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On piecewise isomorphism of some varieties

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Abstract. Two quasi-projective varieties are called *piecewise isomorphic* if they can be stratified into pairwise isomorphic strata. We show that the m-th symmetric power $S^m(\mathbb{C}^n)$ of the complex affine space $\mathbb{C}^n$ is piecewise isomorphic to $\mathbb{C}^{mn}$ and the m-th symmetric power $S^m(\mathbb{CP}^\infty)$ of the infinite dimensional complex projective space is piecewise isomorphic to the infinite dimensional Grassmannian $\mathbf{Gr}(m, \infty)$.

Key Words and Phrases: algebraic varieties, piecewise isomorphism, Grothendieck semiring of varieties

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1. Introduction

Let $K_0(\mathcal{V}_\mathbb{C})$ be the Grothendieck ring of complex quasi-projective varieties. This is the Abelian group generated by the classes $[X]$ of all complex quasi-projective varieties X modulo the relations:

1) $[X] = [Y]$ for isomorphic X and Y;

2) $[X] = [Y] + [X \setminus Y]$ when Y is a Zariski closed subvariety of X.

The multiplication in $K_0(\mathcal{V}_\mathbb{C})$ is defined by the Cartesian product of varieties: $[X_1] \cdot [X_2] = [X_1 \times X_2]$. The class $[\mathbb{A}^1_\mathbb{C}] \in K_0(\mathcal{V}_\mathbb{C})$ of the complex affine line is denoted by $\mathbb{L}$.

Definition 1. *Quasi-projective varieties X and Y are called* piecewise isomorphic *if there exist decompositions $X = \coprod_{i=1}^{s} X_i$ and $Y = \coprod_{i=1}^{s} Y_i$ of X and Y into (Zariski) locally closed subsets such that X_i and Y_i are isomorphic for $i = 1, \ldots, s$.*

If the varieties X and Y are piecewise isomorphic, their classes $[X]$ and $[Y]$ in the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ coincide. There exists the conjecture (or at least the corresponding question) that the opposite also holds: if $[X] = [Y]$, then X and Y are piecewise isomorphic (see [8, 9]).

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It is well-known that the m-th symmetric power $S^m\mathbb{C}^n$ of the affine space $\mathbb{C}^n$ is birationally equivalent to $\mathbb{C}^{mn}$(see e.g. [3]). An explicit birational isomorphism between $S^m\mathbb{C}^n$ and $\mathbb{C}^{mn}$ was constructed in [1]. Moreover the class $[S^m\mathbb{C}^n]$ of the variety $S^m\mathbb{C}^n$ in the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties is equal to the class $[\mathbb{C}^{mn}] = \mathbb{L}^{mn}$ (see e.g. [4, 6]). The conjecture formulated above means that the varieties $S^m\mathbb{C}^n$ and $\mathbb{C}^{mn}$ are piecewise isomorphic. This is well-known for $n = 1$. Moreover $S^m\mathbb{C}$ and $\mathbb{C}^m$ are isomorphic. The fact that indeed $S^m\mathbb{C}^n$ and $\mathbb{C}^{mn}$ are piecewise isomorphic seems to (or must) be known to specialists. Moreover proofs are essentially contained in [4] (Lemma 4.4 proved by Burt Totaro) and [6] (Statement 3). However this fact is not explicitly reflected in the literature. Here we give a proof of this statement.

In [5], it was shown that the Kapranov zeta function $\zeta_{B\mathbb{C}^*}(T)$ of the classifying stack $B\mathbb{C}^* = BGL(1)$ is equal to

$$1 + \sum_{m=1}^{\infty} \frac{\mathbb{L}^{m^2-m}}{(\mathbb{L}^m - \mathbb{L}^{m-1})(\mathbb{L}^m - \mathbb{L}^{m-2})\dots(\mathbb{L}^m - 1)} T^m.$$

Unrigorously speaking this can be interpreted as the class $[S^m B\mathbb{C}^*]$ of the "m-th symmetric power" of the classifying stack $B\mathbb{C}^*$ in the Grothendieck ring $K_0(\mathrm{Stck}_{\mathbb{C}})$ of algebraic stacks of finite type over $\mathbb{C}$ is equal to $\mathbb{L}^{m^2-m}$ times the class $[BGL(m)] = 1/(\mathbb{L}^m - \mathbb{L}^{m-1})(\mathbb{L}^m - \mathbb{L}^{m-2})\dots(\mathbb{L}^m - 1)$ of the classifying stack $BGL(m)$. The natural topological analogues of the classifying stacks $B\mathbb{C}^*$ and $BGL(m)$ are the infinite-dimensional projective space $\mathbb{CP}^\infty$ and the infinite Grassmannian $\mathbf{Gr}(m,\infty)$. We show that the m-th symmetric power $S^m\mathbb{CP}^\infty$ of $\mathbb{CP}^\infty$ and $\mathbf{Gr}(m,\infty)$ are piecewise isomorphic in a natural sense.

Theorem 1. *The varieties $S^m\mathbb{C}^n$ and $\mathbb{C}^{mn}$ are piecewise isomorphic.*

Proof. The proofs which we know in any case are not explicit, we do not know the neccesary partitions of $S^m\mathbb{C}^n$ and $\mathbb{C}^{mn}$. Therefore we prefer to use the language of power structure over the Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties invented in [6]. This language sometimes permits to substitute somewhat envolved combinatorial considerations by short computations (or even to avoid them at all, as it was made in [7]). Since the majority of statements in [6] (including those which could be used to prove Theorem 1) are formulated and proved in the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties, we repeat a part of the construction in the appropriate setting.

The Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties is the semigroup generated by isomorphism classes $\{X\}$ of such varieties modulo the relation $\{X\} = \{X - Y\} + \{Y\}$ for a Zariski closed subvariety $Y \subset X$. The multiplication is defined by the Cartesian product of varieties: $\{X_1\} \cdot \{X_2\} = \{X_1 \times X_2\}$. Classes $\{X\}$ and $\{Y\}$ of two varieties X and Y in $S_0(\mathrm{Var}_{\mathbb{C}})$ are equal if and only if X and Y are piecewise isomorphic. Let $\mathbb{L} \in S_0(\mathrm{Var}_{\mathbb{C}})$ be the class of the affine line. If $\pi : E \to B$ is a Zariski locally trivial fibre bundle with fibre F, one has $\{E\} = \{F\} \cdot \{B\}$. For example if $\pi : E \to B$ is a Zariski locally trivial vector bundle of rank s, one has $\{E\} = \mathbb{L}^s\{B\}$.

A *power structure* over a semiring R is a map $(1 + T \cdot R[[T]]) \times R \to 1 + T \cdot R[[T]]$: $(A(T), m) \mapsto (A(T))^m$, which possesses the properties:

1. $(A(T))^0 = 1$,
2. $(A(T))^1 = A(T)$,
3. $(A(T)B(T))^m = (A(T))^m (B(T))^m$,
4. $(A(T))^{m+n} = (A(T))^m (A(T))^n$,
5. $(A(T))^{mn} = ((A(T))^n)^m$,
6. $(1+T)^m = 1 + mT+$ terms of higher degree,
7. $\left(A(T^\ell)\right)^m = (A(T))^m \,|_{T\mapsto T^\ell}$, $\ell \geq 1$.

In [6], there was defined a power structure over the Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$. Namely, for $A(T) = 1 + \{A_1\}\, T + \{A_2\}\, T^2 + \ldots$ and $\{M\} \in S_0(\mathrm{Var}_{\mathbb{C}})$, the series $(A(T))^{\{M\}}$ is defined as

$$1 + \sum_{k=1}^{\infty} \left(\sum_{\{k_i\}: \sum i k_i = k} \left\{ \left(((\prod_i M^{k_i}) \setminus \Delta) \times \prod_i A_i^{k_i} \right) / \prod_i S_{k_i} \right\} \right) T^k, \tag{1}$$

where Δ is the "large diagonal" in $M^{\Sigma k_i} = \prod\limits_i M^{k_i}$ which consists of $(\sum k_i)$-tuples of points of M with at least two coinciding ones, the group S_{k_i} of permutations on k_i elements acts by permuting corresponding k_i factors in $\prod\limits_i M^{k_i} \supset (\prod\limits_i M^{k_i}) \setminus \Delta$ and the spaces A_i simultaneously. The action of the group $\prod\limits_i S_{k_i}$ on $(\prod\limits_i M^{k_i}) \setminus \Delta$ is free. The properties 1–7 are proved in [6, Theorem 1].

Special role is played by the Kapranov zeta function in the Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$: $\zeta_{\{M\}}(T) := 1 + \sum_{k=1}^{\infty} \{S^k M\} T^k$, where $S^k M$ is the k-th symmetric power M^k / S_k of the variety M. In terms of the power structure one has $\zeta_{\{M\}}(T) = (1 + T + T^2 + \ldots)^{\{M\}}$. Theorem 1 is equivalent to the fact that

$$\zeta_{\mathbb{L}^m}(T) = (1 + \sum_{i=1}^{\infty} \mathbb{L}^{im} T^i). \tag{2}$$

Lemma 1. *Let A_i and M be complex quasi-projective varieties, $A(T) = 1 + \{A_1\}T + \{A_2\}T^2 + \ldots$. Then, for any integer $s \geq 0$,*

$$(A(\mathbb{L}^s T))^{\{M\}} = \left(A(T)^{\{M\}} \right) |_{T \mapsto \mathbb{L}^s T}. \tag{3}$$

Proof. The coefficient at the monomial T^k in the power series $(A(T))^{\{M\}}$ is a sum of the classes of varieties of the form

$$V = \left(((\prod_i M^{k_i}) \setminus \Delta) \times \prod_i A_i^{k_i} \right) / \prod_i S_{k_i},$$

with $\sum ik_i = k$. The corresponding summand $\{\widetilde{V}\}$ in the coefficient at the monomial T^k in the power series $(A(\mathbb{L}^s T))^{\{M\}}$ has the form

$$\widetilde{V} = \left(((\prod_i M^{k_i}) \setminus \Delta) \times \prod_i (\mathbb{L}^{si} \times A_i)^{k_i} \right) / \prod_i S_{k_i}.$$

The natural map $\widetilde{V} \to V$ is a Zariski locally trivial vector bundle of rank sk (see e.g. [10, Section 7, Proposition 7]). This implies that $\{\widetilde{V}\} = \mathbb{L}^{sk} \cdot \{V\}$.

One has $\zeta_{\mathbb{L}}(T) = (1 + \mathbb{L}T + \mathbb{L}^2T^2 + \ldots)$. For all A_i being points, i.e. $\{A_i\} = 1$, one gets

$$\begin{aligned} \zeta_{\mathbb{L}\{M\}}(T) &= (1 + T + T^2 + \ldots)^{\mathbb{L}\{M\}} = ((1 + T + T^2 + \ldots)^{\mathbb{L}})^{\{M\}} \\ &= (1 + \mathbb{L}T + \mathbb{L}^2T^2 + \ldots)^{\{M\}}. \end{aligned}$$

Equation (3) implies that

$$\zeta_{\mathbb{L}\{M\}}(T) = (1 + \mathbb{L}T + \mathbb{L}^2T^2 + \ldots)^{\{M\}} = \zeta_{\{M\}}(\mathbb{L}T).$$

Assuming (2) holds for $m < m_0$ and applying the equation above to $m = m_0 - 1$ one gets

$$\begin{aligned} \zeta_{\mathbb{L}^{m_0}}(T) &= \zeta_{\mathbb{L}^{m_0-1}}(\mathbb{L}T) = (1 + \mathbb{L}^{m_0-1}T + \mathbb{L}^{2(m_0-1)}T^2 + \ldots)_{|T \mapsto \mathbb{L}T} \\ &= (1 + \mathbb{L}^{m_0}T + \mathbb{L}^{2m_0}T^2 + \ldots). \end{aligned}$$

This gives the proof.

Let $\mathbb{CP}^\infty = \varinjlim \mathbb{CP}^N$ be the infinite dimensional projective space and let $\mathbf{Gr}(m, \infty) = \varinjlim \mathbf{Gr}(m, N)$ be the infinite dimensional Grassmannian. (In the both cases the inductive limit is with respect to the natural sequence of inclusions. The spaces $\mathbb{CP}^\infty$ and $\mathbf{Gr}(m, \infty)$ are, in the topological sense, classifying spaces for the groups $\mathbb{C}^* = GL(1; \mathbb{C})$ and $GL(m; \mathbb{C})$ respectively.) The symmetric power $S^m\mathbb{CP}^\infty$ is the inductive limit of the quasi-projective varieties $S^m\mathbb{CP}^N$. For a sequence $X_1 \subset X_2 \subset X_3 \subset \ldots$ of quasi-projective varieties, let $X = \varinjlim X_i (= \bigcup_i X_i)$ be its (inductive) limit. A *partition of the space* X *compatible with the filtration* $\{X_i\}$ is a representation of X as a disjoint union $\coprod_j Z_j$ of (not more than) countably many quasi-projective varieties Z_j such that each X_i is the union of a subset of the strata Z_j and each Z_j is a Zariski locally closed subset in the corresponding X_i.

Theorem 2. *The spaces* $S^m\mathbb{CP}^\infty$ *and* $\mathbf{Gr}(m, \infty)$ *are piecewise isomorphic in the sense that there exist partitions* $S^m\mathbb{CP}^\infty = \coprod_j U_j$ *and* $\mathbf{Gr}(m, \infty) = \coprod_j V_j$ *into pairwise isomorphic quasi-projective varieties* U_j *and* V_j $(U_j \cong V_j)$ *compatible with the filtrations* $\{S^m\mathbb{CP}^N\}_N$ *and* $\{\mathbf{Gr}(m, N)\}_N$.

Proof. The natural partition of $\mathbf{Gr}(m, N)$ consists of the Schubert cells corresponding to the flag $\{0\} \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \ldots$ (see e.g [2, §5.4]). Each Schubert cell is a locally closed subvariety of $\mathbf{Gr}(m, N)$ isomorphic to the complex affine space of certain dimension. This partition is compatible with the inclusion $\mathbf{Gr}(m, N) \subset \mathbf{Gr}(m, N+1)$ and therefore gives a partition of $\mathbf{Gr}(m, \infty)$. The number of cells of dimension n in $\mathbf{Gr}(m, \infty)$ is equal to the number of partitions of n into summands not exceeding m.

Since $\mathbb{CP}^\infty = \mathbb{C}^0 \coprod \mathbb{C}^1 \coprod \mathbb{C}^2 \coprod \ldots$ and $S^p(A \coprod B) = \coprod_{i=0}^{p} S^i A \times S^{p-i} B$, one has

$$S^m \mathbb{CP}^\infty = \coprod_{\{i_0, i_1, i_2, \ldots\}: i_0 + i_1 + i_2 + \ldots = m} \prod_j S^{i_j} \mathbb{C}^j = \coprod_{\{i_1, i_2, \ldots\}: i_1 + i_2 + \ldots \leq m} \prod_j S^{i_j} \mathbb{C}^j,$$

where i_j are non-negative integers. This partition is compatible with the natural filtration $\{\mathbb{CP}^0\} \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \ldots$. The number of parts of dimension n is equal to the number of sequences $\{i_1, i_2, \ldots\}$ such that $\sum_j i_j \leq m$, $\sum_j i_j j = n$. Thus, it coincides with the number of partitions of n into not more than m summands and is equal to the number of n-dimensional Schubert cells in the partition of $\mathbf{Gr}(m, \infty)$. Due to Proposition 1 each part $\prod_j S^{i_j} \mathbb{C}^j$ is piecewise isomorphic to the complex affine space of the same dimension. This concludes the proof.

It would be interesting to find explicit piecewise isomorphisms between the spaces in Theorems 1 and 2.

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