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On Iwaniec-Sbordone spaces on sets which may have infinite measure

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Abstract. We introduce grand Lebesgue spaces on open sets Ω of infinite measure in $\mathbb{R}^n$, controlling the integrability of $|f(x)|^{p-\varepsilon}$ at infinity by means of a weight (depending also on ε); in general, such spaces are different for different ways to introduce dependence of the weight on ε. We prove some properties of these spaces. We introduce also a version of weighted grand Lebesgue spaces, different from the usual ones, for bounded sets, in which together with the passage from p to $p-\varepsilon$ we introduce a weight also depending on ε. In both versions we show that every linear operator bounded in a Lebesgue space with Muckenhoupt weights is also bounded in the corresponding grand Lebesgue space a Muckenhoupt weight (for bounded or unbounded Ω).

Key Words and Phrases: grand Lebesgue spaces, boundedness of linear operators

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1. Introduction

In 1992 T. Iwaniec and C. Sbordone [11] introduced a new type of function spaces which was called *grand Lebesgue spaces* by them. Harmonic analysis related to these spaces and also *small Lebesgue spaces* associated with them, was intensively developed during last years and they continue to attract attention of researchers due to various applications (C. Capone, G. Di Fratto, A. Fiorenza, L. Greco, B. Gupta, T. Iwaniec, P. Jain, G.E. Karadzhov, V. Kokilashvili, P. Koskela, M. Krbec, A. Mercaldo, A. Meskhi, M. Milman, J.M. Rakotoson, C. Sbordone, X. Zhong, e.g.). Introduced in relation to problems of integrability of jacobians, these papers proved to be appropriate in various applications in partial differential equations and variational problems, they were used in the study of maximal functions, extrapolation theory etc, we refer for instance to [1, 3, 4, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19].

Such spaces were introduced and studied for functions defined on an open set of bounded measure (in $\mathbb{R}^n$ or, more generally homogeneous spaces), since the basic idea of their definition is based on the supposition that a function f in grand Lebesgue space must be integrable to the power $p-\varepsilon$ with all (at the least small) $\varepsilon > 0$.

This paper was inspired by recent results for singular operators in weighted grand spaces obtained in [12], [13], [15], [16], [19] for domains with finite measure. We show that grand Lebesgue spaces may be defined on an arbitrary set of infinite measure in $\mathbb{R}^n$, if considered with weight introduced also depending on the parameter ε. Note that the

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spaces introduced in this way prove to be dependent on the way how we define dependence of the weight on ε.

We start with the case of the power weight, "responsible" for the behaviour of functions at infinity, i.e. $\varrho(x) = \left(1+|x|^2\right)^{-\frac{\lambda}{2}}$, which clarifies the matter, and give more examples and details in this case, after which we pass to the case of general weights.

We show some properties of the introduced spaces. We introduce also a new version of weighted grand Lebesgue spaces for bounded sets Ω, which are in general larger than the usual ones (and might be called "*grandgrand Lebesgue spaces*"). For both the versions we prove that *any linear operator bounded in the weighed Lebesgue space $L^p(\Omega, w)$ with Muckenhoupt weights $w \in A_p$ is also bounded in the corresponding grand space with a Muckenhoupt weight*, see a more precise formulation in Theorems 5.2 and 6.1. The main tool is the Stein-Weiss version ([20]) of Riesz-Thorin interpolation theorem with change of measure. An application to singular integral operators is mentioned.

The paper is arranged as follows. In Section 2.1 we give some preliminary consideration with the emphasis on how the dependence of the weight on ε may be introduced. Definition of grand Lebesgue spaces with power weights on sets with infinite measure is given in this section. In Section 3 we consider various examples with power or power-logarithmic behaviour at infinity, which show in a sense the nature of the introduced spaces. In Section 4 we introduce grand Lebesgue spaces on sets with infinite measure in the case of general weight and in Section 5 we prove the theorem on the boundedness of linear operators. In Section 6 we introduce a new version of weighted grand Lebesgue spaces for bounded sets Ω and show that the same result on the boundedness holds in this setting.

2. Spaces $L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda}),\ \Omega \subseteq \mathbb{R}^n$

2.1. Preliminaries

Let $1 < p < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. To introduce grand Lebesgue spaces in the case $|\Omega| = \infty$, it is quite natural to make use of weighted spaces

$$L^p(\Omega, \varrho) := \left\{ f : \int\limits_{\Omega} |f(x)|^p \varrho(x)\, dx < \infty \right\}. \tag{2.1}$$

with a weight ϱ. First, we pay a special attention to the case of power weights

$$\varrho(x) = \langle x\rangle^{-\lambda}, \quad \lambda \in \mathbb{R}^1,$$

where $\langle x\rangle = \sqrt{1+|x|^2}$.

Everywhere in the sequel we do not suppose that Ω should be necessarily unbounded. If Ω is bounded, then everywhere where we use the power weight $\langle x\rangle^{-\lambda}$, it should be omitted.

Let $1 \le q < p$. By the Hölder inequality we have

$$\|f\|_{L^q(\Omega,r)} \le K\|f\|_{L^p(\Omega,\varrho)}, \tag{2.2}$$

where

$$K = \left(\int\limits_{\Omega} \left(\frac{r(x)^p}{\varrho(x)^q} \right)^{\frac{1}{p-q}} dx \right)^{\frac{1}{q}-\frac{1}{p}}. \tag{2.3}$$

Hence, with the Riesz theorem on linear functionals in L^p taken into account, we obtain

$$L^p(\Omega,\varrho) \hookrightarrow L^q(\Omega,r) \iff \int\limits_\Omega \left(\frac{r(x)^p}{\varrho(x)^q}\right)^{\frac{1}{p-q}} dx < \infty. \tag{2.4}$$

In particular, we have

$$L^p(\Omega,\langle x\rangle^{-\lambda}) \hookrightarrow L^{p-\varepsilon}(\Omega,\langle x\rangle^{-\lambda(\varepsilon)}) \iff \lambda(\varepsilon) > \lambda + \frac{n-\lambda}{p}\varepsilon. \tag{2.5}$$

2.2. Restrictions on the choice of $\lambda(\varepsilon)$ for embeddings of the spaces $L^{p-\varepsilon}_{\lambda(\varepsilon)}(\Omega)$

In accordance with (2.5) we put

$$\lambda(\varepsilon) = \lambda + \frac{n-\lambda}{p}\varepsilon + \nu(\varepsilon), \quad \text{where} \quad \nu(\varepsilon) > 0. \tag{2.6}$$

In the sequel we always assume that the small "perturbation" $\nu(\varepsilon)$ satisfies the assumptions

$$\nu \in C([0,p-1]), \quad \nu(\varepsilon) > 0 \text{ for } \varepsilon > 0, \quad \nu(0) = 0; \tag{2.7}$$

some additional assumptions required later will be imposed in their turn.

The following lemma provides a condition on the choice of $\nu(\varepsilon)$, which guarantees monotone narrowing of the space $L^{p-\varepsilon}_{\lambda(\varepsilon)}(\Omega)$ (i.e. quasimonotone increasing of the norm $\|f\|_{L^{p-\varepsilon}_{\lambda(\varepsilon)}(\Omega)}$) when ε decreases.

Lemma 1. *Let $\nu(\varepsilon)$ be a non-negative non-decreasing function on $(0,p-1)$ and $0<\varepsilon_1<\varepsilon_2<p-1$. Then*

$$\|f\|_{L^{p-\varepsilon_2}(\Omega,\langle x\rangle^{\lambda(\varepsilon_2)})} \le k(\varepsilon_1,\varepsilon_2)\|f\|_{L^{p-\varepsilon_1}(\Omega,\langle x\rangle^{\lambda(\varepsilon_1)})}, \tag{2.8}$$

where

$$k(\varepsilon_1,\varepsilon_2) \le \left(\pi^{\frac{n}{2}}\frac{\Gamma\left(\frac{\gamma-n}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}\right)^{\frac{\varepsilon_2-\varepsilon_1}{(p-\varepsilon_1)(p-\varepsilon_2)}}, \quad \gamma = n + \frac{\nu(\varepsilon_2)(p-\varepsilon_1)-\nu(\varepsilon_1)(p-\varepsilon_2)}{\varepsilon_2-\varepsilon_1}. \tag{2.9}$$

In the case $\Omega=\mathbb{R}^n$ the constant $k(\varepsilon_1,\varepsilon_2) = \left(\pi^{\frac{n}{2}}\frac{\Gamma\left(\frac{\gamma-n}{2}\right)}{2\Gamma\left(\frac{\gamma}{2}\right)}\right)^{\frac{\varepsilon_2-\varepsilon_1}{(p-\varepsilon_1)(p-\varepsilon_2)}}$ is sharp.

Proof. The general estimate (2.2) yields (2.8) with

$$k = k(\varepsilon_1,\varepsilon_2) = \left(\int\limits_\Omega \frac{dx}{\langle x\rangle^\gamma}\right)^{\frac{\varepsilon_2-\varepsilon_1}{(p-\varepsilon_1)(p-\varepsilon_2)}}, \tag{2.10}$$

where

$$\gamma = \frac{\lambda(\varepsilon_2)(p-\varepsilon_1)-\lambda(\varepsilon_1)(p-\varepsilon_2)}{\varepsilon_2-\varepsilon_1}. \tag{2.11}$$

In view of (2.6) this is reduced to the value of γ given in (2.9). So the constant (2.10) is finite if $\gamma > n$, which holds, since

$$\frac{\nu(\varepsilon_2)(p-\varepsilon_1)-\nu(\varepsilon_1)(p-\varepsilon_2)}{\varepsilon_2-\varepsilon_1} = \nu(\varepsilon_1)+(p-\varepsilon_1)\frac{\nu(\varepsilon_2)-\nu(\varepsilon_1)}{\varepsilon_2-\varepsilon_1} > 0.$$

To prove (2.9), it suffices to pass to polar coordinates in the right-hand side integral in the inequality

$$k \le \left(\int\limits_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{\gamma}{2}}}\right)^{\frac{\varepsilon_2-\varepsilon_1}{(p-\varepsilon_1)(p-\varepsilon_2)}},$$

and make use of the formula

$$\int\limits_0^\infty \frac{dr}{r^a(1+r^2)^{\frac{b}{2}}} = \frac{\Gamma\left(\frac{1-a}{2}\right)\Gamma\left(\frac{a+b-1}{2}\right)}{2\Gamma\left(\frac{b}{2}\right)}, \tag{2.12}$$

(the integral on the left-hand side is reduced to the beta-function via the change of variables $1+r^2=\frac{1}{t}$); here a and b are arbitrary such that $a<1,\ a+b>1$. ◄

The next lemma shows that if we wish to have the norms $\|f\|_{L^{p-\varepsilon}_{\lambda(\varepsilon)}}$ uniformly bounded in ε in the case where $f \in L^p(\Omega, \langle x\rangle^{-\lambda})$, we have to impose a stronger assumption on the choice of the exponent $\lambda(\varepsilon)$, i.e. the choice of $\nu(\varepsilon)$. This assumption excludes strong exponential decay of $\nu(\varepsilon)$ as $\varepsilon \to 0$.

Lemma 2. *Let $\nu(\varepsilon)$ satisfy the conditions in* (2.7). *Then*

$$\|f\|_{L^{p-\varepsilon}(\Omega,\langle x\rangle^{\lambda(\varepsilon)})} \le k(\varepsilon)\|f\|_{L^p(\Omega,\langle x\rangle^{-\lambda})}, \tag{2.13}$$

where

$$k(\varepsilon) = \left[\pi^{\frac{n}{2}}\frac{\Gamma\left(\frac{p\nu(\varepsilon)}{2\varepsilon}\right)}{\Gamma\left(\frac{n}{2}+\frac{p\nu(\varepsilon)}{2\varepsilon}\right)}\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} \le C\left[\frac{\varepsilon}{\nu(\varepsilon)}\right]^{\frac{\varepsilon}{p^N}}, \tag{2.14}$$

$N = \begin{cases} 1, & if\ \nu(\varepsilon)\ge\varepsilon \\ 2, & if\ \nu(\varepsilon)\le\varepsilon \end{cases}$ *and C do not depend on ε, so that the condition*

$$\sup_{\varepsilon\in(0,p-1)}\left[\frac{\varepsilon}{\nu(\varepsilon)}\right]^{\varepsilon} < \infty, \tag{2.15}$$

is sufficient for the estimate (2.13) *to be uniform in ε.*

Proof. To obtain (2.13)-(2.14), one may formally use estimate (2.8)-(2.9) with the choice $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon$, but the proof of estimate (2.8) assumed the monotonicity of $\nu(\varepsilon)$. Since we do not suppose this now, it just suffices to refer instead to the fact that after repeating calculations in (2.10)-(2.11) with $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon$, we arrive at (2.13) and (2.14).

The inequality in (2.14) follows from the properties of the gamma-function. Indeed, let $z = \frac{p\nu(\varepsilon)}{2\varepsilon}$, then in the case $z \leq 1$ we have

$$k(\varepsilon) \leq \left[\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} \cdot \left[\frac{\Gamma(1+z)}{z}\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} \leq \frac{C}{z^{\frac{\varepsilon}{p(p-\varepsilon)}}} \leq C\left[\frac{\varepsilon}{\nu(\varepsilon)}\right]^{\frac{\varepsilon}{p}},$$

and when $z \geq 1$, we use the asymptotic formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \leq C z^{a-b}, \quad z \to \infty, \tag{2.16}$$

which gives

$$k(\varepsilon) \leq \frac{C}{z^{\frac{\varepsilon}{p(p-\varepsilon)}}} \leq C\left[\frac{\varepsilon}{\nu(\varepsilon)}\right]^{\frac{\varepsilon}{p^2}},$$

so that the boundedness of the function $k(\varepsilon)$ follows from the assumption in (2.15) in both the cases.◀

2.3. Definition of grand Lebesgue spaces with power weights on sets with infinite measure

Let $1 < p < \infty, \theta > 0$ and $\lambda \in \mathbb{R}^1$. We define the grand Lebesgue spaces $L^{p),\theta}(\Omega, \langle x\rangle^{-\lambda})$ on a set Ω which may have infinite measure, as the space of functions $f : \Omega \to \mathbb{R}^1$ with the finite norm

$$\sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega, \langle x\rangle^{-\lambda(\varepsilon)})} < \infty, \tag{2.17}$$

where the extension $\lambda(\varepsilon)$ of the exponent $\lambda = \lambda(0)$ is made by the formula

$$\lambda(\varepsilon) = \lambda + \frac{n-\lambda}{p}\varepsilon + \nu(\varepsilon), \tag{2.18}$$

with the choice of $\nu(\varepsilon)$ according to conditions (2.7). The space $L^{p),\theta}(\Omega, \langle x\rangle^{-\lambda})$ is well posed under every concrete choice of such a $\nu(\varepsilon)$ in view of the embedding in (2.5). However, as simple examples below show, so defined space depends on the choice of $\nu(\varepsilon)$. By this reason, to underline this dependence on ν, we denote

$$L_\nu^{p),\theta}(\Omega, \langle x\rangle^{-\lambda}) = \left\{f : \|f\|_{L_\nu^{p),\theta}(\Omega, \langle x\rangle^{-\lambda})} < \infty\right\}, \tag{2.19}$$

where

$$\|f\|_{L_\nu^{p),\theta}(\Omega, \langle x\rangle^{-\lambda})} := \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int\limits_\Omega \frac{|f(x)|^{p-\varepsilon}}{\langle x\rangle^{\lambda + \frac{n-\lambda}{p}\varepsilon + \nu(\varepsilon)}} dx\right)^{\frac{1}{p-\varepsilon}}.$$

It is easily seen that

$$L_{\nu_1}^{p),\theta}(\Omega, \langle x\rangle^{-\lambda}) \hookrightarrow L_{\nu_2}^{p),\theta}(\Omega, \langle x\rangle^{-\lambda}),$$

if $\nu_2(\varepsilon) \leq \nu_1(\varepsilon)$.

The examples given below illustrate what happens under a possibility of an arbitrary choice of $\nu(\varepsilon)$, but our main result will be given for the case of the linear extension, that is, the choice

$$\nu(\varepsilon) = \alpha\varepsilon, \quad \alpha > 0, \tag{2.20}$$

because just in this case it will be possible to make use of interpolation. In the case (2.20) we denote

$$L^{p),\theta}_{\alpha}(\Omega, \langle x\rangle^{-\lambda}) := L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda})\Big|_{\nu(\varepsilon)=\alpha\varepsilon},$$

without a danger of confusing notation.

Lemma 3. *Under the choice* (2.18) *and* (2.7)*, the space* $L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda}))$, $1 < p < \infty$, $\lambda \in \mathbb{R}^1$, $\theta > 0$, *is a Banach space with respect to the norm* (2.17)*.*

The proof follows the standard scheme as in the case of bounded domains. For the completeness of presentation we give this proof in Section 7,

Definition 1. *The space* $L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda}))$ *will be called properly defined, if the function* $\nu(\varepsilon)$*, used in its definition, satisfies condition* (2.15)*.*

Theorem 2.1. *Let* $1 < p < \infty$ *and* $\theta \geq 0$*. For a properly defined space* $L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda}))$ *there holds the continuous embedding*

$$L^{p,\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda})) \hookrightarrow L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda})). \tag{2.21}$$

Theorem 2.1 is an immediate consequence of Lemma 2.

3. Some examples of functions in $L^{p),\theta}_{\nu}(\mathbb{R}^n, \langle x\rangle^{-\lambda})$

Consider the function

$$f(x) = \frac{1}{|x|^a(1+|x|^2)^{\frac{b}{2}}}, \quad x \in \mathbb{R}^n, \tag{3.1}$$

with a power behaviour at the origin and infinity.

Lemma 4. *Let*

$$ap \leq n, \qquad (a+b)p + \lambda \geq n. \tag{3.2}$$

then for the function (3.1) *and every* $\varepsilon > 0$ *the formula is valid*

$$\int\limits_{\mathbb{R}^n} |f(x)|^{p-\varepsilon}\langle x\rangle^{-\lambda(\varepsilon)}dx = |\mathbb{S}^{n-1}|\frac{\Gamma\left(\frac{n-a(p-\varepsilon)}{2}\right)\Gamma\left(\frac{(a+b)(p-\varepsilon)+\lambda(\varepsilon)-n}{2}\right)}{2\Gamma\left(\frac{b(p-\varepsilon)+\lambda(\varepsilon)}{2}\right)}, \tag{3.3}$$

where $|\mathbb{S}^{n-1}|$ *is the area measure of the unit sphere* $\mathbb{S}^{n-1}$ *in* $\mathbb{R}^n$*.*

Proof. Passing to polar coordiantes we get

$$\int\limits_{\mathbb{R}^n} |f(x)|^{p-\varepsilon}\langle x\rangle^{-\lambda(\varepsilon)}dx = |\mathbb{S}^{n-1}| \int\limits_0^\infty \frac{dr}{r^{a(p-\varepsilon)+1-n}(1+r^2)^{\frac{b(p-\varepsilon)+\lambda(\varepsilon)}{2}}}.$$

Formula (2.12) with $a = a(p-\varepsilon)+1-n, b = b(p-\varepsilon)+\lambda(\varepsilon)$, yields (3.3), with conditions $a < 1,\ a+b > 1$, necessary for (2.12), leading to the conditions

$$ap - \varepsilon < n, \qquad (a+b)p - \varepsilon + \lambda(\varepsilon) > n. \tag{3.4}$$

It is easily seen that conditions (3.2) are sufficient for (3.4) to be fulfilled, when $\varepsilon > 0$. This is obvious for the first condition in (3.4), and to check the sufficiency of the inequality $(a+b)p + \lambda \geq n$ for the validity of the second condition in (3.4), rewrite this second condition in the form

$$[(a+b)p+\lambda]\frac{p-\varepsilon}{p} - \lambda\frac{p-\varepsilon}{p} + \lambda(\varepsilon) > n.$$

It suffices to check the last condition in "the worst" case $(a+b)p+\lambda = n$. In this case it takes the form $\lambda(\varepsilon) - \lambda\frac{p-\varepsilon}{p} - \frac{n\varepsilon}{p} > 0$, which fulfills.◄

If $\nu(\varepsilon)$ satisfies condition (2.15), then from the embedding (2.21) it follows that $\frac{1}{|x|^a(1+|x|^2)^{\frac{b}{2}}} \in L^{p),\theta}_{\nu}(\mathbb{R}^1_+, \langle x\rangle^{-\lambda})$ under the condition of type (3.2) with strict inequalities:

$$ap < n, \qquad (a+b)p + \lambda > n. \tag{3.5}$$

We are now interested in the limiting cases $ap = n$ and/or $(a+b)p + \lambda = n$, i.e. the examples

$$f_1(x) = \frac{(1+|x|^2)^{\frac{\lambda}{2p}}}{|x|^{\frac{n}{p}}}, \qquad f_2(x) = \frac{1}{1+|x|^2)^{\frac{n-\lambda}{2p}}}, \tag{3.6}$$

corresponding to the choices $a = \frac{n}{p},\ b = -\frac{\lambda}{p}$ and $a = \frac{n-\lambda}{p},\ b = -0$, respectively. We will consider simultaneously also the following example with a power-logarithmic behaviour, also in the limiting case $a = \frac{n}{p}$. the last example for simplicity will be considered in for $n = 1$ and $\Omega = (0, \infty)$:

$$f_3(x) = \frac{1}{(x\ln x)^{\frac{1}{p}}}, \qquad x > 1.$$

Lemma 5. *Let $1 < p < \infty, \theta > 0$ and $\nu(\varepsilon)$ satisfy the conditions in (2.7). Then*
i) $f_1, f_2 \in L^{p),\theta}_{\nu}(\mathbb{R}^n, \langle x\rangle^{-\lambda})$, *if and only if*

$$\sup_{\varepsilon\in(0,p-1)} \frac{\varepsilon^\theta}{\nu(\varepsilon)} < \infty, \tag{3.7}$$

and $\theta \geq 1$ in the case of $f_1(x)$ and θ is arbitrary (> 0) in the case of $f_2(x)$;
ii) $f_3 \in L^{p),\theta}_{\nu}((1,\infty), \langle x\rangle^{-\lambda})$, *if and only if*

$$\sup_{\varepsilon\in(0,p-1)} \frac{\varepsilon^{(\theta-1)p}}{[\nu(\varepsilon)]^\varepsilon} < \infty. \tag{3.8}$$

Proof. *i)* In the case $a = \frac{n}{p},\ b = -\frac{\lambda}{p}$ equality (3.3) takes the form

$$\int\limits_{\mathbb{R}^n} \frac{dx}{|x|^{a(p-\varepsilon)}(1+|x|^2)^{\frac{b(p-\varepsilon)+\lambda(\varepsilon)}{2}}} = |\mathbb{S}^{n-1}| \frac{\Gamma\left(\frac{n\varepsilon}{2p}\right)\Gamma\left(\frac{\nu(\varepsilon)}{2}\right)}{2\Gamma\left(\frac{\frac{n\varepsilon}{p}+\nu(\varepsilon)}{2}\right)},$$

and then by properties of the gama-function we obtain

$$\varepsilon^\theta \int\limits_{\mathbb{R}^n} \frac{dx}{|x|^{a(p-\varepsilon)}(1+|x|^2)^{\frac{b(p-\varepsilon)+\lambda(\varepsilon)}{2}}} \sim C\varepsilon^\theta \left[\frac{n}{\nu(\varepsilon)} + \frac{p}{\varepsilon}\right], \qquad a = \frac{n}{p}, \quad b = -\frac{\lambda}{p}.$$

the case of $f_2(x)$ is similarly treated.

ii) Let $\lambda = 0$ for simplicity. Simple calculations yield:

$$\int\limits_1^\infty \frac{dx}{(x\ln x)^{\frac{p-\varepsilon}{p}}(1+x^2)^{\frac{\lambda(\varepsilon)}{2}}} \sim \int\limits_1^\infty \frac{dx}{x^{\lambda(\varepsilon)+\frac{p-\varepsilon}{p}}(\ln x)^{\frac{p-\varepsilon}{p}}} = \int\limits_0^\infty t^{\frac{\varepsilon}{p}-1} e^{-\nu(\varepsilon)t}\, dt,$$

and then

$$\varepsilon^\theta \int\limits_1^\infty \frac{dx}{(x\ln x)^{\frac{p-\varepsilon}{p}}(1+x^2)^{\frac{\lambda(\varepsilon)}{2}}} \sim \frac{\varepsilon^\theta \Gamma\left(\frac{\varepsilon}{p}\right)}{[\nu(\varepsilon)]^{\frac{\varepsilon}{p}}} \sim \frac{\varepsilon^{\theta-1}}{[\nu(\varepsilon)]^{\frac{\varepsilon}{p}}}.$$

Consequently, $\frac{1}{(x\ln x)^{\frac{1}{p}}} \in L_\nu^{p,\theta}(1,\infty)$, if and only if ν satisfies condition (3.8).◄

Thus we see that the belonging of these important model examples to the space $L_\nu^{p,\theta}(\Omega, \langle x\rangle^{-\lambda})$ is possible, if the function $\nu(\varepsilon)$ is chosen satisfying restriction (3.7) or (3.8). Comparing these restrictions with condition (2.15) appeared earlier, note that

$$(3.7) \quad \Longrightarrow \quad (2.15) \quad (\text{for every} \quad \theta \in \mathbb{R}^1)$$

$$(2.15) \quad \Longrightarrow \quad (3.8) \quad (\text{for} \quad \theta \geq 1).$$

4. Grand Lebesgue spaces on sets with infinite measure; the case of general weights

Let now $L^p(\Omega, \varrho)$ be a space of form (2.1) with an arbitrary weight. When passing to $p - \varepsilon$, the corresponding weighted grand Lebesgue space may be introduced via the weighted Lebesgue space in two ways:

$$\text{either} \quad L^{p-\varepsilon}(\Omega, \varrho^{1+\alpha\varepsilon}) \quad \text{or} \quad L^{p-\varepsilon}(\Omega, \langle\cdot\rangle^{-\alpha\varepsilon}\varrho), \tag{4.1}$$

with a parameter $\alpha > 0$. However, these two choices lead to different restriction on the choice of the parameter α and (or) the weight ϱ. Correspondingly to (4.1), by $\varrho_{\alpha,\varepsilon}(x)$ we denote one of the two following choices

$$\varrho_{\alpha,\varepsilon}(x) = \varrho^{1+\alpha\varepsilon}(x) \quad \text{or} \quad \varrho_{\alpha,\varepsilon}(x) = \varrho(x)\cdot\langle x\rangle^{-\alpha\varepsilon}. \tag{4.2}$$

Definition 2. *The space $L^{p),\theta}_{\alpha}(\Omega,\varrho)$ is defined as the set of functions f with the finite norm*

$$\|f\|_{L^{p),\theta}_{\alpha}(\Omega,\varrho)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^{\theta} \int\limits_{\Omega} |f(x)|^{p-\varepsilon} \varrho_{\alpha,\varepsilon}(x)\, dx \right)^{\frac{1}{p-\varepsilon}}, \tag{4.3}$$

where for the "extension" $\varrho_{\alpha,\varepsilon}$ of the weight we admit one of the two possibilities (4.2).

To make this definition well posed, we are interested in the embedding

$$L^{p-\varepsilon_2}(\Omega,\varrho_{\alpha,\varepsilon_2}) \hookrightarrow L^{p-\varepsilon_1}(\Omega,\varrho_{\alpha,\varepsilon_1}), \tag{4.4}$$

with $0<\varepsilon_1<\varepsilon_2<p-1$. By (2.2)-(2.3), this embedding holds if and only if

$$\int\limits_{\Omega} \varrho^{1+\alpha p}(x)\, dx < \infty, \qquad \int\limits_{\Omega} \frac{\varrho(x)}{\langle x\rangle^{\alpha p}} dx < \infty, \tag{4.5}$$

correspondingly to the choice in (4.2). The first condition is in a sense more restrictive being an assumption on the choice of the weight ϱ itself. The second looks more preferable, see for instance, Remark 1 below. Note also that if the first condition holds with some $\alpha=\alpha_0$, then the second one holds with every $\alpha > \frac{\alpha_0 p}{\alpha_0 n+1}$. By this reason, in the main result of this section we choose the second possibility $\varrho_{\alpha,\varepsilon}(x) = \varrho(x)\cdot\langle x\rangle^{-\alpha\varepsilon}$ (but we return to the first possibility in Section 6) and give the following definition for the case of unbounded domains.

Definition 3. *In the case Ω is unbounded, we call a positive number α admissible for the weight ϱ, if the condition*

$$\int\limits_{\Omega} \frac{\varrho(x)\, dx}{(1+|x|)^{\alpha}} < \infty,$$

holds with this α.

By $A_p = A_p(\Omega), 1\le p<\infty$, we denote the Muckenhoupt class of weights on Ω. It is known that

$$\varrho \in A_p \quad \Longrightarrow \quad \int\limits_{\Omega} \frac{\varrho(x)\, dx}{(1+|x|)^{np}} < \infty. \tag{4.6}$$

Remark 1. *By* (4.6), *all $\alpha \ge n$ are admissible for all the weights $\varrho \in A_p$.*

5. Boundedness of linear operators in weighted grand Lebesgue spaces

5.1. Preliminaries: Riesz-Thorin-Stein-Weiss interpolation theorem

The short proof of the boundedness of linear operators in weighted grand Lebesgue spaces given in the next subsection is based on the following version of Riesz-Thorin theorem with change of measure due to E. Stein and G. Weiss ([20], see also [2]).

In application to the case of weighted spaces $L^{p_0}(\Omega,\varrho_0)$ and $L^{p_1}(\Omega,\varrho_1)$ of form (2.1) it runs as follows.

Theorem 5.1. *Let T be a linear operator defined on $L^{p_0}(\Omega,\varrho_0)\cup L^{p_1}(\Omega,\varrho_1)$. If*

$$\|Tf\|_{L^{p_0}(\Omega,\varrho_0)}\le M_0\|f\|_{L^{p_0}(\Omega,\varrho_0)} \quad \textit{and} \quad \|Tf\|_{L^{p_1}(\Omega,\varrho_1)}\le M_1\|f\|_{L^{p_1}(\Omega,\varrho_1)},$$

then also $\|Tf\|_{L^p(\Omega,\varrho)}\le M\|f\|_{L^p(\Omega,\varrho)}$, where $M\le M_0^{1-t}M_1^t$,

$$\frac{1}{p}=\frac{1-t}{p_0}+\frac{t}{p_1} \qquad \textit{and} \qquad \varrho=\varrho_0^{\frac{(1-t)p}{p_0}}\varrho_1^{\frac{tp}{p_1}}, \qquad 0<t<1. \tag{5.1}$$

We need some definitions.

Definition 4. *Let $W_p=W_p(\Omega)$ be a class of weights on Ω, depending on the parameter $p\in[1,\infty)$. We say that a linear operator T belongs to the class $\mathcal{B}(\Omega,W_p)$, if it is bounded in the space $L^p(\Omega,\varrho)$ for every $\varrho\in W_p$.*

Definition 5. *A class $W_p=W_p(\Omega), p\in(1,\infty)$ of weights on Ω will be called allowable, if it possesses properties*

$$\varrho\in W_p \implies \varrho\in W_{p-\varepsilon} \quad \textit{for some} \ \ \varepsilon>0, \tag{5.2}$$

$$\varrho\in W_p \implies \varrho^{1+\varepsilon}\in W_p \quad \textit{for some} \ \ \varepsilon>0, \tag{5.3}$$

$$\varrho_1,\varrho_2\in W_p \implies \varrho_1^t\varrho_2^{1-t}\in W_p \quad \textit{for every} \quad t\in[0,1]. \tag{5.4}$$

As it is well known, the Muckenhoupt class $A_p, 1<p<\infty$ is allowable in the sense of Definition 5.

In the next section we will use the following simple fact.

Lemma 6. *For every $\varrho\in W_p$ and $\alpha\in\mathbb{R}^1$ there exists a $\delta>0$ such that $\varrho(x)\left(1+|x|^2\right)^{\frac{\alpha\varepsilon}{2}}\in W_{p-\varepsilon}$ for all $0<\varepsilon<\delta$.*

Proof. By (5.2) there exists an $\varepsilon_1>0$ such that $\varrho\in W_{p-\varepsilon}$ for all $0<\varepsilon\le\varepsilon_1$. Then we choose and fix any $\varepsilon_2>0$ such that $\varrho^{1+\varepsilon_2}\in W_{p-\varepsilon}$, which is possible by (5.3). The weight $\varrho(x)\langle x\rangle^{\alpha\varepsilon}$ may be represented in the form

$$\varrho(x)\langle x\rangle^{\alpha\varepsilon}=\left(\varrho^{1+\varepsilon_2}\right)^t\left(\langle x\rangle^{\alpha\varepsilon\frac{1+\varepsilon_2}{\varepsilon_2}}\right)^{1-t},$$

where $t=\frac{1}{1+\varepsilon_2}\in(0,1)$. As is well known, $\langle x\rangle^\alpha$ belongs to W_p if and only if $\alpha\in(0,n(p-1))$. Under the choice $\varepsilon\le\frac{\varepsilon_2}{1+\varepsilon_2}$ we have $\varepsilon\frac{1+\varepsilon_2}{\varepsilon_2}\le 1$. Then also $\alpha\varepsilon\frac{1+\varepsilon_2}{\varepsilon_2}\in(0,n(p-1))$ and consequently $\langle x\rangle^{\alpha\varepsilon\frac{1+\varepsilon_2}{\varepsilon_2}}\in W_{p-\varepsilon}$, after which it remains to refer to the property (5.4).◀

5.2. The main statement

We prove the following theorem (note that it is new also for bounded sets Ω). Recall that from now on the space $L^{p),\theta}_\alpha(\Omega,\varrho)$ is defined with the second choice in (4.2); in the case where Ω is a bounded set, one may obviously take $\alpha=0$.

The following properties of the class W_p are well known (see for instance [5], [6]):

Theorem 5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $1 < p < \infty$ and W_p an allowable class of weights. If*

$$T \in \mathcal{B}(L^p(\Omega), W_p) \cap \mathcal{B}(L^{p-\varepsilon_0}(\Omega), W_{p-\varepsilon_0}),$$

for some $\varepsilon_0 \in (0, p-1)$, then T is also bounded in the weighted grand Lebesgue space $L^{p),\theta}_{\alpha}(\Omega, \varrho)$, where $\varrho \in W_p$ and α is arbitrary in the case of a bounded set Ω and an arbitrary positive admissible number in the case of an unbounded set.

Proof. Let ε_0 be any fixed number in the interval $(0, \delta)$, where δ is a number from Lemma 6. We have

$$\|Tf\|_{L^{p),\theta}_{\alpha}(\Omega,\varrho)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L^{p-\varepsilon}(\Omega,\rho\cdot\langle x\rangle^{-\alpha\varepsilon})} = \max\{A, B\}, \tag{5.5}$$

where

$$A = \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L^{p-\varepsilon}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon})}, \quad B = \sup_{\varepsilon_0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L^{p-\varepsilon}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon})}. \tag{5.6}$$

Estimation of A. By the assumption of the theorem and Lemma 6 we have

$$\|Tf\|_{L^p(\Omega,\varrho)} \leq M_1 \|f\|_{L^p(\Omega,\varrho)} \quad \text{and} \quad \|Tf\|_{L^{p-\varepsilon_0}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon_0})} \leq M_2 \|f\|_{L^{p-\varepsilon_0}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon_0})} \tag{5.7}$$

We then apply the interpolation Theorem 5.1 with

$$p_0 = p, \; p_1 = p - \varepsilon_0, \quad \varrho_0 = \varrho, \text{ and } \; \varrho_1 = \varrho \cdot \langle x\rangle^{-\alpha\varepsilon_0},$$

to obtain the boundedness of T in $L^{p-\varepsilon}(\Omega, \varrho\cdot\langle x\rangle^{-\alpha\varepsilon})$, uniform in $\varepsilon \in [0, \varepsilon_0]$. From the corresponding interpolation relation $\frac{1-t}{p} + \frac{t}{p-\varepsilon_0} = \frac{1}{p-\varepsilon}$ we have $t = \frac{\varepsilon}{\varepsilon_0}\frac{p-\varepsilon_0}{p-\varepsilon}$. Substituting this into the equality (5.1) for the weight, we expect that we must obtain the weight $\varrho \cdot \langle x\rangle^{-\alpha\varepsilon}$, corresponding to the exponent $p - \varepsilon$, which is true:

$$\varrho = \varrho^{\frac{(1-t)(p-\varepsilon)}{p}} \varrho^{\frac{t(p-\varepsilon)}{p-\varepsilon_0}} \cdot \langle x\rangle^{-\alpha\varepsilon_0 \frac{t(p-\varepsilon)}{p-\varepsilon_0}} = \varrho \cdot \langle x\rangle^{-\alpha\varepsilon}. \tag{5.8}$$

Thus by Theorem 5.1 we have the uniform estimate

$$\|Tf\|_{L^{p-\varepsilon}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon})} \leq M \|f\|_{L^{p-\varepsilon}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon})}, \tag{5.9}$$

where M does not depend on ε. Hence

$$A \leq M \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon})} \leq M \|f\|_{L^{p),\theta}_{\alpha}(\Omega,\varrho)}. \tag{5.10}$$

Estimate for B is obtained by Hölder inequality with the exponents $q = \frac{p-\varepsilon_0}{p-\varepsilon} > 1$, $q' = \frac{p-\varepsilon_0}{\varepsilon-\varepsilon_0}$:

$$\|Tf\|_{L^{p-\varepsilon}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon})} \leq C(\varepsilon) \|Tf\|_{L^{p-\varepsilon_0}(\Omega,\varrho\cdot\langle x\rangle^{-\alpha\varepsilon_0})},$$

where

$$C(\varepsilon) = \left\{ \int\limits_{\Omega} \varrho(x) \cdot \langle x\rangle^{-\alpha p} dx \right\}^{\frac{\varepsilon-\varepsilon_0}{(p-\varepsilon)(p-\varepsilon_0)}},$$

is a bounded function of $\varepsilon \in [\varepsilon_0, p-1)$. Consequently,

$$B \leq C_1 \varepsilon_0^{\frac{\theta}{p-\varepsilon_0}} \|Tf\|_{L^{p-\varepsilon_0}(\Omega, \varrho \cdot \langle x \rangle^{-\alpha\varepsilon_0})} \leq C_1 \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L^{p-\varepsilon}(\Omega, \varrho \cdot \langle x \rangle^{-\alpha\varepsilon})} = C_1 A,$$

where C_1 does not depend on ε. Then by (5.10)

$$\|Tf\|_{L^{p),\theta}_{\varrho,\alpha}(\Omega)} \leq \max\{1, C_1\} M \|f\|_{L^{p),\theta}_{\alpha}(\Omega,\varrho)},$$

which completes the proof.◀

Corollary 1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $1 < p < \infty$. If*

$$T \in \mathcal{B}(L^p(\Omega), A_p) \cap \mathcal{B}(L^{p-\varepsilon_0}(\Omega), A_{p-\varepsilon_0}),$$

for some $\varepsilon_0 \in (0, p-1)$, then T is also bounded in the weighted grand Lebesgue space $L^{p),\theta}_{\alpha}(\Omega, \varrho)$, where $\varrho \in A_p$ and α is arbitrary in the case of a bounded set Ω and an arbitrary positive admissible number in the case of an unbounded set.

Remark 2. *Theorem 5.2 in case of bounded sets was proved in [16] for the case where T is the the Hilbert transform operator and in [13] for more general singular type operators; we followed the proof in [16]. Note that in [16], [13] there was given a complete characterization of weights for the boundedness of singular operators in weighted grand spaces: is bounded in the space $L^{p),\theta}_{\alpha}(\Omega, \varrho)$ if and only if $\varrho \in A_p$. In the next section, for bounded sets Ω we prove a theorem similar to 5.2 for another version of weighted grand spaces, where together with the passage from p to $p - \varepsilon$ we introduce a weight also depending on ε.*

Corollary 2. *Let $1 < p < \infty$. Every Calderón-Zygmund singular integral operator*

$$\int_{\mathbb{R}^n} K(x,y) f(y) dy,$$

with the so called standard kernel (in the sense of Coifman-Meyer) is bounded in the weighted grand Lebesgue space $L^{p),\theta}_{\alpha}(\mathbb{R}^n, \varrho)$, $\varrho \in A_p$, for every admissible $\alpha > 0$.

We also single out a special case of the singular operator

$$Sf(x) := \frac{1}{\pi} \int_{\mathbb{R}^1} \frac{f(t)dt}{x-t}.$$

Corollary 3. *The operator S is bounded in $L^{p),\theta}_{\alpha}(\mathbb{R}^1, \varrho)$ for every $\varrho \in A_p$ and admissible α; in the case of a power weight $\varrho(x) = |x|^{-\lambda}$ or $\varrho(x) = (1+|x|)^{-\lambda}$ we may take $\alpha > \frac{1-\lambda}{p}$ and the operator S is bounded in the space $L^{p),\theta}_{\alpha}(\mathbb{R}^1, \varrho)$ if and only if $-1-p < \lambda < 1$.*

The "if part" of this corollary is an immediate consequence of Theorem 5.2, while the "only if part" needs to be proved. The proof of the necessity of the condition $1-p < \lambda < 1$ is obtained via the choice of counterexamples. We dwell on the case of $\varrho(x) = (1+|x|)^{\gamma} \sim \langle x \rangle^{\gamma}$; the construction of counterexamples for $\varrho(x) = |x|^{\gamma}$ follows the same lines.

Suppose that $\lambda > 1$. The function $\left(1+x^2\right)^{\frac{\lambda-1}{2}}$ belongs to the space $L_\alpha^{p),\theta}(\mathbb{R}^1,\langle x\rangle^\lambda)$ by Lemma 5, but it grows at infinity and the operator S does not exist on this element of $L_{\lambda,k}^{p),\theta}(\mathbb{R}^1)$ and consequently may not be bounded on this space. When $\lambda = 1$ we modify this counterexample by taking $f(x) = \frac{1}{\ln^\gamma(e+|x|)}$, where $\frac{1}{p} < \gamma < 1$. Then it is easy to check that the condition $\gamma p > 1$ yields $f \in L_\alpha^{p),\theta}(\mathbb{R}^1,\langle x\rangle)^{-1}$, but this function with $\gamma < 1$ is not integrable and consequently the operator S does not exist on this element of the space.

Let $\lambda < 1 - p$. In this case we choose $f(x) = \frac{1}{(1+x^2)^{\frac{1+a}{2}}}$. It belongs to the space $L_\alpha^{p),\theta}(\mathbb{R}^1,\langle x\rangle^{-\lambda})$ for every positive $a > \frac{(1-p)-\lambda}{p}$, but the singular integral of it cannot belong to this space. Indeed the function

$$Sf(x) = \frac{1}{\pi}\int\limits_{\mathbb{R}^1} \frac{dt}{\langle t\rangle^{1+a}(t-x)} = \frac{2x}{\pi}\int\limits_0^\infty \frac{dt}{\langle t\rangle^{1+a}(t^2-x^2)},$$

is known to be decaying at infinity not faster than $\frac{1}{\langle x\rangle}$. To see this, make the change of variables $\langle t\rangle = \frac{\langle x\rangle}{s}$ in the last integral which transforms it to the form

$$Sf(x) = \frac{2x}{\pi\langle x\rangle^{a+1}}\int\limits_0^{\langle x\rangle} \frac{s^a\,ds}{1-s} \sim \frac{C}{\langle x\rangle}, \quad C \neq 0.$$

In the remaining limiting case $\lambda = 1 - p$ it suffices to refer to the counterexample $f(x) = \frac{1}{\langle x\rangle \ln^\mu(e+|x|)}$, $\frac{1}{p} < \mu < 1$ which belongs to the space $L_\alpha^{p),\theta}(\mathbb{R}^1,\langle x\rangle^{1-p})$ when $\frac{1}{p} < \mu$, but is not integrable on $\mathbb{R}^1$ if $\mu < 1$.

6. Another insight on weighted grand spaces on bounded sets Ω

Let now Ω be a bounded open set in $\mathbb{R}^n$. In the "usual" definition of the weighted grand Lebesgue space via the norm

$$\|f\|_{L^{p),\theta}(\Omega,w)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon^\theta\int\limits_\Omega |f(x)|^{p-\varepsilon}w(x)\,dx\right)^{\frac{1}{p-\varepsilon}}, \tag{6.1}$$

the Lebesgue exponent p is changed from p to $p-\varepsilon$, but the weight is not changed. Meanwhile, the weight is a characteristic of the space as much important as the exponent p. Why not to change the weight itself, making it depending on ε as well? The arguments developed in the preceding sections suggest that such an approach to weighted grand Lebesgue spaces is quite possible.

6.1. The grand spaces $\mathcal{L}^{p)}(\Omega,w)$

We may define a weighted grand Lebesgue space $\mathcal{L}^{p)}(\Omega,w)$ with a change of the weight by the norm

$$\|f\|_{\mathcal{L}^{p),\theta}(\Omega,w)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon^\theta\int\limits_\Omega |f(x)|^{p-\varepsilon}w_\varepsilon(x)\,dx\right)^{\frac{1}{p-\varepsilon}}, \tag{6.2}$$

where an "extension" $w_\varepsilon(x)$ of $w(x)$ may be chosen in this or other way. The properties (5.2)-(5.3) of Muckenhoupt weights and the idea of interpolation with change of measure prompt us to choose this extension in the form

$$w_\varepsilon(x) := w(x)^{1+\beta\varepsilon}, \qquad \beta \neq 0.$$

To make the above definition well-posed, we are interested in the embedding

$$L^p(\Omega, w) \hookrightarrow L^{p-\varepsilon}(\Omega, w^{1+\beta\varepsilon}).$$

In view of (2.4) the above embedding is equivalent to the condition

$$\int\limits_\Omega w(x)^{1+\beta p}\, dx < \infty,$$

so that we have to assume that the weight w has the property that the set

$$E_w := \left\{ \beta \in \mathbb{R}^1 \backslash \{0\} : \int\limits_\Omega w(x)^{1+\beta p}\, dx < \infty \right\},$$

is non-empty. We will call numbers $\beta \in E_w$ appropriate for the weight w. By (5.2)-(5.4), the set E_w is non empty for all $w \in A_p$.

With this extension of the weight w, the grand Lebesgue space will depend on the choice of β, so we denote

$$\mathcal{L}_\beta^{p),\theta}(\Omega, w) = \left\{ f : \|f\|_{\mathcal{L}_\beta^{p),\theta}(\Omega,w)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{L}_\beta^{p),\theta}(\Omega,w)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int\limits_\Omega |f(x)|^{p-\varepsilon} w(x)^{1+\beta\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}},$$

and $\beta \neq 0$ is an appropriate number for w.

Simple examples show that so defined grand space $\mathcal{L}_\beta^{p)}(\Omega, w)$ is in general larger than the usual grand Lebesgue space $L^{p)}(\Omega, w)$; the words "in general" mean that this fact depends on the weight w and on the sign of β: take $\Omega = (0,1),\ w(x) = x^\lambda, -1 < \lambda < p-1$ and $f(x) = \frac{1}{x^{\frac{1+\lambda}{p}}}$. then it is easily checked that

$$f \in \mathcal{L}_\beta^{p),1}(\Omega, w), \qquad \text{but} \qquad f \notin L^{p),1}(\Omega, w),$$

when $\beta > 0$ and λ is close to -1, and $\beta < 0$ and λ is close to $p-1$ (so in this way we have a kind of *grandgrand* Lebesgue space).

6.2. On the boundedness of the linear operators in the space $\mathcal{L}^{p)}_{\beta}(\Omega, w)$

Similarly to Theorem 5.2, the following statement also holds.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$ and $W_p = W_p(\Omega)$ an allowable class of weights. If*

$$T \in \mathcal{B}(L^p(\Omega), W_p) \cap \mathcal{B}(L^{p-\varepsilon_0}(\Omega), W_{p-\varepsilon_0}),$$

for some $\varepsilon_0 \in (0, p-1)$, then T is also bounded in the weighted grand Lebesgue space $\mathcal{L}^{p),\theta}_{\beta}(\Omega, w)$, where $w \in W_p$ and β is an arbitrary number appropriate for the weight w.

Proof. The proof follows the same lines as for Theorem 5.2, so we dwell only on the usage of the interpolation theorem with change of measure for the estimation of

$$A := \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L^{p-\varepsilon}(\Omega, w^{1+\beta\varepsilon})}.$$

By the assumption of the theorem and Lemma 6 we have

$$\|Tf\|_{L^p(\Omega,w)} \leq M_1\|f\|_{L^p(\Omega,w)} \quad \text{and} \quad \|Tf\|_{L^{p-\varepsilon_0}(\Omega,w^{1+\beta\varepsilon_0})} \leq M_2\|f\|_{L^{p-\varepsilon_0}(\Omega,w^{1+\beta\varepsilon_0})}, \tag{6.3}$$

for ε_0 sufficiently small. We apply the interpolation Theorem 5.1 with

$$p_0 = p, \; p_1 = p - \varepsilon_0, \quad w_0 = w, \text{ and } \quad w_1 = w^{1+\beta\varepsilon_0},$$

to obtain the boundedness of T in $L^{p-\varepsilon}(\Omega, w^{1+\beta\varepsilon_0})$, uniform in $\varepsilon \in [0, \varepsilon_0]$. We substitute the parameter $t = \frac{\varepsilon}{\varepsilon_0}\frac{p-\varepsilon_0}{p-\varepsilon}$, from the interpolation relation $\frac{1-t}{p} + \frac{t}{p-\varepsilon_0} = \frac{1}{p-\varepsilon}$, into the equality (5.1) for the weight and obtain the weight, corresponding to $p - \varepsilon$:

$$\varrho_0^{\frac{1-t}{p_0}(p-\varepsilon)} \varrho_1^{\frac{t}{p_1}(p-\varepsilon)} = w^{\left[\frac{1-t}{p} + \frac{t}{p-\varepsilon_0} + \frac{\beta t\varepsilon_0}{p-\varepsilon_0}\right](p-\varepsilon)} = w^{1+\beta t\frac{p-\varepsilon}{p-\varepsilon_0}} = w^{1+\beta\varepsilon},$$

what was naturally expected. It remains then to apply Theorem 5.1.◄

7. Appendix

7.1. Proof of Lemma 3

We follow the scheme of the proof presented in [15], p.8. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda})$, so that for every $\delta > 0$ there exists a natural number N such that $\|f_n - f_m\|_{L^{p),\theta}_{\nu}(\Omega,\langle x\rangle^{-\lambda})} < \delta$ for all $n, m \geq N$. We have to show that $\{f_n\}$ converges to a certain function $f \in L^{p),\theta}_{\nu}(\Omega, \langle x\rangle^{-\lambda})$. We have

$$\|f_n - f_m\|_{L^{p),\theta}_{\nu}(\Omega,\langle x\rangle^{-\lambda})} < \delta \Rightarrow$$

$$\sup_{0<\varepsilon<p-1} \left(\varepsilon^{\theta} \int\limits_{\Omega} |f_n(x) - f_m(x)|^{p-\varepsilon} \langle x\rangle^{-\lambda(\varepsilon)} dx\right)^{\frac{1}{p-\varepsilon}} < \delta \Rightarrow$$

$$\left(\varepsilon^\theta \int\limits_\Omega |f_n(x)-f_m(x)|^{p-\varepsilon}\langle x\rangle^{-\lambda(\varepsilon)}dx\right)^{\frac{1}{p-\varepsilon}} < \delta,$$

for all $n \geq N,\ m \geq N$ and $\varepsilon > 0$. Hence

$$\left(\int\limits_\Omega |f_n(x)-f_m(x)|^{p-\varepsilon}\langle x\rangle^{-\lambda(\varepsilon)}dx\right)^{\frac{1}{p-\varepsilon}} < \varepsilon^{-\frac{\theta}{p-\varepsilon}}\delta,\ \forall n,m \geq N.$$

This means that the sequence $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L_\nu^{p-\varepsilon}(\Omega,\langle x\rangle^{-\lambda(\varepsilon)})$ for every $\varepsilon > 0$. By the completeness of the space $L_\nu^{p-\varepsilon}(\Omega,\langle x\rangle^{-\lambda(\varepsilon)})$ the sequence $\{f_n\}_{n=1}^\infty$ converges to a function $f \in L_{\lambda(\varepsilon)}^{p-\varepsilon}(\Omega)$. Let us show that the sequence $\{f_n\}_{n=1}^\infty$ converges in $L_\nu^{p),\theta}(\Omega,\langle x\rangle^{-\lambda})$ to this function f. Let η be an arbitrary small positive number. We have

$$\|f-f_n\|_{L_\nu^{p),\theta}(\Omega,\langle x\rangle^{-\lambda})} = \sup_{0<\varepsilon<p-1}\left(\varepsilon^\theta \int\limits_\Omega |f(x)-f_n(x)|^{p-\varepsilon}\langle x\rangle^{-\lambda(\varepsilon)}dx\right)^{\frac{1}{p-\varepsilon}}.$$

Then by the definition of sup, for the number $\frac{\eta}{2} > 0$ there exists a number $\varepsilon_1 > 0$ such that

$$\sup_{0<\varepsilon<p-1}\left(\varepsilon^\theta \int\limits_\Omega |f(x)-f_n(x)|^{p-\varepsilon}\langle x\rangle^{-\lambda(\varepsilon)}dx\right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\left(\varepsilon_1^\theta \int\limits_\Omega |f(x)-f_n(x)|^{p-\varepsilon_1}\langle x\rangle^{-\lambda(\varepsilon_1)}dx\right)^{\frac{1}{p-\varepsilon_1}} + \frac{\eta}{2}.$$

Therefore,

$$\|f-f_n\|_{L_\nu^{p),\theta}(\Omega,\langle x\rangle^{-\lambda})} \leq \|f-f_n\|_{L_{\lambda(\varepsilon_1)}^{p-\varepsilon_1}}\varepsilon_1^{\frac{\theta}{p-\varepsilon_1}} + \frac{\eta}{2}.$$

Since the sequence $\{f_n\}_{n=1}^\infty$ converges to f in $L_\nu^{p-\varepsilon_1}(\Omega,\langle x\rangle^{-\lambda(\varepsilon_1)})$, $\varepsilon_1 > 0$, for $\frac{\eta}{2}\varepsilon_1^{-\frac{\theta}{p-\varepsilon_1}} > 0$ there exists a natural number N such that for all $n \geq N$ there holds the inequality $\|f-f_n\|_{L_\nu^{p-\varepsilon_1}(\Omega,\langle x\rangle^{-\lambda(\varepsilon_1)})} < \frac{\eta}{2}\varepsilon_1^{-\frac{\theta}{p-\varepsilon_1}}$ and then $\|f-f_n\|_{L_\nu^{p),\theta}(\Omega,\langle x\rangle^{-\lambda})} < \frac{\eta}{2}\varepsilon_1^{-\frac{\theta}{p-\varepsilon_1}}\varepsilon_1^{\frac{\theta}{p-\varepsilon_1}} + \frac{\eta}{2} = \eta$, which completes the proof.

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References

[1] B. Gupta A. Fiorenza and P. Jain. The maximal theorem in weighted grand Lebesgue spaces. *Studia Math.*, 188(2):123–133, 2008.

[2] J. Berg and J. Löfström. *Interpolation spaces. An Introduction.* Springer, Berlin, 1976.

[3] A. Fiorenza C. Capone and G.E. Karadzhov. Grand Orlicz spaces and global integrability of the Jacobian. *Math. Scand.*, 102(1):131–148, 2008.

[4] C. Capone and A. Fiorenza. On small Lebesgue spaces. *J. Function Spaces and Applications*, 3:73–89, 2005.

[5] J. Duoandikoetxea. *Fourier Analysis.* "Graduate Studies", Amer. Math. Soc., 2001.

[6] E.M. Dyn'kin and B.B. Osilenker. *Weighted estimates for singular integrals and their applications.* Math. Anal., vol. **21** Itogi nauki i techniki VINITI Akad. Nauk SSSR, (Russian), Moscow: Nauka, 1983.

[7] A. Fiorenza. Duality and reflexivity in grand Lebesgue spaces. *Collect. Math.*, 51(2):131–148, 2000.

[8] A. Fiorenza and G. E. Karadzhov. Grand and small lebesgue spaces and their analogs. *Journal for Analysis and its Applications*, 23(4):657–681, 2004.

[9] A. Fiorenza and J. M. Rakotoson. Petits espaces de Lebesgue et leurs applications. *C.R.A.S. t*, 333:1–4, 2001.

[10] G. Di Fratta and A. Fiorenza. A direct approach to the duality of grand and small Lebesgue spaces. *Nonlinear Analysis: Theory, Methods and Applications*, 70(7):2582–2592, 2009.

[11] T. Iwaniec and C. Sbordone. On the integrability of the Jacobian under minimal hypotheses. *Arch. Rational Mech. Anal.*, 119:129–143, 1992.

[12] V. Kokilashvili. Boundedness criterion for the Cauchy singular integral operator in weighted grand Lebesgue spaces and application to the Riemann problem. *Proc. A. Razmadze Math. Inst.*, 151:129–133, 2009.

[13] V. Kokilashvili. Boundedness criteria for singular integrals in weighted grand lebesque spaces. *J. Math. Sci.*, 170:20–33, 2010.

[14] V. Kokilashvili. The Riemann boundary value problem for analytic functions in the frame of grand $L^{p)}$ spaces. *Bull. Georgian Nat. Acad. Sci.*, 4(1):5–7, 2010.

[15] V. Kokilashvili. Weighted problems for operators of harmonic analysis in some Banach function spaces. *Lecture Course at Summer School HART2010, Lisbon, June 21–25*, http://www.math.ist.utl.pt/ hart2010/kokilashvili.pdf., 2010.

[16] V. Kokilashvili and A. Meskhi. A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces. *Georgian Math. J.*, 16(3):547–551, 2009.

[17] V. Kokilashvili and S. Samko. Boundedness of weighted singular integral operators on a Carleson curves in Grand Lebesgue spaces. *In* ICNAAM 2010: Intern. Conf. Numer. Anal. Appl. Math. AIP Confer. Proc., 1281:490–493, 2010.

[18] T. Iwaniec L. Greco and C. Sbordone. Inverting the p-harmonic operator. *Manuscripta Math.*, 92:249–258, 1997.

[19] A. Meskhi. Maximal functions and singular integrals in Morrey spaces associated with grand Lebesgue spaces. *Proc. A. Razmadze Math. Inst.*, 151:139–143, 2009.

[20] E.M. Stein and G. Weiss. Interpolation of operators with change of measures. *Trans. Amer. Math. Soc.*, 87:159–172, 1958.

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