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On the Best Approximation of Certain Classes of Periodic Functions by Trigonometric Polynomials

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Abstract. We obtain the estimates for the best approximation in the uniform metric of the classes of 2π-periodic functions whose (ψ, β)-derivatives have a given majorant ω of the modulus of continuity. It is shown that the estimates obtained here are asymptotically exact under some natural conditions on the parameters ψ, ω and β defining the classes.

Key Words and Phrases: Best approximation, Modulus of continuity, Asymptotic formula, (ψ, β)-derivative, Convolution

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1. Introduction

Let L be the space of 2π-periodic functions summable over the period with the norm $\|f\|_1 = \int_{-\pi}^{\pi} |f(t)|\,dt$ and let C be the space of 2π-periodic continuous functions f with the norm $\|f\|_C = \max\limits_t |f(t)|$. Suppose $f \in L$ and

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{1}$$

is its Fourier series. Suppose also that $\psi(k)$ is an arbitrary numerical sequence and β is a fixed real number ($\beta \in \mathbb{R}$). If the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)}\left(a_k \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k \sin\left(kx + \frac{\beta\pi}{2}\right)\right)$$

is the Fourier series of a certain function $\varphi \in L$, then φ is called (see, e.g., [10, 11]) the (ψ, β)-derivative of the function f and is denoted by f^{ψ}_{β}. The set of continuous functions $f(x)$ having (ψ, β)-derivatives such that $f^{\psi}_{\beta} \in H_\omega$, where

$$H_\omega = \{\varphi \in C: \ |\varphi(t') - \varphi(t'')| \leqslant \omega(|t' - t''|) \ \ \forall t', t'' \in \mathbb{R}\},$$

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and $\omega(t)$ is a fixed modulus of continuity is usually denoted by $C^{\psi}_{\beta}H_{\omega}$.

For $\psi(k) = k^{-r}$, $r > 0$, the classes $C^{\psi}_{\beta}H_{\omega}$ become the well-know Weyl-Nagy classes $W^{r}_{\beta}H_{\omega}$ which, in turn, for $\beta = r$ coincide with the Weyl classes $W^{r}_{r}H_{\omega}$ (see, e.g., [11, Chap. 3, Sec. 4, 6]). For natural numbers r and $\beta = r$ we obtain the classes of periodic functions whose r-th derivatives are in the class H_{ω}.

Let $\mathfrak{M}$ be the set of all continuous functions $\psi(t)$ convex downwards for $t \geqslant 1$ and satisfying the condition $\lim\limits_{t\to\infty} \psi(t) = 0$.

If $\psi \in \mathfrak{M}'$, where

$$\mathfrak{M}' := \mathfrak{M}'(\beta) = \{\psi :\ \psi \in \mathfrak{M} \text{ when } \sin\frac{\beta\pi}{2} = 0 \text{ or}$$

$$\psi \in \mathfrak{M} \text{ and } \int_1^{\infty} \frac{\psi(t)}{t}\, dt < \infty \text{ when } \sin\frac{\beta\pi}{2} \neq 0\},$$

then the classes $C^{\psi}_{\beta}H_{\omega}$ coincide with the classes of functions $f(x)$, which are representable by the convolutions

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x+t)\Psi_{\beta}(t)\, dt, \quad \varphi \in H^0_{\omega}, \quad x \in \mathbb{R}, \tag{2}$$

(see, e.g., [10, p. 31]), where $H^0_{\omega} = \{\varphi \in H_{\omega} :\ \int_{-\pi}^{\pi} \varphi(t)\, dt = 0\}$, and $\Psi_{\beta}(t)$ is a summable function, whose Fourier series have the form $\sum\limits_{k=1}^{\infty} \psi(k)\cos(kt + \beta\pi/2)$.

The set $\mathfrak{M}$ is very inhomogeneous in the rate of convergence of functions $\psi(t)$ to zero as $t \to \infty$. This is why it was suggested in [10, pp. 115, 116] (see also [13, Subsec. 1.3]) to select subsets $\mathfrak{M}_0$ and $\mathfrak{M}_C$ from $\mathfrak{M}$ as follows:

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M} :\ 0 < \mu(t) \leqslant K < \infty, \quad \forall t \geqslant 1\},$$

$$\mathfrak{M}_C = \{\psi \in \mathfrak{M} :\ 0 < K_1 \leqslant \mu(t) \leqslant K_2 < \infty, \quad \forall t \geqslant 1\},$$

where $\mu(t) = \mu(\psi; t) = \frac{t}{\eta(t)-t}$, $\eta(t) = \eta(\psi; t) = \psi^{-1}(\psi(t)/2)$, $\psi^{-1}(\cdot)$ is the inverse function of $\psi(\cdot)$, and K, K_1, K_2 are positive constants (possibly dependent on $\psi(\cdot)$). The function $\mu(\psi; t)$ is called the modulus of half-decay of the function $\psi(t)$. It is obvious that $\mathfrak{M}_C \subset \mathfrak{M}_0$. Typical representatives of the set $\mathfrak{M}_C$ are the functions t^{-r}, $r > 0$, representatives of the set $\mathfrak{M}_0 \setminus \mathfrak{M}_C$ are the functions $\ln^{-\alpha}(t+1)$, $\alpha > 0$. Let $\mathfrak{M}'_0 = \mathfrak{M}' \cap \mathfrak{M}_0$. Natural representatives of the set $\mathfrak{M}'_0$ are the functions $\ln^{-\alpha}(t+1)$, $\alpha > 1$. It is easy to see that if $\beta = 2l$, $l \in \mathbb{Z}$, the set $\mathfrak{M}'_0$ coincide with $\mathfrak{M}_0$. Moreover, since for all $\psi \in \mathfrak{M}_C$

$$\int_n^{\infty} \frac{\psi(t)}{t}\, dt \leqslant K\psi(n), \quad n \in \mathbb{N}, \tag{3}$$

(see [11, p. 204]) then $\mathfrak{M}_C \subset \mathfrak{M}'_0$. Throughout the paper we denote the positive constants that may be different in different relations by K, K_i, $i = 1, 2$.

Let us denote the best approximation of the classes $C^{\psi}_{\beta}H_{\omega}$ by trigonometric polynomials $t_{n-1}(\cdot)$ of order not more than $n-1$ by $E_n(C^{\psi}_{\beta}H_{\omega})$, that is

$$E_n(C^{\psi}_{\beta}H_{\omega}) = \sup_{f\in C^{\psi}_{\beta}H_{\omega}} \inf_{t_{n-1}} \|f(\cdot)-t_{n-1}(\cdot)\|_C. \qquad (4)$$

As is shown in [10, p. 330] if $\omega(t)$ is an arbitrary modulus of continuity and $\psi\in\mathfrak{M}_C$, $\beta\in\mathbb{R}$ or $\psi\in\mathfrak{M}'_0$, $\beta=0$, then the following estimates hold for the quantity $E_n(C^{\psi}_{\beta}H_{\omega})$:

$$K_1\psi(n)\omega(1/n)\leqslant E_n(C^{\psi}_{\beta}H_{\omega})\leqslant K_2\psi(n)\omega(1/n). \qquad (5)$$

When $\psi(k)=k^{-r}$, $r>0$, $\beta\in\mathbb{R}$, the orders of decrease of quantity (4) have been known earlier [3] (see also [15, p. 508]).

It should be noted that unlike order estimates, exact values for the quantity $E_n(C^{\psi}_{\beta}H_{\omega})$ have been found for $\psi(k)=k^{-r}$, $r\in\mathbb{N}$, $\beta=r$ and for the convex upwards modulus of continuity by Korneichuk [5] (see also [6, p. 319], [2, p. 344]). The similar problem on the class of real-valued functions defined on the entire real axis and having the r-th continuous derivatives $f^{(r)}$ such that $\omega(f^{(r)};t)\leqslant\omega(t)$, $t\in[0,\infty)$, is solved in the paper of Ganzburg [4].

The aim of the present work is to study the rate of decrease of quantity (4) when $\psi\in\mathfrak{M}'_0$ and $\beta\in\mathbb{R}$.

2. Main Results

The following statements are true.

Theorem 1. *Let $\psi\in\mathfrak{M}'_0$, $\beta\in\mathbb{R}$ and let $\omega(t)$ be an arbitrary modulus of continuity. Then, as $n\to\infty$,*

$$E_n(C^{\psi}_{\beta}H_{\omega}) = \frac{\theta_n(\omega)}{\pi}\Big|\sin\frac{\beta\pi}{2}\Big|\int_0^{1/n}\psi\Big(\frac{1}{t}\Big)\frac{\omega(t)}{t}\,dt + O(1)\psi(n)\omega(1/n), \qquad (6)$$

where $\theta_n(\omega)\in[2/3,1]$ and $O(1)$ is a quantity uniformly bounded in n and β. If $\omega(t)$ is a convex upwards modulus of continuity, then $\theta_n(\omega)=1$.

We give an example of functions ψ and ω for which (6) is an asymptotic formula.

Example 1. *Let $\psi(t)=\ln^{-\gamma}(t+1)$, $\gamma>1$, $\beta\neq 2l$, $l\in\mathbb{Z}$ and*

$$\omega(t)=\begin{cases}0, & t=0,\\ \ln^{-\alpha}\left(\frac{1}{t}+1\right), & t>0,\quad 0<\alpha\leqslant 1.\end{cases}$$

Then by virtue of (6) the following asymptotic formula holds as $n\to\infty$:

$$E_n(C^{\psi}_{\beta}H_{\omega}) = \ln^{-(\gamma+\alpha)}(n+1)\left(\frac{1}{\pi(\gamma+\alpha-1)}\Big|\sin\frac{\beta\pi}{2}\Big|\ln n + O(1)\right),$$

where $O(1)$ is a quantity uniformly bounded in n and β.

Note that if

$$\lim_{n\to\infty}\frac{|\psi'(n)|n}{\psi(n)}=0,\quad \psi'(n):=\psi'(n+),\tag{7}$$

and

$$\lim_{n\to\infty}\frac{\omega'(1/n)}{\omega(1/n)n}=0,\quad \omega'(1/n):=\omega'(1/n+),\tag{8}$$

then equalities

$$\lim_{n\to\infty}\frac{\psi(n)\omega(1/n)}{\int_0^{1/n}\psi(\frac{1}{t})\frac{\omega(t)}{t}\,dt}=\lim_{n\to\infty}\frac{|\psi'(n)|n}{\psi(n)}+\lim_{n\to\infty}\frac{\omega'(1/n)}{\omega(1/n)n}=0,$$

are valid.

Therefore from Theorem 1 we obtain

Corollary 1. *Assume that $\psi\in\mathfrak{M}'_0$, $\beta\neq 2l$, $l\in\mathbb{Z}$, $\omega(t)$ is a convex upwards modulus of continuity and conditions* (7) *and* (8) *are fulfilled. Then the following asymptotic formula holds as $n\to\infty$:*

$$E_n(C^\psi_\beta H_\omega)=\frac{1}{\pi}\Big|\sin\frac{\beta\pi}{2}\Big|\int_0^{1/n}\psi\Big(\frac{1}{t}\Big)\frac{\omega(t)}{t}\,dt+O(1)\psi(n)\omega(1/n),$$

where $O(1)$ is a quantity uniformly bounded in n and β.

The functions ψ and ω from Example 1 can serve as an example of the functions which satisfy conditions (7) and (8), respectively.

Relation (6) implies that if $\psi\in\mathfrak{M}'_0$ and

$$\Big|\sin\frac{\beta\pi}{2}\Big|\int_0^{1/n}\frac{\omega(t)}{t}\,dt=O(1)\omega(1/n),\quad \beta\in\mathbb{R},\tag{9}$$

or $\psi\in\mathfrak{M}_C$ (see (3)), then

$$E_n(C^\psi_\beta H_\omega)=O(1)\psi(n)\omega(1/n).$$

Taking into account that function $\psi(t)$ is monotonically decreasing for $t\geqslant 1$ and using the estimate

$$E_n(C^\psi_\beta H_\omega)\geqslant K\psi(n)\omega(1/n),\quad \forall\psi\in\mathfrak{M}',\tag{10}$$

(see [10, pp. 329, 330]), by virtue of relation (6) we arrive at the following statement:

Corollary 2. *Let $\beta\in\mathbb{R}$ and let $\omega(t)$ be an arbitrary modulus of continuity. If $\psi\in\mathfrak{M}_C$ or $\psi\in\mathfrak{M}'_0$ and $\omega(t)$ satisfies condition* (9)*, then*

$$K_1\psi(n)\omega(1/n)\leqslant E_n(C^\psi_\beta H_\omega)\leqslant K_2\psi(n)\omega(1/n),\tag{11}$$

where K_1 and K_2 are positive constants.

Thus, estimates (5) obtained by Stepanets [10, p. 330] (see also [11, Chap. 5, Sec. 22; Chap. 7, Sec. 4]) for the arbitrary modulus of continuity $\omega(t)$ and for $\psi\in\mathfrak{M}_C$, $\beta\in\mathbb{R}$ or for $\psi\in\mathfrak{M}'_0$, $\beta=0$, hold also in the case when $\psi\in\mathfrak{M}'_0$, $\beta\neq 0$ and $\omega(t)$ satisfies condition (9). For example, the function $\omega(t)=t^\alpha$, $0<\alpha\leqslant 1$, satisfies (9).

3. Proof of Theorem 1

Suppose that all conditions of the theorem are satisfied. Let us carry out the proof in two stages.

1. We shall find an upper estimate for $E_n(C^\psi_\beta H_\omega)$.

We set

$$U^\psi_{n-1}(f;x)=\frac{a_0}{2}+\sum_{k=1}^{n-1}\Big(1-\frac{\psi(n)}{\psi(k)}\frac{k^2}{n^2}\Big)(a_k\cos kx+b_k\sin kx),\quad n\in\mathbb{N},\tag{12}$$

where a_k and b_k are the Fourier coefficients of a function $f\in C^\psi_\beta H_\omega$. Show that for the quantity

$$\mathcal{E}_n(C^\psi_\beta H_\omega)=\sup_{f\in C^\psi_\beta H_\omega}\|f(\cdot)-U^\psi_{n-1}(f;\cdot)\|_C,$$

the inequality

$$\mathcal{E}_n(C^\psi_\beta H_\omega)\leqslant\frac{1}{\pi}\Big|\sin\frac{\beta\pi}{2}\Big|\int_0^{1/n}\psi\Big(\frac{1}{t}\Big)\frac{\omega(t)}{t}\,dt+O(1)\psi(n)\omega(1/n),\tag{13}$$

is true. Since

$$E_n(C^\psi_\beta H_\omega)\leqslant\mathcal{E}_n(C^\psi_\beta H_\omega),\tag{14}$$

then the required upper estimate for $E_n(C^\psi_\beta H_\omega)$ follows from (13).

For further reasoning, we need the one statement, which follows from the results of book [10, p. 65]. We will give a few notations before formulating it. Let f be a summable function, whose Fourier series have the form (1). Further, let $\lambda_n=\{\lambda_1(u),\lambda_2(u),\dots,\lambda_n(u)\}$ be a collection of continuous functions on $[0,1]$ such that $\lambda(k/n)=\lambda^{(n)}_k$, $k=\overline{0,n}$, $n\in\mathbb{N}$, where $\lambda^{(n)}_k$ are elements of the triangular matrix $\Lambda=\|\lambda^{(n)}_k\|$, $k=\overline{1,n}$, $\lambda^{(n)}_0=1$, that determine a polynomial of the form

$$U_n(f;x;\Lambda)=\frac{a_0}{2}+\sum_{k=1}^{n}\lambda^{(n)}_k(a_k\cos kx+b_k\sin kx),\quad n\in\mathbb{N}.\tag{15}$$

The following statement is true:

Lemma A [10, p. 65]. *Suppose that $f\in C^\psi_\beta H_\omega$ and $\tau_n(u)$ is the continuous function defined by relation*

$$\tau_n(u)=\tau_n(u;\lambda;\psi)=\begin{cases}(1-\lambda_n(u))\psi(1)nu, & 0\leqslant u\leqslant\frac{1}{n},\\ (1-\lambda_n(u))\psi(nu), & \frac{1}{n}\leqslant u\leqslant 1,\\ \psi(nu), & u\geqslant 1,\end{cases}\tag{16}$$

and such that its Fourier transform

$$\widehat{\tau}_n(t):=\widehat{\tau}_n(t;\beta)=\frac{1}{\pi}\int_0^\infty\tau_n(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)\,du,\quad \beta\in\mathbb{R},$$

is summable on the whole real line, i.e. $\int_{-\infty}^{\infty}|\widehat{\tau}_n(t)|\,dt<\infty$. *Then at any point* x *the following equality holds*:

$$f(x)-U_n(f;x;\Lambda)=\int_{-\infty}^{\infty}f_\beta^\psi\Big(x+\frac{t}{n}\Big)\widehat{\tau}_n(t)\,dt,\quad n\in\mathbb{N}. \tag{17}$$

Using Lemma A, let us show that

$$f(x)-U_{n-1}^\psi(f;x)=\int_{-\infty}^{\infty}f_\beta^\psi\Big(x+\frac{t}{n}\Big)\widehat{\tau}_n(t)\,dt,\quad \forall f\in C_\beta^\psi H_\omega,\quad n\in\mathbb{N}, \tag{18}$$

where $\widehat{\tau}_n(t)$ is the Fourier transform of the function

$$\tau_n(u)=\tau_n(u;\psi)=\begin{cases}\psi(n)u^2, & 0\leqslant u\leqslant 1,\\ \psi(nu), & u\geqslant 1.\end{cases} \tag{19}$$

Since polynomial (12) can be represented in the form

$$U_{n-1}^\psi(f;x)=\frac{a_0}{2}+\sum_{k=1}^{n}\lambda^\psi(k/n)(a_k\cos kx+b_k\sin kx),$$

where $\lambda^\psi(k/n)$ are the values of continuous function

$$\lambda^\psi(u)=\lambda_n^\psi(u)=\begin{cases}1-\frac{\psi(n)}{\psi(1)}\frac{u}{n}, & 0\leqslant u\leqslant \frac{1}{n},\\ 1-\frac{\psi(n)}{\psi(nu)}u^2, & \frac{1}{n}\leqslant u\leqslant 1,\end{cases} \tag{20}$$

at the points $u=k/n$ and

$$\tau_n(u)=\tau_n(u;\psi)=\begin{cases}(1-\lambda^\psi(u))\psi(1)nu, & 0\leqslant u\leqslant\frac{1}{n},\\ (1-\lambda^\psi(u))\psi(nu), & \frac{1}{n}\leqslant u\leqslant 1,\\ \psi(nu), & u\geqslant 1,\end{cases}$$

then it follows from Lemma A that for proving (18) it is sufficient to establish the inequality

$$\int_{-\infty}^{\infty}|\widehat{\tau}_n(t)|\,dt<\infty. \tag{21}$$

With this aim we put

$$\mu_n(u)=\begin{cases}\psi(n)(u^2-u), & 0\leqslant u\leqslant 1,\\ 0, & u\geqslant 1,\end{cases}\qquad \nu_n(u)=\tau_n(u)-\mu_n(u).$$

Integrating twice by parts, we get

$$\widehat{\mu}_n(t):=\widehat{\mu}_n(t;\beta)=\frac{1}{\pi}\int_0^\infty\mu_n(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)\,du=\frac{O(1)}{t^2},\quad t>0,$$

which yields

$$\int_{|t|\geqslant 1} |\widehat{\mu}_n(t)|\,dt < \infty. \tag{22}$$

It is obvious that

$$\int_{|t|\leqslant 1} |\widehat{\mu}_n(t)|\,dt < \infty. \tag{23}$$

Taking (22), (23) together and using the estimates

$$\int_{-\infty}^{\infty} |\widehat{\nu}_n(t)|\,dt < \infty \quad \forall \psi \in \mathfrak{M}_0',$$

(see, e.g., [11, p. 174]) and

$$|\widehat{\tau}_n(t)| \leqslant |\widehat{\mu}_n(t)| + |\widehat{\nu}_n(t)|,$$

we obtain (21).

Furthermore, since the function $\tau_n(u)$ satisfies all conditions of Lemma 3 from [14] according to which

$$\tau_n(u) = \int_{-\infty}^{\infty} \cos\Big(ut + \frac{\beta\pi}{2}\Big)\widehat{\tau}_n(t)\,dt, \quad u \geqslant 0,$$

we have

$$\int_{-\infty}^{\infty} \widehat{\tau}_n(t)\,dt = \frac{\tau_n(0)}{\cos\frac{\beta\pi}{2}} = 0, \quad \beta \neq 2l-1, \quad l \in \mathbb{Z}.$$

If $\beta = 2l-1$, $l \in \mathbb{Z}$, the equality $\int_{-\infty}^{\infty} \widehat{\tau}_n(t)\,dt = 0$ is obvious, because $\widehat{\tau}_n(t)$ is odd. Hence, starting from (18) we can write

$$f(x) - U_{n-1}^{\psi}(f;x) = \int_{-\infty}^{\infty} \Big(f_\beta^\psi\Big(x + \frac{t}{n}\Big) - f_\beta^\psi(x)\Big)\widehat{\tau}_n(t)\,dt \quad \forall f \in C_\beta^\psi H_\omega, \quad n \in \mathbb{N}. \tag{24}$$

Since $f_\beta^\psi \in H_\omega^0$ and, as it is not hard to see, for every $\varphi \in H_\omega^0$ function $\varphi_1(u) = \varphi(u+h)$, $h \in \mathbb{R}$, also belongs to H_ω^0, then using the notation

$$\delta(t,\varphi) = \varphi(t) - \varphi(0),$$

it follows from (24) that

$$\mathcal{E}_n(C_\beta^\psi H_\omega) \leqslant \sup_{\varphi\in H_\omega^0} \bigg|\int_{-\infty}^{\infty} \Big(\varphi\Big(\frac{t}{n}\Big) - \varphi(0)\Big)\widehat{\tau}_n(t)\,dt\bigg| = \sup_{\varphi\in H_\omega^0} \bigg|\int_{-\infty}^{\infty} \delta\Big(\frac{t}{n},\varphi\Big)\widehat{\tau}_n(t)\,dt\bigg|. \tag{25}$$

Now we shall simplify the integral in the right-hand side of (25) without loss of its principal value. The following relations are true:

$$\int_{-\infty}^{\infty} \delta\Big(\frac{t}{n},\varphi\Big)\widehat{\tau}_n(t)\,dt =$$

$$= \frac{\cos\frac{\beta\pi}{2}}{\pi}\int_{-\infty}^{\infty}\delta\Big(\frac{t}{n},\varphi\Big)\int_0^{\infty}\tau_n(u)\cos ut\,du\,dt - \frac{\sin\frac{\beta\pi}{2}}{\pi}\int_{-\infty}^{\infty}\delta\Big(\frac{t}{n},\varphi\Big)\int_0^{\infty}\tau_n(u)\sin ut\,du\,dt =$$

$$= \frac{\cos\frac{\beta\pi}{2}}{\pi}\int_{-\infty}^{\infty}\delta\Big(\frac{t}{n},\varphi\Big)\int_0^{\infty}\tau_n(u)\cos ut\,du\,dt - \frac{\sin\frac{\beta\pi}{2}}{\pi}\Bigg(\int_{|t|\geqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\int_0^{\infty}\tau_n(u)\sin ut\,du\,dt$$

$$+\int_{|t|\leqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\int_0^{1}\tau_n(u)\sin ut\,du\,dt + \int_{|t|\leqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\int_1^{\infty}\psi(nu)\sin ut\,du\,dt\Bigg). \qquad (26)$$

Integrating by parts, taking into account the equality $\tau_n(0) = \tau_n(\infty) = 0$ and assuming that $\psi'(u) := \psi'(u+)$, we have

$$\int_0^{\infty}\tau_n(u)\cos ut\,du = -\frac{1}{t}\int_0^{\infty}\tau_n'(u)\sin ut\,du =$$

$$= -\frac{2\psi(n)}{t}\int_0^{1}u\sin ut\,du - \frac{n}{t}\int_1^{\infty}\psi'(nu)\sin ut\,du, \qquad (27)$$

and similarly

$$\int_0^{\infty}\tau_n(u)\sin ut\,du = \frac{2\psi(n)}{t}\int_0^{1}u\cos ut\,du + \frac{n}{t}\int_1^{\infty}\psi'(nu)\cos ut\,du. \qquad (28)$$

Combining (26)–(28), we obtain

$$\int_{-\infty}^{\infty}\delta\Big(\frac{t}{n},\varphi\Big)\widehat{\tau}_n(t)\,dt =$$

$$= -\frac{\sin\frac{\beta\pi}{2}}{\pi}\int_{|t|\leqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\int_1^{\infty}\psi(nu)\sin ut\,du\,dt + r_n(\psi,\varphi,\beta),\ \ \varphi\in H_\omega^0,\ \ n\in\mathbb{N}, \qquad (29)$$

where

$$r_n(\psi,\varphi,\beta) = \frac{\cos\frac{\beta\pi}{2}}{\pi}\Bigg(-2\psi(n)\int_{-\infty}^{\infty}\delta\Big(\frac{t}{n},\varphi\Big)\frac{1}{t}\int_0^{1}u\sin ut\,du\,dt -$$

$$-n\int_{-\infty}^{\infty}\delta\Big(\frac{t}{n},\varphi\Big)\frac{1}{t}\int_1^{\infty}\psi'(nu)\sin ut\,du\,dt\Bigg) -$$

$$-\frac{\sin\frac{\beta\pi}{2}}{\pi}\Bigg(2\psi(n)\int_{|t|\geqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\frac{1}{t}\int_0^{1}u\cos ut\,du\,dt +$$

$$+n\int_{|t|\geqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\frac{1}{t}\int_1^{\infty}\psi'(nu)\cos ut\,du\,dt +$$

$$+\int_{|t|\leqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\int_0^{1}\tau_n(u)\sin ut\,du\,dt\Bigg) = \frac{\cos\frac{\beta\pi}{2}}{\pi}\sum_{i=1}^{2}J_{i,n} - \frac{\sin\frac{\beta\pi}{2}}{\pi}\sum_{i=3}^{5}J_{i,n}. \qquad (30)$$

Let us show that

$$r_n(\psi,\varphi,\beta)=O(1)\psi(n)\omega(1/n). \tag{31}$$

Since for $t\in[-1,1]$ the quantity

$$\frac{1}{t}\int_0^1 u\sin ut\,du,$$

is bounded by a constant, then using the inequality $|\delta(t,\varphi)|\leqslant\omega(|t|)$, we get

$$J_{1,n}=-2\psi(n)\int_{|t|\geqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\frac{1}{t}\int_0^1 u\sin ut\,du\,dt+O(1)\psi(n)\omega(1/n). \tag{32}$$

To estimate the integral in (32) we establish the following auxiliary statements.

Lemma 1. *On every interval* $(x_k^{(i)},x_{k+1}^{(i)})$, $x_k^{(i)}=(2k-1+i)\pi/2a$, $i=0,1$, $k\in\mathbb{N}$, $a>0$, *the function*

$$\int_x^\infty\frac{1}{t}\int_0^a u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt,\qquad x>0,\qquad s\geqslant 1,$$

has at least one zero.

Proof. We will give a proof of the lemma only for the case $i=0$, because the proof in case $i=1$ is similar. On the basis of the estimate $\big|\int_x^\infty\frac{\sin t}{t}\,dt\big|\leqslant\frac{2}{x}$, $x>0$ (see, e.g., [1, p. 5], [9, p. 343]) it is simple to see that the integral

$$\int_x^\infty\frac{u^s\sin ut}{t}\,dt=u^s\int_{ux}^\infty\frac{\sin t}{t}\,dt,$$

converges uniformly with respect to $u\in[0,a]$, $a>0$. Therefore, changing the order of integration, we obtain

$$S(x):=\int_x^\infty\frac{1}{t}\int_0^a u^s\sin ut\,du\,dt=\int_0^a u^s\int_x^\infty\frac{\sin ut}{t}\,dt\,du.$$

Making the change of variables and integrating by parts, we have

$$S(x)=\int_0^a u^s\int_{ux}^\infty\frac{\sin t}{t}\,dt\,du=\frac{1}{s+1}\bigg(a^{s+1}\int_{ax}^\infty\frac{\sin t}{t}\,dt+\int_0^a u^s\sin ux\,du\bigg)=$$

$$=\frac{1}{s+1}\bigg(a^{s+1}\int_{ax}^\infty\frac{\sin t}{t}\,dt-a^s\frac{\cos ax}{x}+\frac{s}{x}\int_0^a u^{s-1}\cos ux\,du\bigg).$$

Hence, taking into account the equation

$$\int_{ax}^\infty\frac{\sin t}{t}\,dt=\frac{\cos ax}{ax}-\int_{ax}^\infty\frac{\cos t}{t^2}\,dt,$$

we get

$$S(x)=\frac{1}{s+1}\bigg(-a^{s+1}\int_{ax}^{\infty}\frac{\cos t}{t^2}\,dt+\frac{s}{x^{s+1}}\int_0^{ax}u^{s-1}\cos u\,du\bigg). \tag{33}$$

On every interval $(t_j,t_{j+1}),\ t_j=(2j+1)\pi/2,\ j=0,1,\dots,$ the function $\int_x^\infty \frac{\cos t}{t^2}\,dt$ vanishes with a change of sign at some point $\widetilde{x}_j$. Since

$$\int_{\pi/2}^{\infty}\frac{\cos t}{t^2}\,dt=-\int_{\pi/2}^{\infty}\frac{\sin t}{t}\,dt<0,$$

then for any $k\in\mathbb{N}$

$$\operatorname{sign}\int_{(2k-1)\pi/2}^{\infty}\frac{\cos t}{t^2}\,dt=(-1)^k. \tag{34}$$

Further, we have

$$\int_0^{(2k-1)\pi/2}u^{s-1}\cos u\,du=\alpha_0+\sum_{j=1}^{k-1}\alpha_j,$$

where

$$\alpha_0=\int_0^{\pi/2}u^{s-1}\cos u\,du,\qquad \alpha_j=\int_{(2j-1)\pi/2}^{(2j+1)\pi/2}u^{s-1}\cos u\,du.$$

If $k=1$, then

$$\operatorname{sign}\int_0^{(2k-1)\pi/2}u^{s-1}\cos u\,du=\operatorname{sign}\alpha_0=1. \tag{35}$$

Let $k=2,3,\dots$ Since the function u^{s-1} does not decrease $(s\geqslant 1)$ for $u\geqslant 0$, we can write

$$|\alpha_0|<|\alpha_j|\leqslant|\alpha_{j+1}|,\qquad j\geqslant 1,$$

and respectively

$$\operatorname{sign}\int_0^{(2k-1)\pi/2}u^{s-1}\cos u\,du=\operatorname{sign}\int_{(2k-3)\pi/2}^{(2k-1)\pi/2}u^{s-1}\cos u\,du=(-1)^{k+1},\quad k=2,3,\cdots. \tag{36}$$

Taking account of (33)–(36), we have

$$\operatorname{sign}S\bigg(\frac{2k-1}{2a}\pi\bigg)=(-1)^{k+1},\quad k\in\mathbb{N},\quad a>0. \tag{37}$$

The function $S(x)$ is continuous for any $x>0$. Therefore, it follows from (37) that on every interval (x_k,x_{k+1}), where $x_k=(2k-1)\pi/2a,\ k\in\mathbb{N},\ a>0$, the function $S(x)$ has the required zero. Lemma 1 is proved. ◄

Lemma 2. *Let $\varphi\in H_\omega,\ 1\leqslant a\leqslant n,\ n\in\mathbb{N}$ and $s\geqslant 1$. Then for $i=0,1$, the following estimate holds*:

$$\int_{|t|\geqslant 1}\Big(\varphi\Big(\frac{t}{n}\Big)-\varphi(0)\Big)\frac{1}{t}\int_0^{a/n}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt=O(1)\omega(1/n), \tag{38}$$

where $O(1)$ is a quantity uniformly bounded in n, φ, a and s.

Proof. Making the change of variables, we get

$$\int_{|t|\geqslant 1}\Big(\varphi\Big(\frac{t}{n}\Big)-\varphi(0)\Big)\frac{1}{t}\int_0^{a/n}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt=$$

$$=\frac{1}{n^{s+1}}\int_{|t|\geqslant 1/n}(\varphi(t)-\varphi(0))\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt,\quad i=0,1. \tag{39}$$

Let us denote by $t_k^{(i)}$ the zero of function

$$\int_x^{\infty}\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt,\quad i=0,1,$$

on interval $(x_k^{(i)},x_{k+1}^{(i)})$, $x_k^{(i)}=\frac{2k-1+i}{2a}\pi$, which exists according to Lemma 1. Using the notation $\delta(t)=\varphi(t)-\varphi(0)$, we have

$$\Big|\int_{1/n}^{\infty}\delta(t)\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt\Big|=\Big|\int_{1/n}^{t_1^{(i)}}\delta(t)\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt+$$

$$+\sum_{k=1}^{\infty}\int_{t_k^{(i)}}^{t_{k+1}^{(i)}}\big(\delta(t)-\delta(t_k^{(i)})\big)\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt\Big|\leq$$

$$\leqslant\omega(t_1^{(i)})\int_{1/n}^{t_1^{(i)}}\frac{1}{t}\Big|\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\Big|\,dt+\omega(\Delta_i)\int_{t_1^{(i)}}^{\infty}\frac{1}{t}\Big|\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\Big|\,dt, \tag{40}$$

where $\Delta_i=\sup\limits_k(t_{k+1}^{(i)}-t_k^{(i)})$. Since $t_1^{(i)}<\frac{2\pi}{a}$ and $\Delta_i<\frac{2\pi}{a}$, it follows from (40) that

$$\Big|\int_{1/n}^{\infty}\delta(t)\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt\Big|<\omega\Big(\frac{2\pi}{a}\Big)\int_{1/n}^{\infty}\frac{1}{t}\Big|\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\Big|\,dt. \tag{41}$$

After integrating by parts it is easy to see, that

$$\Big|\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\Big|\leqslant\frac{2a^s}{t},\quad t>0,\quad i=0,1. \tag{42}$$

From (41) and (42) follows the inequality

$$\Big|\int_{1/n}^{\infty}\delta(t)\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt\Big|<2a^s\omega\Big(\frac{2\pi}{a}\Big)\int_{1/n}^{\infty}\frac{dt}{t^2}=2a^s\omega\Big(\frac{2\pi}{a}\Big)n\leqslant$$

$$\leqslant 2a^s\Big(\frac{2\pi n}{a}+1\Big)\omega\Big(\frac{1}{n}\Big)n<8a^{s-1}\pi n^2\omega\Big(\frac{1}{n}\Big)\leqslant 8\pi n^{s+1}\omega\Big(\frac{1}{n}\Big),\ i=0,1. \tag{43}$$

The estimate

$$\int_{-\infty}^{-1/n}\delta(t)\frac{1}{t}\int_0^{a}u^s\sin\Big(ut+\frac{i\pi}{2}\Big)\,du\,dt=O(1)n^{s+1}\omega(1/n),\quad i=0,1, \tag{44}$$

is similarly proved. Comparing relations (43), (44) and (39), we obtain (38). Lemma 2 is proved. ◀

Applying Lemma 2 to the integral in (32) and, at the same time, to $J_{3,n}$, we have

$$J_{1,n} = O(1)\psi(n)\omega(1/n), \tag{45}$$

$$J_{3,n} = O(1)\psi(n)\omega(1/n). \tag{46}$$

In the monograph [11, pp. 212, 216, see relations (4.26′) and (4.42), (4.45), (4.46)] it is shown, that

$$J_{2,n} = O(1)\psi(n)\omega(1/n), \quad \forall\psi \in \mathfrak{M}_0, \tag{47}$$

and

$$J_{4,n} = O(1)\psi(n)\omega(1/n) \quad \forall\psi \in \mathfrak{M}_0', \quad \beta \neq 2l, \quad l \in \mathbb{Z}. \tag{48}$$

Since $|\tau_n(u)| \leqslant \psi(n)$, $u \in [0,1]$, it is clear that

$$J_{5,n} = O(1)\psi(n)\omega(1/n). \tag{49}$$

Comparing (30), (45)–(49), we arrive at (31). Then from (29) for any function $\varphi \in H_\omega^0$ and $n \in \mathbb{N}$, we obtain

$$\int_{-\infty}^{\infty} \delta\Big(\frac{t}{n},\varphi\Big)\widehat{\tau}_n(t)\,dt = -\frac{\sin\frac{\beta\pi}{2}}{\pi}\int_{|t|\leqslant 1}\delta\Big(\frac{t}{n},\varphi\Big)\int_1^\infty \psi(nu)\sin ut\,du\,dt + O(1)\psi(n)\omega(1/n) =$$

$$= -\frac{\sin\frac{\beta\pi}{2}}{\pi}\int_0^1\bigg(\delta\Big(\frac{t}{n},\varphi\Big) - \delta\Big(-\frac{t}{n},\varphi\Big)\bigg)\int_1^\infty \psi(nu)\sin ut\,du\,dt+$$
$$+O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}_0', \quad \beta \in \mathbb{R}. \tag{50}$$

Since

$$\int_1^\infty \psi(nu)\sin ut\,du > 0, \quad t \in (0,1], \quad \psi \in \mathfrak{M}', \quad \beta \neq 2l,\ l \in \mathbb{Z}, \tag{51}$$

(see, e.g., [12, p. 143]) and

$$\int_0^1 \omega\Big(\frac{2t}{n}\Big)\int_1^\infty \psi(nu)\sin ut\,du\,dt =$$

$$= \int_0^{1/n}\psi\Big(\frac{1}{t}\Big)\frac{\omega(t)}{t}\,dt + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}_0', \quad \beta \neq 2l,\ l \in \mathbb{Z}, \tag{52}$$

(see [8, p. 528]), from (25) and (50) we obtain (13). Putting together inequalities (13) and (14) we find a required estimate for quantity (4):

$$E_n(C_\beta^\psi H_\omega) \leqslant \frac{1}{\pi}\Big|\sin\frac{\beta\pi}{2}\Big|\int_0^{1/n}\psi\Big(\frac{1}{t}\Big)\frac{\omega(t)}{t}\,dt + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}_0', \quad \beta \in \mathbb{R}. \tag{53}$$

2. Let us find a lower bound for $E_n(C_\beta^\psi H_\omega)$.

Let $\varphi_n(t)$ be an odd $2\pi/n$-periodic function defined on $[0,\pi/n]$ by the equalities

$$\varphi_n(t) = \begin{cases} \frac{c_\omega}{2}\omega(2t), & t\in[0,\pi/2n], \\ \frac{c_\omega}{2}\omega(\frac{2\pi}{n}-2t), & t\in[\pi/2n,\pi/n], \end{cases}$$

where $c_\omega = 1$ if $\omega(t)$ is a convex upwards modulus of continuity and $c_\omega = 2/3$ otherwise. As shown in [10, pp. 83–85] if $\omega(t)$ is an arbitrary modulus of continuity, then

$$|\varphi_n(t') - \varphi_n(t'')| \leqslant \omega(|t'-t''|), \quad t', t'' \in [-\pi/2n, \pi/2n].$$

This implies that

$$|\varphi_n(t') - \varphi_n(t'')| \leqslant \omega(|t'-t''|), \quad t', t'' \in \mathbb{R},$$

and, hence, $\varphi_n \in H_\omega$. We denote by $f^*(\cdot)$ the function from the set $C^\psi_\beta H_\omega$, $\psi\in\mathfrak{M}'$, whose (ψ,β)-derivative $f^{*\psi}_\beta(t)$ coincides with the function $\varphi_n(t)$ on a period. By relations (2), such a function $f^*(\cdot)$ exists.

In virtue of formula (3.4) from the book [10, Chap. 2, Subsec. 3.1] the following equality holds for any $f\in C^\psi_\beta H_\omega$, $\psi\in\mathfrak{M}'$:

$$f(x) - U_{n-1}(f;x;\Lambda) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^\psi_\beta(x+t)\bigg(\sum_{k=1}^\infty \psi(k)\cos\Big(kt+\frac{\beta\pi}{2}\Big) - $$
$$-\sum_{k=1}^{n-1}\lambda_k^{(n)}\psi(k)\cos\Big(kt+\frac{\beta\pi}{2}\Big)\bigg)\,dt, \quad x\in\mathbb{R}, \quad n\in\mathbb{N}, \tag{54}$$

where $U_{n-1}(f;x;\Lambda)$ is a trigonometric polynomial of the form (15), such that $\lambda_n^{(n)} = 0$. Since function $\varphi_n(t)$ is odd $2\pi/n$-periodic, the equalities

$$\int_{-\pi}^{\pi}\varphi_n(t)\sin kt\,dt = 0, \quad k=1,2,\ldots,n-1, \quad n\geqslant 2, \tag{55}$$

(see, e.g., [6, p. 159]) and

$$\varphi_n\Big(\frac{i\pi}{n}+t\Big) = (-1)^i\varphi_n(t), \quad i\in\mathbb{Z},$$

hold. Then, using relation (54) for $f^*(\cdot)$, we obtain

$$f^*\big(\frac{i\pi}{n}\big) - U_{n-1}\big(f^*;\frac{i\pi}{n};\Lambda\big) =$$

$$= \frac{(-1)^i}{\pi}\int_{-\pi}^{\pi}\varphi_n(t)\bigg(\sum_{k=1}^\infty\psi(k)\cos\Big(kt+\frac{\beta\pi}{2}\Big) - \sum_{k=1}^{n-1}\lambda_k^{(n)}\psi(k)\cos\Big(kt+\frac{\beta\pi}{2}\Big)\bigg)\,dt =$$
$$= \frac{(-1)^i}{\pi}\int_{-\pi}^{\pi}\varphi_n(t)\sum_{k=1}^\infty\psi(k)\cos\Big(kt+\frac{\beta\pi}{2}\Big)\,dt =$$

$$= \frac{(-1)^i}{\pi} \sin\frac{\beta\pi}{2} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=n}^{\infty} \psi(k) \sin kt\, dt, \quad i \in \mathbb{Z}, \quad n = 2, 3, \cdots . \tag{56}$$

It is obvious from this that there exist $2n$ points $t_i = \frac{i\pi}{n}$, $i = 0, 1, \ldots, 2n-1$, on the period $[0, 2\pi)$ at which the difference

$$f^*(x) - U_{n-1}(f^*; x; \Lambda),$$

takes values with alternating signs. Then by the de la Vallée Poussin theorem [7] (see also [10, p. 312], [11, p. 491]), we find

$$E_n(f^*) \geqslant \frac{1}{\pi} \Big| \sin\frac{\beta\pi}{2} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=n}^{\infty} \psi(k) \sin kt\, dt \Big|, \quad \psi \in \mathfrak{M}', \tag{57}$$

where

$$E_n(f^*) = \inf_{t_{n-1}} \|f^*(\cdot) - t_{n-1}(\cdot)\|_C, \quad n \in \mathbb{N}.$$

From (56) and (57) it follows, in particular, that

$$E_n(f^*) \geqslant |f^*(0) - U_{n-1}(f^*; 0; \Lambda)|, \quad n = 2, 3, \cdots . \tag{58}$$

Inequality (58) is satisfied for triangular matrix $\Lambda = \|\lambda_k^{(n)}\|$, $k = \overline{1, n}$, such that $\lambda_n^{(n)} = 0$. Let's define its remaining elements in the following way:

$$\lambda_k^{(n)} = \lambda^{\psi}(k/n), \quad k = \overline{1, n-1}, \quad n \in \mathbb{N},$$

where $\lambda^{\psi}(\cdot)$ is defined by (20). Since in this case

$$U_{n-1}(f^*; 0; \Lambda) = U_{n-1}^{\psi}(f^*; 0),$$

then from (58) we obtain, taking the inequality $E_n(C_{\beta}^{\psi} H_{\omega}) \geqslant E_n(f^*)$ into account,

$$E_n(C_{\beta}^{\psi} H_{\omega}) \geqslant |f^*(0) - U_{n-1}^{\psi}(f^*; 0)|, \quad n = 2, 3, \ldots, \ \psi \in \mathfrak{M}'. \tag{59}$$

By virtue of (24) and (50)

$$f^*(0) - U_{n-1}^{\psi}(f^*; 0) = \int_{-\infty}^{\infty} \left(f^{*\psi}_{\ \beta}\Big(\frac{t}{n}\Big) - f^{*\psi}_{\ \beta}(0) \right) \widehat{\tau}_n(t)\, dt =$$

$$= -\frac{\sin\frac{\beta\pi}{2}}{\pi} \int_0^1 \left(\varphi_n\Big(\frac{t}{n}\Big) - \varphi_n\Big(-\frac{t}{n}\Big) \right) \int_1^{\infty} \psi(nu) \sin ut\, du\, dt + O(1)\psi(n)\omega(1/n) =$$

$$= -c_{\omega} \frac{\sin\frac{\beta\pi}{2}}{\pi} \int_0^1 \omega\Big(\frac{2t}{n}\Big) \int_1^{\infty} \psi(nu) \sin ut\, du\, dt + O(1)\psi(n)\omega(1/n), \ \psi \in \mathfrak{M}_0'. \tag{60}$$

Combining (51), (52), (59) and (60), we arrive at the desired estimate

$$E_n(C_{\beta}^{\psi} H_{\omega}) \geqslant \frac{c_{\omega}}{\pi} \Big| \sin\frac{\beta\pi}{2} \Big| \int_0^{1/n} \psi\Big(\frac{1}{t}\Big) \frac{\omega(t)}{t}\, dt + O(1)\psi(n)\omega(1/n), \ \psi \in \mathfrak{M}_0', \ \beta \neq 2l, \ l \in \mathbb{Z}. \tag{61}$$

From (53) and (61) we obtain formula (6). Theorem 1 is proved.

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