))^2\left\{\int_0^\infty x^{1-\lambda}f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)\,dy\right\}^{\frac{1}{2}}. \tag{3.3}$$

If the constant factor $(C_\lambda(\alpha))^2$ in (3.1) is not the best possible, then it is known from (3.3) that the constant factor $C_\lambda(\alpha)$ in (2.1) is also not the best possible. This is a contradiction. Therefore the inequality (3.1) is equivalent to (2.1). It is obvious that the equality in (3.1) holds if and only if $f(x) = 0$. The proof of Theorem is completed.◄

Similarly, we have the following result.

Theorem 3.2. *With the assumptions as Theorem 3.1, then*

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda - y^\lambda|}f(x)\,dx\right\}^2 dy \leq \ (C_\lambda(\alpha))^2\int_0^\infty x^{1-\lambda}f^2(x)\,dx, \tag{3.4}$$

where $C_\lambda(\alpha)$is defined by (1.10), and $C_\lambda(\alpha)$in (3.4) is the best possible, and the equality in (3.4) holds if and only if $f(x) = 0$. And the inequality (3.4) is equivalent to (2.7) .

Its proof is similar to one of Theorem 3.1. Hence it is omitted.

Similarly, we can establish also some new inequalities which they are respectively equivalent to the inequalities (2.6) and (2.9). They are omitted here.

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