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The relation between different norms of algebraic polynomials in the regions of complex plane

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Abstract. In this work, we study Bernstein-Zigmund type and Nikolskii type estimations for the arbitrary algebraic polynomial in regions of the complex plane.

Key Words and Phrases: Algebraic polynomial, Quasiconformal mapping, Quasicircle

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1. Introduction and Main results

Let G be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, and let X and Y be norm spaces of functions defined in G and let $\wp_n$ denote the set of arbitrary algebraic polynomials $P_n(z)$, $\deg P_n = n$, $n = 0, 1, 2, ...,$. Our goal is to find the estimate

$$\|P_n^{(k)}\|_X \leq A(k, n, G)\|P_n\|_Y,$$

for all polynomials $P_n \in \wp_n$ and all $k = 0, 1, 2, ...$, where $A(k, n, G)$ is a constant depending on k, n and G in general.

The comparison of norms of polynomials with itself and itself with derivation of polynomials have been studied by many mathematicians (see, for example, [1], [2], [3], [4], [5], [10], [12]).

Let σ be the two-dimensional Lebesque measure and $h(z)$ is a weight function in G. Let $A_p(h, G)$, $p > 0$, denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|_{A_p} := \|f\|_{A_p(h,G)} := \left(\iint_G h(z) \, |f(z)|^p \, d\sigma_z \right)^{1/p} < \infty,$$

and $A_p(1, G) \equiv A_p(G)$.

Let $w = \varphi(z)$ $(w = \Phi(z))$ be the conformal mapping of G $(\Omega := C\overline{G})$ onto the unit disc $B := B(0, 1) := \{w : |w| < 1\}$ $(\Delta := \Delta(0, 1) := \{z : |z| > 1\})$ normalized by $\varphi(0) = 0$, $\varphi'(0) > 0$ $(\Phi(\infty) = \infty$, $\Phi'(\infty) > 0)$ and let $\psi := \varphi^{-1} (\Psi := \Phi^{-1})$.

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Definition 1. *[13] A bounded Jordan region G is called a k -quasidisk, $0 \leq k < 1$, if any conformal mapping ψ can be extended to a K -quasiconformal, $K = \frac{1+k}{1-k}$, homeomorphizm of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In that case the curve $L := \partial G$ is called a K -quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is k -quasidisk (k -quasicircle) with some $0 \leq k < 1$.*

Throughout this paper, $c, c_1, c_2, \ldots$ are positive constants (in general, different in different relations), which depend on G in general.

Theorem 1. *Let G be a k -quasidisk, $0 \leq k < 1$. Then for arbitrary $P_n \in \wp_n$ and any $m = 0, 1, 2, \ldots$ we have*

$$\|P_n^{(m)}\|_{C(\overline{G})} \leq c_1 n^{(m+\frac{2}{p})(1+k)} \|P_n\|_{A_p(G)}, \ p > 1. \tag{1.1}$$

Theorem 2. *Let G be a k -quasidisk, $0 \leq k < 1$. Then for arbitrary $P_n \in \wp_n$ and any $m = 0, 1, 2, \ldots$ we have*

$$\|P_n^{(m)}\|_{A_p(G)} \leq c_2 n^{\left[(m+1)-\frac{2}{p}\right](1+k)} \|P_n\|_{A_2(G)}, \ p > 2. \tag{1.2}$$

Theorem 3. *Let G be a k -quasidisk, $0 \leq k < 1$. Then for arbitrary $P_n \in \wp_n$ and any $1 < p \leq q < \infty$ we have*

$$\|P_n\|_{A_q(G)} \leq c_3 n^{2(\frac{1}{p}-\frac{1}{q})(1+k)} \|P_n\|_{A_p(G)}. \tag{1.3}$$

Theorems 1, 2 and 3 are fine, since we can know coefficients quasiconformality of taking regions. Note that the result which is similar to the (3) was obtained in [14] in the case of $1 < p < q \leq \infty$. But, the dependence on n and k of the right side of the inequality was not clearly expressed like (3).

Now, we define the class of regions under functional conditions, such that the coefficients quasiconformality of this regions are hard to define, but we can define these regions in according to other parameters.

Definition 2. *We say that $G \in Q_\alpha$, $0 < \alpha \leq 1$, if*

a) L is a quasicircle,

b) $\Phi \in Lip\alpha$, $z \in \overline{\Omega}$.

Theorem 4. *Let $G \in Q_\alpha$. Then, for arbitrary $P_n \in \wp_n$ and any $m = 0, 1, 2, \ldots$ we have*

$$\|P_n^{(m)}\|_{C(\overline{G})} \leq c_4 \begin{cases} n^{\delta(m+\frac{2}{p})}, & \alpha < \frac{1}{2} \\ n^{\frac{1}{\alpha}(m+\frac{2}{p})}, & \alpha \geq \frac{1}{2} \end{cases} \|P_n\|_{A_p(G)}, \quad p > 1, \tag{1.4}$$

where $\delta = \delta(G)$, $1 \leq \delta \leq 2$, is a certain number.

Theorem 5. *Let $G \in Q_\alpha$. Then, for arbitrary $P_n \in \wp_n$ and any $m = 0, 1, 2, \ldots$ we have*

$$\|P_n^{(m)}\|_{A_2(G)} \leq c_5 \begin{cases} n^{\delta \cdot m}, & \alpha < \frac{1}{2} \\ n^{\frac{m}{\alpha}}, & \alpha \geq \frac{1}{2} \end{cases} \|P_n\|_{A_2(G)}, \tag{1.5}$$

where $\delta = \delta(G)$, $1 \leq \delta \leq 2$, is a certain number.

Theorem 6. *Let $G \in Q_\alpha$. Then, for arbitrary $P_n \in \wp_n$ and any $1 < p \leq q < \infty$ we have*

$$\|P_n\|_{A_q(G)} \leq c_6 \begin{cases} n^{2\delta(\frac{1}{p}-\frac{1}{q})}, & \alpha < \frac{1}{2} \\ n^{\frac{2}{\alpha}(\frac{1}{p}-\frac{1}{q})}, & \alpha \geq \frac{1}{2} \end{cases} \|P_n\|_{A_p(G)}, \tag{1.6}$$

where $\delta = \delta(G)$, $1 \leq \delta \leq 2$, is a certain number.

2. Some auxiliary results

Throughout this paper, we denote that "$a \prec b$" and "$a \asymp b$" are equivalent to $a \leq b$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 respectively.

Let G is a quasidisk. Then there exists a quasiconformal reflection $y(.)$ across L such that $y(G) = \Omega$, $y(\Omega) = G$ and $y(.)$ fixes the points of L. The quasiconformal reflection $y(.)$ is such that it satisfies the following condition [7], [9, p.26];

$$\begin{aligned} |y(\zeta) - z| &\asymp |\zeta - z|, \ z \in L, \ \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ \left|y_{\overline{\zeta}}\right| &\asymp |y_\zeta| \asymp 1, \ \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ \left|y_{\overline{\zeta}}\right| &\asymp |y(\zeta)|^2, \ |\zeta| < \varepsilon, \ \left|y_{\overline{\zeta}}\right| \asymp |\zeta|^{-2}, \ |\zeta| > \frac{1}{\varepsilon}. \end{aligned} \tag{2.1}$$

For $t > 0$, let $L_t := \{z : |\varphi(z)| = t, \ if \ t < 1, \ \ |\Phi(z)| = t, \ if \ t > 1\}$, $G_t := intL_t$, $\Omega_t := extL_t$.

For $R > 1$, we denote $L^* := y(L_R)$, $G^* := intL^*$, $\Omega^* := extL^*$; $w = \Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty) = \infty$, $\Phi_R'(\infty) > 0$; $\Psi_R := \Phi_R^{-1}$; For $t > 1$, let $L_t^* := \{z : |\Phi_R(z)| = t\}$, $G_t^* := intL_t^*$, $\Omega_t^* := extL_t^*$.

According to [8], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L)$ we have

$$\begin{aligned} d(z, L) &\asymp d(t, L_R) \asymp d(z, L_R^*), \\ |\Phi_R(z)| &\leq |\Phi_R(t)| \leq 1 + c(R - 1). \end{aligned} \tag{2.2}$$

Lemma 1. *[6] Let G be a quasidisk, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then,*

a) The statements $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \prec |z_1 - z_3|$ *, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ *is a constant, depending on* G *and* k.

Lemma 2. *Let* G *be a* k *-quasidisk for some* $0 \leq k < 1$. *Then*

$$|\Psi(w_1) - \Psi(w_2)| \succ |w_1 - w_2|^{1+k},$$

for all $w_1, w_2 \in \overline{\Omega}'$.

This fact follows from an appropriate result for the mapping $f \in \sum(k)$[13, p.287] and estimation for the $\Psi^{'}$[9, Th.2.8].

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ defined as the following:

$$h(z) = h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \tag{2.3}$$

where $\gamma_j > -2$ for $j = \overline{1, m}$, and $h_0(z)$ is uniformly separated from zero in G:

$$h_0(z) \geq c > 0, \ \forall z \in G.$$

Lemma 3. *[2] Let* G *be a quasidisk and* $P_n(z)$, $\deg P_n \leq n, n = 1, 2, ...,$ *is an arbitrary polynomial and weight function* $h(z)$ *satisfies the condition(2.3). Then for any* $R > 1$, $p > 0$ *and* $n = 1, 2, ...$

$$\|P_n\|_{A_p(h, G_{1+c(R-1)})} \leq c_3 R^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \tag{2.4}$$

where c, c_3 *are independent of* n *and* G.

In particular, in case of $h(z) \equiv 1$, we get

$$\|P_n\|_{A_p(G_{1+c(R-1)})} \leq c_4 R^{n+\frac{1}{p}} \|P_n\|_{A_p(G)}. \tag{2.5}$$

This result is the integral analog of the familiar lemma of Bernstein-Walsh[15, p.101] for the case $A_p(G)$ -norm and, shows that the order $A_p(G)$ -norm of arbitrary polynomials is taken from the region G and $G_{1+1/n}$ which are both identical.

Lemma 4. *Let* G *be a quasidisk and* $P_n(z)$, $\deg P_n \leq n, n = 1, 2, ...,$ *is an arbitrary polynomial. Then for any* $R = 1 + \frac{c}{n}$ *,* $n = 1, 2, ...,$ *and* $m = 0, 1, 2, ...,$ *there exists a* $c_1 := c_1(G, c) > 0$ *such that*

$$\left\|P_n^{(m)}\right\|_{C(\overline{G})} \leq c_1 \left\|P_n^{(m)}\right\|_{C(\overline{G^*})}. \tag{2.6}$$

Proof. For any fixed number $m=0,1,2,...,\ m\leq n$, we put

$$F(z):=F(z,m,n,R):=\frac{P_n^{(m)}(z)}{[\Phi_R(z)]^{n+1-m}},\ z\in\Omega^*.$$

Obviously, the function $F(z)$ is analytic in Ω^*, continuous on $\overline{\Omega^*}$, $F(\infty)=0$ and $|F(z)|=\left|P_n^{(m)}(z)\right|$ for $z\in L^*$. Then, the maximum modulus principle yields

$$|F(z)|\leq\max_{z\in L^*}|F(z)|=\max_{z\in L^*}\left|P_n^{(m)}(z)\right|.$$

So,

$$\left|P_n^{(m)}(z)\right|\leq|\Phi_R(z)|^{n+1-m}\left\|P_n^{(m)}\right\|_{C(\overline{G^*})},\quad z\in\overline{\Omega^*}.$$

According to (2.2), for the $z\in L$, we get

$$|\Phi_R(z)|^{n+1-m}\leq[1+c(R-1)]^{n+1-m}=\left[1+\frac{c}{n}\right]^{n+1-m}\prec 1.$$

Since $z\in L$ is arbitrary, then

$$\left\|P_n^{(m)}\right\|_{C(\overline{G})}\prec\left\|P_n^{(m)}\right\|_{C(\overline{G^*})}. \blacktriangleleft$$

3. Proof of Theorems

3.1. Proof of Theorems 1 and 4

Proof. As L is a quasicircle, then for arbitrary $z\in G^*$, we can write the following integral representation for $P_n(z)$[9]:

$$P_n^{(m)}(z)=-\frac{(m+1)!}{\pi}\iint\limits_G\frac{P_n(\zeta)y_{\overline{\zeta}}(\zeta)}{(y(\zeta)-z)^{m+2}}d\sigma_\zeta,\quad z\in\overline{G^*}.$$

Applying the Minkowski inequality, we have

$$\begin{aligned}|P_n^{(m)}(z)| &\leq \frac{(m+1)!}{\pi}\left[\iint\limits_G|P_n(\zeta)|^pd\sigma_\zeta\right]^{\frac{1}{p}}\times\\ &\times\left[\iint\limits_G\frac{|y_{\bar\zeta}|^q}{|y(\zeta)-z|^{q(m+2)}}d\sigma_\zeta\right]^{\frac{1}{q}}\prec\\ &\prec\left[\iint\limits_G\frac{|y_{\bar\zeta}|^q}{|y(\zeta)-z|^{q(m+2)}}d\sigma_\zeta\right]^{\frac{1}{q}}\|P_n\|_{A_p(G)},\ \frac{1}{p}+\frac{1}{q}=1. \qquad (3.1)\end{aligned}$$

Let us set

$$J^q(z) := \iint\limits_G \frac{|y_{\bar{\zeta}}|^q}{|y(\zeta)-z|^{q(m+2)}} d\sigma_\zeta.$$

For $\varepsilon > 0$, we put $U_\varepsilon(z) := \{\zeta : |\zeta - z| < \varepsilon\}$; without loss of generality, we may take $U_\varepsilon := U_\varepsilon(0) \subset G^*$. For arbitrary fixed $z \in L^*$, we have

$$\begin{aligned} J^q(z) &= \iint\limits_{U_\varepsilon} \frac{|y_{\bar{\zeta}}|^q}{|y(\zeta)-z|^{q(m+2)}} d\sigma_\zeta + \\ &\quad + \iint\limits_{G\setminus U_\varepsilon} \frac{|y_{\bar{\zeta}}|^q}{|y(\zeta)-z|^{q(m+2)}} d\sigma_\zeta = \\ &= : J_1(z) + J_2(z). \end{aligned} \tag{3.2}$$

Let us estimate $J_1(z)$. According to (2.1), $|y_{\bar{\zeta}}| \asymp |y(\zeta)|^2$ for all $\zeta \in U_\varepsilon$, because of $|\zeta - z| \geq \varepsilon$, $|y(\zeta) - z| \asymp |y(\zeta)|$ for $z \in L^*$ and $\zeta \in U_\varepsilon$, then we can find

$$\begin{aligned} J_1(z) &= \iint\limits_{U_\varepsilon} \frac{|y_{\bar{\zeta}}|^q}{|y(\zeta)-z|^{q(m+2)}} d\sigma_\zeta \asymp \\ &\asymp \iint\limits_{U_\varepsilon} \frac{|y(\zeta)|^{2q}}{|y(\zeta)|^{q(m+2)}} d\sigma_\zeta = \\ &= \iint\limits_{U_\varepsilon} \frac{d\sigma_\zeta}{|y(\zeta)|^{qm}} \prec 1. \end{aligned} \tag{3.3}$$

For the estimation of $J_2(z)$, first of all, we note that the Jacobian $\pounds_y := |y_\zeta|^2 - \left|y_{\overline{\zeta}}\right|^2$ of the reflection $y(\zeta)$ satisfies the following inequality

$$\begin{aligned} \left|y_{\overline{\zeta}}\right| &= \left[\frac{\pounds_y \left|y_{\overline{\zeta}}\right|^2}{|y_\zeta|^2 - \left|y_{\overline{\zeta}}\right|^2}\right]^{\frac{1}{2}} = \left[\frac{\pounds_y}{\left(|y_\zeta|^2 / \left|y_{\overline{\zeta}}\right|^2\right) - 1}\right]^{\frac{1}{2}} \leq \\ &\leq \left(\frac{\chi^2}{1-\chi^2}\right)^{\frac{1}{2}} |\pounds_y|^{\frac{1}{2}} \prec |\pounds_y|^{\frac{1}{2}}, \end{aligned}$$

where $\chi := \frac{K_1 - 1}{K_1 + 1}$. Consequently, $|\pounds_y| \succ |y_{\bar{\zeta}}|^2$. Then, after carrying out the change of variable we obtain for the $J_2(z)$:

$$J_2(z) \prec \iint\limits_{G\setminus U_\varepsilon} \frac{|y_{\bar{\zeta}}|^q}{|y(\zeta)-z|^{q(m+2)}} d\sigma_\zeta \prec$$

$$
\begin{aligned}
&\prec \iint\limits_{y(G\setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z|^{q(m+2)}} \prec \\
&\prec \iint\limits_{|\zeta - z|\geq d(z,L)} \frac{d\sigma_\zeta}{|\zeta - z|^{q(m+2)}} \\
&\prec \quad (d(z,L))^{2-q(m+2)}. \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
J^q(z) &\prec \quad 1 + (d(z,L_R))^{2-q(m+2)} \prec \\
&\prec \quad (d(z,L))^{2-q(m+2)}, \ \forall z \in L^*. \qquad (3.5)
\end{aligned}
$$

Using (2.2), from (3.1),(3.2)-(3.5) for all $z \in L^*$ and $t \in L$ such that $|z-t| = d(z,L)$, we obtain

$$|P_n^{(m)}(z)| \prec d^{-(m+\frac{2}{p})}(t,L_R)\, \|P_n\|_{A_p(G)}\,, \ \forall z \in L^*. \qquad (3.6)$$

If $G \in Q_\alpha$, according to [9] and [11] we have

$$d\,(t, L_R) \succ (R-1)^\mu \succ n^{-\mu}, \qquad (3.7)$$

where $\mu = \frac{1}{\alpha}$, if $\alpha \geq \frac{1}{2}$ and $\mu = \delta$, if $\alpha < \frac{1}{2}$, $\delta = \delta(\alpha, G)$, $1 \leq \delta \leq 2$, is a certain number.

If G is a quasidisk, taking Lemma 2 into account, we get

$$d(z, L_R) = |\zeta - z| = |\Psi(\tau) - \Psi(w)| \geq |\tau - w|^{1+k} \succ n^{-(1+k)}. \qquad (3.8)$$

Consequently, according to Lemma 4, we obtain

$$\|P_n^{(m)}\|_{C(\overline{G})} \prec \left\{ \begin{array}{ll} n^{\delta(m+\frac{2}{p})}, & \alpha < \frac{1}{2} \\ n^{\frac{1}{\alpha}(m+\frac{2}{p})}, & \alpha \geq \frac{1}{2} \end{array} \right. \|P_n\|_{A_p(G)}, \quad p > 1,$$

and

$$\|P_n^{(m)}\|_{C(\overline{G})} \prec n^{(1+k)\cdot(m+\frac{2}{p})} \|P_n\|_{A_p(G)}, \quad p > 1.$$

The proof or theorems 1 and 4 is completed.◀

3.2. Proof of Theorem 5

Proof. Since L is a quasicircle, we conclude that any L_R, $R = 1 + cn^{-1}$ is also a quasicircle. Therefore, we can construct a K_1-quasiconformal reflection y_R, $y_R(0) = \infty$ across L_R that satisfies conditions (2.1) described for $y_R(\zeta)$. By using $y_R(\zeta)$ constructed in this way, we can write the following integral representations for $P_n(z)$

$$P_n^{(m)}(z) = -\frac{(m+1)!}{\pi} \iint\limits_{G_R} \frac{P_n(\zeta) y_{R,\overline{\zeta}}(\zeta)}{(y_R(\zeta) - z)^{m+2}} d\sigma_\zeta, \quad z \in G. \qquad (3.9)$$

Applying the Hölder inequality, we get

$$|P_n^{(m)}(z)|^2 \leq \left[\frac{(m+1)!}{\pi}\right]^2 \iint\limits_{G_R} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta \times \iint\limits_{G_R} \frac{|P_n(\zeta)|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta .$$

After integration from over a region G, we have

$$\begin{aligned} &\iint\limits_{G} |P_n^{(m)}(z)|^2 d\sigma_z \leq \left[\frac{(m+1)!}{\pi}\right]^2 \times \\ &\times \iint\limits_{G} \left[\iint\limits_{G_R} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta \cdot \iint\limits_{G_R} \frac{|P_n(\zeta)|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta \right] d\sigma_z \leq \\ \leq\ & \sup_{z\in\overline{G}} \iint\limits_{G_R} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta \times \\ &\times \sup_{\zeta\in\overline{G_R}} \iint\limits_{G} \frac{d\sigma_z}{|y_R(\zeta)-z|^{m+2}} \times \|P_n\|^2_{A_2(G_R)} =: \\ =\ & : A_R(z)\times B_R(z)\times \|P_n\|^2_{A_2(G_R)}. \end{aligned} \tag{3.10}$$

Let us estimate

$$A_R(z) := \sup_{z\in\overline{G}} \iint\limits_{G_R} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta . \tag{3.11}$$

For $\varepsilon > 0$, $U_\varepsilon(z) := \{\zeta : |\zeta - z| < \varepsilon\}$, we can assume without loss of generality that $U_\varepsilon := U_\varepsilon(0) \subset G^*$. For arbitrary fixed $z \in L$, we have

$$\begin{aligned} \iint\limits_{G_R} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta &= \iint\limits_{U_\varepsilon} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta + \\ &+ \iint\limits_{G_R\setminus U_\varepsilon} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta =: \\ &= : J_1 + J_2. \end{aligned} \tag{3.12}$$

Let us estimate J_1. According to (2.1), $|y_{R,\bar{\zeta}}| \asymp |y_R(\zeta)|^2$ for all $\zeta \in U_\varepsilon$, because of $|\zeta - z| \geq \varepsilon$, $|y_R(\zeta) - z| \asymp |y_R(\zeta)|$ for $z \in L$ and $\zeta \in U_\varepsilon$, then we can find

$$J_1 = \iint\limits_{U_\varepsilon} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}} d\sigma_\zeta \asymp$$

$$\begin{aligned} &\asymp \iint\limits_{U_\varepsilon} \frac{|y_R(\zeta)|^4}{|y_R(\zeta)|^{m+2}} d\sigma_\zeta = \\ &= \iint\limits_{U_\varepsilon} \frac{d\sigma_\zeta}{|y_R(\zeta)|^{m+2}} \prec 1. \end{aligned} \tag{3.13}$$

For the estimation of J_2 , first of all, we note that the Jacobian $\pounds_{y_R} := |y_{R,\zeta}|^2 - \left|y_{R,\overline{\zeta}}\right|^2$ of the reflection $y_R(\zeta)$ satisfies the following inequality

$$\begin{aligned} \left|y_{R,\overline{\zeta}}\right| &= \left[\frac{\pounds_{y_R}\left|y_{R,\overline{\zeta}}\right|^2}{|y_{R,\zeta}|^2 - \left|y_{R,\overline{\zeta}}\right|^2}\right]^{\frac{1}{2}} = \\ &= \left[\frac{\pounds_{y_R}}{\left(|y_{R,\zeta}|^2 / \left|y_{R,\overline{\zeta}}\right|^2\right) - 1}\right]^{\frac{1}{2}} \leq \\ &\leq \left(\frac{\chi^2}{1-\chi^2}\right)^{\frac{1}{2}} |\pounds_{y_R}|^{\frac{1}{2}} \prec |\pounds_{y_R}|^{\frac{1}{2}}, \end{aligned}$$

where $\chi := \frac{K_1-1}{K_1+1}$. Consequently, $|\pounds_{y_R}| \succ |y_{R,\bar{\zeta}}|^2$. Then, analogous to the estimate for J_1, after carrying out the change of variable, we obtain for the J_2:

$$\begin{aligned} J_2 &\prec \iint\limits_{G_R \setminus U_\varepsilon} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta) - z|^{m+2}} d\sigma_\zeta \prec \\ &\prec \iint\limits_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z|^{m+2}} \prec \\ &\prec \iint\limits_{|\zeta - z| \geq d(z, L_R)} \frac{d\sigma_\zeta}{|\zeta - z|^{m+2}} \prec \\ &\prec (d(z, L_R))^{-m}. \end{aligned} \tag{3.14}$$

$$A_R(z) \prec 1 + (d(z, L_R))^{-m} \prec (d(z, L_R))^{-m}, \ \forall z \in L. \tag{3.15}$$

Next, analogous to the estimate $A_R(z)$, for the $B_R(z)$, we get

$$B_R(z) \prec (d(z, L_R))^{-m}, \ \forall z \in L. \tag{3.16}$$

From (3.10), (3.15), and (3.16), we obtain

$$\iint\limits_{G} |P_n^{(m)}(z)|^2 d\sigma_z \prec (d(z, L_R))^{-2m} \|P_n\|^2_{A_2(G_R)}, \quad \forall z \in L.$$

Since $G \in Q_\alpha$, according to [9] and [11], we have

$$d\,(z, L_R) \succ (R-1)^{\mu} \succ n^{-\mu},$$

where $\mu = \frac{1}{\alpha}$, if $\alpha \geq \frac{1}{2}$ and $\mu = \delta$, if $\alpha < \frac{1}{2}$, $\delta = \delta(\alpha, G)$, $1 \leq \delta \leq 2$, is a certain number.

Consequently, according to Lemma 3, we obtain

$$\|P_n^{(m)}\|_{A_2(G)} \prec n^{\mu\cdot m}\|P_n\|_{A_2(G)},$$

and we completed the proof.◄

3.3. Proof of Theorem 2

Proof. In a similar way, analogous to (3.9), we can write the following integral representation

$$P_n^{(m)}(z) = -\frac{(m+1)!}{\pi}\iint\limits_{G_R}\frac{P_n(\zeta)y_{R,\bar{\zeta}}(\zeta)}{(y_R(\zeta)-z)^{m+2}}d\sigma_\zeta,\ z \in G.$$

Applying the Hölder inequality, we get

$$\begin{aligned}|P_n^{(m)}(z)| \quad &\leq \quad \frac{(m+1)!}{\pi}\left(\iint\limits_{G_R}\frac{|P_n(\zeta)|^2}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta\right)^{1/2} \times \\ &\times\left(\iint\limits_{G_R}\frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta\right)^{1/2}.\end{aligned}$$

Then

$$\iint\limits_{G}|P_n^{(m)}(z)|^p d\sigma_z \leq \left\{\frac{(m+1)!}{\pi}\right\}^p \times$$

$$\times\iint\limits_{G}\left(\iint\limits_{G_R}\frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta\right)^{p/2}\left(\iint\limits_{G_R}\frac{|P_n(\zeta)|^2}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta\right)^{p/2} d\sigma_z \leq$$

$$\leq \left\{\frac{(m+1)!}{\pi}\right\}^p \sup_{z\in\overline{G}}\left\{\iint\limits_{G_R}\frac{|y_{R,\bar{\zeta}}|^2}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta\right\}^{p/2} \times$$

$$\times\iint\limits_{G}\left[\left(\iint\limits_{G_R}\frac{|P_n(\zeta)|^2}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta\right)^{p/2}\right]d\sigma_z =: A_R^1(z)\times B_R^1(z). \tag{3.17}$$

Estimations for $A_R^1(z)$, we can find analogous estimation $A_R(z)$. In this case, for $z \in L$ we get

$$A_R^1(z) = \sup_{\zeta \in \overline{G}} \left\{ \iint\limits_{G_R} \frac{|y_{R,\bar{\zeta}}|^p d\sigma_z}{|y_R(\zeta) - z|^{m+2}} \right\}^{p/2} \prec (d(z, L_R))^{-\frac{mp}{2}}.$$

Applying the generalized Minkowski inequality for the estimation $B_R^1(z)$ for $z \in L$, we get

$$\begin{aligned} B_R^1(z) &= \iint\limits_G \left[\left(\iint\limits_{G_R} \frac{|P_n(\zeta)|^2}{|y_R(\zeta) - z|^{m+2}} d\sigma_\zeta \right)^{p/2} \right] d\sigma_z \leq \\ &\leq \left[\iint\limits_{G_R} |P_n(\zeta)|^2 \left(\iint\limits_G \frac{d\sigma_z}{|y_R(\zeta) - z|^{\frac{m+2}{2}p}} \right)^{2/p} d\sigma_\zeta \right]^{p/2} \leq \\ &\leq \sup_{\zeta \in \overline{G_R}} \left\{ \iint\limits_G \frac{d\sigma_z}{|y_R(\zeta) - z|^{\frac{m+2}{2}p}} \right\} \times \|P_n\|_{A_2(G_R)}^p \prec \\ &\prec (d(z, L_R))^{2 - \frac{m+2}{2}p} \times \|P_n\|_{A_2(G_R)}^p . \end{aligned}$$

Then, from (3.17) we have

$$\iint\limits_G |P_n^{(m)}(z)|^p d\sigma_z \prec (d(z, L_R))^{2-(m+1)p} \times \|P_n\|_{A_2(G_R)}^p . \tag{3.18}$$

Let $\zeta \in L_R$ such that $d(z, L_R) = |\zeta - z|$, $z \in L$. Taking Lemma 2 into account, we get

$$d(z, L_R) = |\zeta - z| = |\Psi(\tau) - \Psi(w)| \geq |\tau - w|^{1+k} \succ n^{-(1+k)}. \tag{3.19}$$

According to Lemma 3, from (3.17), (3.18) and (3.19), we complete the proof.

$$\|P_n^{(m)}\|_{A_p(G)} \prec n^{(1+k)\left[(m+1) - \frac{2}{p}\right]} \|P_n\|_{A_2(G)}. \blacktriangleleft$$

3.4. Proof of Theorems 6 and 3

Proof. According to Lemma 4 and (3.6), in case of $m = 0$, we obtain

$$\begin{aligned} \|P_n\|_{A_q(G)} &= \left(\iint_G |P_n(z)|^q \, d\sigma_z \right)^{1/q} = \\ &= \left(\iint_G |P_n(z)|^{q-p} \, |P_n(z)|^p \, d\sigma_z \right)^{1/q} \leq \end{aligned}$$

$$
\begin{aligned}
&\leq \max_{z\in\overline{G}} |P_n(z)|^{1-\frac{p}{q}} \left(\iint_G |P_n(z)|^p \, d\sigma_z\right)^{1/q} \prec \\
&\prec \max_{z\in\overline{G^*}} |P_n(z)|^{1-\frac{p}{q}} \|P_n\|_{A_q(G)}^{\frac{p}{q}} \prec \\
&\prec d^{2(\frac{1}{q}-\frac{1}{p})}(t, L_R) \|P_n\|_{A_p(G)} .
\end{aligned}
$$

Thus, using (3.7) and (3.8) respectively, we completed the proofs.◀

We note that the Theorems 1 -6 are sharp. For the Theorems 4 and 1, this is easy to see on the example $P_n(z) = \sum_{j=0}^{n}(j+1)z^j,\ G = B,\ m = 0,\ p = 2$ and $\alpha = 1$.(Theorem 4) and $k = 0$ (Theorem 1). In this case

$$
\|P_n\|_{C(\overline{G})} = \frac{(n+1)(n+2)}{2}; \quad \|P_n\|_{A_2(G)} = \sqrt{\frac{\pi(n+1)(n+2)}{2}}.
$$

Then, we have

$$
\begin{aligned}
\|P_n\|_{C(\overline{G})} &= \frac{(n+1)(n+2)}{2} = \\
&= \frac{(n+1)(n+2)}{2}\sqrt{\frac{2}{\pi(n+1)(n+2)}} \|P_n\|_{A_2(G)} \geq \\
&\geq \sqrt{\frac{(n+1)(n+2)}{2\pi}} \|P_n\|_{A_2(G)} \geq \\
&\geq \sqrt{\frac{1}{2\pi}}\, n \|P_n\|_{A_2(G)} .
\end{aligned}
$$

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