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TOPICAL REVIEWER

A survey of spectra of parametrized Banach space complexes

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Abstract. In the present paper we summarize all key details of noncommutative spectral theory. A general framework of noncommutative spectral mapping properties have been proposed. As the main tool of our approach we use the known constructions over parametrized Banach space complexes. In this abstract form our framework can be applied to many different categories to develop the relevant spectral theory in that category. To implement this idea we demonstrate how spectral mapping properties can be obtained for representations of a Banach Lie algebra.

Key Words and Phrases: Slodkowski spectra, parametrized Banach space complexes, quasinilpotent Lie algebra, Taylor spectrum
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1. Introduction

A possible realization of a functional algebra as an algebra of linear operators is the well known problem of noncommutative functional calculus. By realization as an operator algebra we mean a continuous algebra homomorphism from a noncommutative functional algebra $\mathcal{F}$ into the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators acting on a certain complex Banach space X. The functional calculus problem goes back to I. M. Gelfand (see [50, 2.2.15]) and creates the foundation of the whole theory of Banach algebras. The known Gelfand's result asserts that if D is an open subset of the complex plane $\mathbb{C}$ containing the spectrum $\mathrm{sp}_A(a)$ of an element a of a complex Banach algebra A, then there is a unital continuous algebra homomorphism $\Gamma : \mathcal{O}(D) \to A$ from the Fréchet algebra $\mathcal{O}(D)$ of all holomorphic functions on the domain D into the Banach algebra A, sending the coordinate function z to the element a. In this case we write $f(a)$ instead of $\Gamma(f)$, where $f \in \mathcal{O}(D)$, and we say that $f(a)$ is a holomorphic function of a.

A similar problem for several commuting operators is much more complicated and can be presented by the following way. Let $A = \mathcal{B}(X)$, and let $a = (a_1, \dots, a_n)$ be a family of commuting operators in A. The latter family automatically involves the canonical algebra homomorphism $\mathcal{P}_n \to A$, $p \mapsto p(a)$, where $\mathcal{P}_n$ is the algebra of all complex polynomials in n variables $z_1, \dots, z_n$. For which domains $D \subseteq \mathbb{C}^n$ the canonical algebra homomorphism $\mathcal{P}_n \to A$ can be extended up to a continuous algebra homomorphism $\mathcal{O}(D) \to A$, that is, whether X tuns out to be a Banach $\mathcal{O}(D)$-module? The problem was completely solved by J. L. Taylor [66] in terms of the joint spectrum $\sigma_{\mathrm{t}}(a)$ (now called Taylor spectrum) [65] of the operator family a. The joint spectrum $\sigma_{\mathrm{t}}(a)$ is defined as a set of those $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that the Koszul complex $\mathrm{Kos}(X, a - \lambda)$ generated by the operator family $a - \lambda = (a_1 - \lambda_1, \dots, a_n - \lambda_n)$ fails to be exact. If a domain $D \subseteq \mathbb{C}^n$ contains Taylor spectrum $\sigma_{\mathrm{t}}(a)$ then required homomorphism $\Gamma : \mathcal{O}(D) \to A$ exists and again we could write $f(a)$ instead of $\Gamma(f)$, $f \in \mathcal{O}(D)$. Thus the function $f(a)$ of the operator family a is obtained by means of the functional calculus or the homomorphism Γ. The basic property of the holomorphic functional calculus is the spectral mapping theorem $\sigma_{\mathrm{t}}(f(a)) = f(\sigma_{\mathrm{t}}(a))$, where $f = (f_1, \dots, f_m) : D \to \mathbb{C}^m$ is a holomorphic mapping. The latter property is closely related to the functional calculus problem itself and it really demonstrates how the homomorphism $\Gamma : \mathcal{O}(D) \to A$ acts in. Taylor proposed two different methods to solve the multivariable functional calculus problem. The first one is based on so called an abstract version of the Cauchy-Weyl integral [64]. The constructions developed in [64] turned out to be very progressive in various questions of spectral theory like spectral decompositions of an operator family and invariant subspace problem. All these aspects of the theory were kindly collected in the monograph [74] by F.-H. Vasilescu. In the Hilbert space case, more canonical integral representation of Taylor's functional calculus have been obtained in [73] and later improved by V. Muller (see [55, 4.30]).

The second method [66] has used topological homology and allows to present a general framework of the functional calculus problem even for a noncommutative operator family. The arguments suggested in [66] by somewhat simplified and were nicely exposed in the monograph [49] by Helemskii. Further advancement of the homological ideas to study functional calculus associated with the algebras $\mathcal{O}(D)$, where D are domains in the Stein space U, were reflected in the monograph by Eschmeier and Putinar [44]. That is an essentially general case than the Taylor functional calculus, where it is assumed $U = \mathbb{C}^n$.

Although far investigation of the commutative functional problem, almost nothing were done toward noncommutative case. Taylor's approach to the noncommutative holomorphic functional calculus proposed in [66], remained undeveloped. The main reason of that uncertainty probably was the construction of noncommutative algebras of holomorphic functions. In this concern Taylor in [66] had considered a different completions of the free algebra. Analytic versions of the algebras with polynomial identities were developed in [53], [54] by D. Luminet. Noncommutative generalizations of the function theory based on quantum groups are reflected in the papers [42], [63], [71], [72] by L. L. Vaskman, S. D. Sinelshikov and D. L. Shklyarov. Noncommutative generalization of the algebras of holomorphic functions on complex varieties have been developed in author's papers [10]-[41], and in the papers [58], [59] by A. Yu. Pirkovskii. Noncommutative versions of Taylor type spectra were independently developed in the papers [1], [3], [4], [22]-[35], [46], [56] by E. Boasso, A. Laratonda, D. Beltiţa, A.S. Fainstein, S. Ott, and the author. Let us also notice dissertations [57] by S. Ott, and [20] by the author, and also the monograph [2] by D. Beltiţa and M. Şabac, which are dedicated exactly to these matters. As the basic concept of investigations are mainly considered various spectra of representations of a finite dimensional Lie algebra based on parametrized (over Lie characters of the considered Lie algebra) Koszul complexes of representations. It turns out that all desirable properties of these spectra are obtained for representations of a nilpotent Lie algebra. These investigations on the spectral theory put forward a hypothesis on existence of Taylor's holomorphic functional calculus for a certain class of noncommutative operator families generating finite dimensional nilpotent Lie algebras. The main result of the paper [14] has confirmed this hypothesis. That is the operator families generating supernilpotent Lie subalgebras.

The goal of the paper is to present a survey of noncommutative spectral theory within a general framework of spectra of parametrized Banach space complexes. We propose a general theory of Taylor type joint spectra based upon infinite parametrized Banach space complexes. Let Ω be a topological space, $\mathfrak{X} = \{X_n : n \in \mathbb{Z}\}$ a family of Banach spaces, and let $\mathfrak{d} = \{d_n : n \in \mathbb{Z}\}$, $d_n : \Omega \to \mathcal{L}(X_{n+1}, X_n)$ be a family of continuous mappings such that $(\mathfrak{X}, \mathfrak{d}(\lambda))$ (here $\mathfrak{d}(\lambda) = \{d_n(\lambda)\}$) is a chain Banach space complex

$$\cdots \longleftarrow X_{n-1} \overset{d_{n-1}(\lambda)}{\longleftarrow} X_n \overset{d_n(\lambda)}{\longleftarrow} X_{n+1} \longleftarrow \cdots,$$

for each $\lambda \in \Omega$. In this case, we say that the family $(\mathfrak{X}, \mathfrak{d}) = \{(\mathfrak{X}, \mathfrak{d}(\lambda)) : \lambda \in \Omega\}$ of Banach space complexes is a parametrized at Ω chain Banach space complex, or briefly Ω-Banach complex. Slodkowski (in particular, Taylor) spectrum $\sigma(\mathfrak{X}, \mathfrak{d})$ (of type π or δ) of this complex is defined to be the set of those points $\lambda \in \Omega$ such that the Banach space complex $(\mathfrak{X}, \mathfrak{d}(\lambda))$ admits nontrivial homologies in a certain places (see Definition 3.1). Similarly, it is defined spectra of a cochain complex, and the class of all Slodkowski spectra is denoted by $\mathfrak{S}$. Using the chain and cochain versions of Banach space complexes, each spectrum $\sigma \in \mathfrak{S}$ associates its dual spectrum $\sigma^* \in \mathfrak{S}$ (see Section 3). It is proved that (see Theorem 3.1) if $(\mathfrak{X}^*, \mathfrak{d}^*)$ is the dual to $(\mathfrak{X}, \mathfrak{d})$ complex, then

$$\sigma(\mathfrak{X}^*, \mathfrak{d}^*) = \sigma^*(\mathfrak{X}, \mathfrak{d}),$$

for all $\sigma \in \mathfrak{S}$. Furthermore

$$\sigma(\mathfrak{X}, \mathfrak{d}) = \sigma(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}),$$

where $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ is the ultrapower of the Ω-Banach complex $(\mathfrak{X}, \mathfrak{d})$. If Y is a projective Banach space then

$$\sigma(\mathfrak{X}, \mathfrak{d}) = \sigma(\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d}))),$$

(see Theorems 3.2 and 3.3). One of the central properties of spectra is the Projection Property which is investigated in Section 4. If β is a bounded endomorphism of Ω-Banach complex $(\mathfrak{X}, \mathfrak{d})$, then the cone $\mathrm{Con}_\beta(\mathfrak{X}, \mathfrak{d})$ of the endomorphism turns into $\Omega \times \mathbb{C}$-Banach complex. Consider the canonical projection $\Pi : \Omega \times \mathbb{C} \to \Omega$. Then we have the projection property

$$\sigma(\mathfrak{X}, \mathfrak{d}) = \Pi(\sigma(\mathrm{Con}_\beta(\mathfrak{X}, \mathfrak{d}))),$$

whenever $(\mathfrak{X}, \mathfrak{d})$ has a stable behavior with respect to the spectrum σ (see Theorem 4.1). In Section 5, a general scheme of spectral mapping properties for π type Slodkowski spectra has been presented. A couple $(\mathfrak{X}, \mathfrak{d})$ and $(\mathfrak{Y}, \overline{\mathfrak{d}})$ of Banach space complexes parametrized at the spaces Ω and Λ, respectively, are connected with a certain Banach space bicomplex, and a spectral mapping $f : \Omega \to \Lambda$ acting from the parameter set Ω into another set Λ has been introduced. We prove (see Theorems 5.1, 5.2) the forward

$$f(\sigma(\mathfrak{X}, \mathfrak{d})) \subseteq \sigma(\mathfrak{Y}, \overline{\mathfrak{d}}),$$

and backward

$$f(\sigma(\mathfrak{X}, \mathfrak{d})) \supseteq \sigma(\mathfrak{Y}, \overline{\mathfrak{d}}),$$

spectral mapping theorems under certain restrictions on the connecting bicomplex. This is a general framework of spectral theory for parametrized Banach space complexes. Further, the proposed framework is applied to the Banach space representations of Banach-Lie algebras.

We fix a Banach module (X, α) over Banach-Lie algebra E, where $\alpha : E \to \mathcal{B}(X)$ is a bounded Lie representation of the Lie algebra E on a Banach space X. The module X generates (co)chain (in the general case, it is an infinite) Banach space complex $C_\bullet(\alpha)$ (resp., $C^\bullet(\alpha)$) (see Section 6), which is a Koszul type complex. The space of Lie characters (equipped with $*$-weak topology) of the considered Banach-Lie algebra E is denoted by $\Delta(E)$. Thus there is a parametrized at the space $\Delta(E)$ (co)chain Banach space complex

$$\mathcal{C}_\bullet(\alpha) = \{C_\bullet(\alpha - \lambda) : \lambda \in \Delta(E)\},$$

(resp., $\mathcal{C}^\bullet(\alpha) = \{C^\bullet(\alpha - \lambda) : \lambda \in \Delta(E)\}$), whose spectra are denoted by $\sigma(\alpha)$ and they are called Slodkowski spectra of the Lie representation α. These spectra with their chain and cochain versions are connected with the relation $\sigma(\alpha) = \sigma^*(\alpha^*)$, where $\alpha^* : E^{\mathrm{op}} \to \mathcal{B}(X^*)$, $\alpha^*(a) = \alpha(a)^*$ is the dual to α Lie representation, or (X^*, α^*) is the dual to X module. The union of all spectra $\sigma(\alpha_{\mathfrak{U}})$

of ultrapowers $\alpha_{\mathfrak{U}} : E \to \mathcal{B}(X_{\mathfrak{U}})$ taken over all countably incomplete ultrafilters $\mathfrak{U}$ is called the *ultraspectrum* $\sigma_{\mathrm{u}}(\alpha)$ of the representation α. In Section 6 we establish a connection between the approximate point spectrum, ultrapoint spectrum and ultraspectrum of the Lie representation α. In particular, it is shown that the ultraspectrum of a solvable Banach-Lie algebra representation is always nonempty.

For the spectral mapping properties it is important to deal with the quasinilpotent Lie algebras. That is the case when all operators of the adjoint representation of a Banach-Lie algebra E are quansinilpotents. For the finite dimensional E, such algebra is nilpotent. If F is a closed ideal of finite codimension in a quasinilpotent Lie algebra E, and (X, α) is a Banach E-module, then $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$. Moreover, if $\alpha([E, E])$ consists of quasinilpotent operators, then

$$\sigma(\alpha)|_F = \sigma(\alpha|_F).$$

In particular, the projection property onto Lie subalgebras is satisfied for a finite dimensional nilpotent Lie algebra and arbitrary Slodkowski spectrum, that is, $\sigma(\alpha)|_F = \sigma(\alpha|_F)$ for any Lie subalgebra F of a finite dimensional nilpotent Lie algebra E, and $\sigma \in \mathfrak{S}$ (see Theorem 7.1). In the infinite dimensional case, we have the inclusion $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$ for a finite dimensional Lie subalgebra F of a quasinilpotent Lie algebra E, whenever E is a projective Banach space.

In order to have a comprehensive spectral mapping property for Banach-Lie algebra representations, it is necessary to postulate noncommutative functions in elements of a Lie algebra and their actions over the Banach module, that is, noncommutative functional calculus. Pursuing our central goal, we fix a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ as a space of noncommutative variables, and consider a locally convex algebra $\mathcal{A}_{\mathfrak{g}}$ with the fixed Lie homomorphism $\pi : \mathfrak{g} \to \mathcal{A}_{\mathfrak{g}}$, whose motivation is the algebra of noncommutative functions in elements of $\mathfrak{g}$. Without going into the specifics of the functional calculus problem (for instance, holomorphic functional calculus), we introduce the property to be dominating of the algebra $\mathcal{A}_{\mathfrak{g}}$ over the $\mathfrak{g}$-module (X, α). Namely, let $\mathcal{B}$ be another locally convex algebra with the Lie homomorphism $\alpha : \mathfrak{g} \to \mathcal{B}$. For instance, $\mathcal{B} = \mathcal{B}(X)$ and α is the Lie representation of $\mathfrak{g}$ on the Banach space X. Note that the Lie homomorphism α taken together with the adjoint representation of $\mathfrak{g}$, defines a new Lie homomorphism (see Subsection 7.1) $\theta : \mathfrak{g} \to M_k(\mathcal{B})$ into the matrix algebra $M_k(\mathcal{B})$ over $\mathcal{B}$, where $k = 2^{\dim(\mathfrak{g})}$. More precisely, $\theta(u)$ is represented as the uppertriangular matrix whose main diagonal consists of the same element $\alpha(u)$, where $u \in \mathfrak{g}$. For the commutative Lie algebra $\mathfrak{g}$, the Lie homomorphism θ is reduced to the diagonal inflation of α. The algebra $\mathcal{A}_{\mathfrak{g}}$ is said to be *dominating over the pair* $(\mathcal{B}, \alpha)$ (resp., over the module (X, α) if $\mathcal{B} = \mathcal{B}(X)$) and we write $\mathcal{A}_{\mathfrak{g}} \succ (\mathcal{B}, \alpha)$ (resp., $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$), if there exists a continuous algebra homomorphism

$$\theta|^{\mathcal{A}_{\mathfrak{g}}} : \mathcal{A}_{\mathfrak{g}} \to M_k(\mathcal{B}),$$

extending θ (in the sense that $\theta|^{\mathcal{A}_{\mathfrak{g}}} \cdot \pi = \theta$) such that the range of the inverse closed subalgebra in $\mathcal{A}_{\mathfrak{g}}$ generated by $\pi(\mathfrak{g})$ turns out to be dense in the range of the representation $\theta|^{\mathcal{A}_{\mathfrak{g}}}$ itself. In the commutative case, the latter property is reduced to the problem whether the Lie homomorphism $\alpha : \mathfrak{g} \to \mathcal{B}$ might be extended up to a continuous algebra homomorphism

$$\alpha|^{\mathcal{A}_{\mathfrak{g}}} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}, \quad \alpha|^{\mathcal{A}_{\mathfrak{g}}} \cdot \pi = \alpha,$$

with the property stated above, that is, there is $\mathcal{A}_{\mathfrak{g}}$-calculus in $\mathcal{B}$. Certainly, the existence of a continuous algebra homomorphism $\theta|^{\mathcal{A}_{\mathfrak{g}}}$ in the noncommutative case defines $\mathcal{A}_{\mathfrak{g}}$-calculus $\alpha|^{\mathcal{A}_{\mathfrak{g}}} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}$ in the algebra $\mathcal{B}$, automatically. But the reverse is not true in general, mostly it depends on the structure of the nilpotent Lie algebra $\mathfrak{g}$, and the algebra $\mathcal{A}_{\mathfrak{g}}$ itself. Thus the noncommutative effect in that statement of the problem is appeared in replacement of the original algebra $\mathcal{B}$ by the matrix algebra $M_k(\mathcal{B})$ over $\mathcal{B}$, which reminds us the main idea of the quantization of a locally convex space. Recall that in quantum functional analysis [43], [51], [75] or the theory of quantum space, the quantizations are considered instead of locally convex spaces, that is, locally convex topologies in the matrix space over the original linear space which are compatible with the main quantum (or matrix) operations. It turns out that the property to be dominating is quite distinctly described within the framework of quantum spaces [9]. In that concern, the author has several published papers [10], [13], [17], [36], [37] related to the theory of quantum spaces itself, and its applications to the quantum moment problem [16], [35].

Now let (X, α) be a Banach $\mathfrak{g}$-module, and let $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$. Then each point λ of the spectrum $\sigma(\alpha)$ of the representation α admits unique extension $\lambda|^{\mathcal{A}_{\mathfrak{g}}}$ up to a continuous character of the algebra $\mathcal{A}_{\mathfrak{g}}$, that is, $\lambda|^{\mathcal{A}_{\mathfrak{g}}} \in \operatorname{Spec} \mathcal{A}_{\mathfrak{g}}$ (see Corollary 8.4). In this case, we write $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}} \subseteq \operatorname{Spec} \mathcal{A}_{\mathfrak{g}}$. In particular, $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$ denotes the the set of all restrictions $\lambda|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$ of those extended functionals from the spectrum onto the Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$. In Subsection 8.2, the forward spectral mapping theorem has been proven. Namely, if $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$ is a normed Lie subalgebra whose completion is a projective Banach space then the inclusion $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ is true for all π-type Slodkowski spectra σ. The assumption on the completion of the Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$ to be projective has purely technical nature and can easily be satisfied in practice (see [33]). The reverse inclusion on spectral mapping demands additional assumptions. The elements of the normed Lie algebra $\mathfrak{F}$ have to be appeared themselves as noncommutative functions in elements of the Lie algebra $\mathfrak{g}$. Such property turned out the splitness of an element $a \in \mathcal{A}_{\mathfrak{g}}$ over the $\mathfrak{g}$-module X (see Definition 8.2). Roughly speaking, for each $\lambda \in \sigma(\alpha)$ certain power of the operator $(\theta - \lambda)|^{\mathcal{A}_{\mathfrak{g}}}(a)$ splits the complex $C^{\bullet}(\alpha - \lambda)$. In particular, the action $(\theta - \lambda)|^{\mathcal{A}_{\mathfrak{g}}}(a)$ over cohomologies of the complex $C^{\bullet}(\alpha - \lambda)$ turn out nilpotent. If S is a subset in $\mathcal{A}_{\mathfrak{g}}$ of splitting over the Banach $\mathfrak{g}$-module (X, α) elements generating a quasinilpotent normed Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$, whose completion is a projective Banach space, then

$$\sigma_{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}},$$

for all π-type Slodkowski spectra σ (see Theorem 8.3). Let us note that if the Lie subalgebra $\mathfrak{F}$ generated by the set S is finite dimensional, then $\mathfrak{F}$ is automatically projective Banach space, and in addition $\mathfrak{F}$ is a nilpotent Lie algebra. Then $\sigma_{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ and we have the equality

$$\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}},$$

for all Slodkowski spectra $\sigma \in \mathfrak{S}$. We emphasize that in the case of polynomial Lie subalgebra $\mathfrak{F}$ and Taylor spectrum σ_{t} the latter result was obtained by A. S. Fainshtein in [46].

The main technical details of this theory were published in author's papers [22] - [29], [32], [33]. The elements of noncommutative Fredholm theory has been considered in [24]. Noncommutative subspectra and regularity in unital Banach algebras were considered in [38]. It was proved that there is a correspondence between them which in turn involves the radical in the class of Banach algebras equipped with a subspectrum. Note that Slodkowski spectra are the main examples of subspectra in the noncommutative case.

2. Preliminaries

In this section we briefly recall the main tools (or methods) and notations which we use throughout the whole text.

If S is a set then 2^S denotes the set of all subsets of S. All considered linear spaces (in particular, algebras, modules) are complex. The identity operator on a linear space X is denoted by 1_X. For the norm of a normed space X and for its dual space we use notations $\|\cdot\|_X$ and X^*

respectively. The unit set of X is denoted by ball X, whereas $\widehat{X}$ indicates the norm-completion of X. Let A, B, C and D be arbitrary sets such that $C \subseteq B \subseteq A$, and let $f : B \to D$ be a function. Then, $f|_C$ denotes the restriction of f onto C, and if f is extended up to a function $g : A \to D$ by some rule then we write $g = f|^A$. For any set of functions $\mathcal{F}$ defined on B with values in D we set $\mathcal{F}|_C = \{f|_C : f \in \mathcal{F}\}$ and $\mathcal{F}|^A = \left\{f|^A : f \in \mathcal{F}\right\}$.

The Jacobson radical of $A \in \mathbf{LCA}$ is denoted by $\operatorname{Rad} A$ as usual. The space of all continuous characters (multiplicative linear functionals) of A furnished with the $*$-weak topology is denoted by $\operatorname{Spec} A$. If $A \in \mathbf{BA}$ and $n \in \mathbb{N}$ then Banach algebra of all $n \times n$-matrices (with the max-norm) over the algebra A is denoted by $\mathcal{M}_n(A)$. The left (resp., right) multiplication operator on an associative algebra A is denoted by L_a (resp., R_a), that is, $L_a(x) = ax$ (resp., $R_a(x) = xa$), where $a, x \in A$. The unit element of A is denoted by 1_A.

2.1. Complexes of linear spaces

Let $\mathbf{BS}$ (resp., $\mathbf{BA}$) be the category of all Banach spaces (resp., unital associative algebras), $\mathbf{FS}$ (resp., $\mathbf{FA}$) the category of all Fréchet spaces (resp., unital associative algebras) and let $\mathbf{LCS}$ (resp., $\mathbf{LCA}$) be the category of all Hausdorff complete locally convex spaces (shortly, l.c.s.) (resp., unital associative algebras with jointly continuous multiplication, shortly, l.c.a.). Let $X, Y \in \mathbf{LCS}$. The space of all continuous linear operators $X \to Y$ is denoted by $\mathcal{L}(X, Y)$ and let $\mathcal{L}(X) = \mathcal{L}(X, X)$ be the algebra of all continuous linear operators on X. If $X \in \mathbf{BS}$ then $\mathcal{L}(X) \in \mathbf{BA}$ with respect to the operator norm and $\mathcal{L}(X) = \mathcal{B}(X)$ is the algebra of all bounded linear operators acting on X. The kernel and the image of an operator $T \in \mathcal{L}(X, Y)$ are denoted by $\ker(T)$ and $\operatorname{im}(T)$ respectively. We use conventional denotation $X \widehat{\otimes} Y$ for the projective tensor product. Let $T : X \to Y$ be a morphism. The morphism T is called to be a (co)*retraction* if $TS = 1$ $(ST = 1)$ for a certain morphism $S : Y \to X$. In this case Y is called a (co)*retract* of X. The algebra of all complex continuous functions on a compact topological space Ω is denoted by $C(\Omega)$. It is well known that $C(\Omega) \in \mathbf{BA}$ with respect to the sup-norm.

Let Φ be a subcategory in $\mathbf{LCS}$. A chain complex in Φ is the pair $(\mathfrak{X}, \mathfrak{d})$, where $\mathfrak{X} = \{X_n : n \in \mathbb{Z}\}$ are objects and $\mathfrak{d} = \{d_n : n \in \mathbb{Z}\}$ are morphisms from Φ, such that $d_{n-1} d_n = 0$ for all n. We also write $(\mathfrak{X}, \mathfrak{d})$ as a sequence:

$$\cdots \longleftarrow X_{n-1} \overset{d_{n-1}}{\longleftarrow} X_n \overset{d_n}{\longleftarrow} X_{n+1} \longleftarrow \cdots .$$

A cochain complex in Φ is defined as a sequence

$$\cdots \to X^{n-1} \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \to \cdots ,$$

of objects and morphism from Φ such that $d^n d^{n-1} = 0$ for all n. The category of all (co)chain complexes in Φ is denoted by $\underline{\Phi}$ $(\overline{\Phi})$. If $\Phi = \mathbf{BS}$ then we say that $(\mathfrak{X}, \mathfrak{d})$ is a Banach space complex. The quotient space $H_n(\mathfrak{X}, \mathfrak{d}) = \ker(d_{n-1}) / \operatorname{im}(d_n)$ (resp., $H^n(\mathfrak{X}, \mathfrak{d}) = \ker(d^n) / \operatorname{im}\left(d^{n-1}\right)$), $n \in \mathbb{Z}$, is called (co)homology of the complex $(\mathfrak{X}, \mathfrak{d})$. The complex $(\mathfrak{X}, \mathfrak{d})$ is said to be nonnegative if $X_n = \{0\}$ (resp., $X^n = \{0\}$) for all n, $n < 0$. Note that each (co)chain complex $(\mathfrak{X}, \mathfrak{d})$ makes into a cochain (resp., chain) complex $\left(\overline{\mathfrak{X}}, \overline{\mathfrak{d}}\right)$ (resp., $(\underline{\mathfrak{X}}, \underline{\mathfrak{d}})$) by setting $\overline{X}^n = X_{-n}$ and $\overline{d}^n = d_{-n}$, $n \in \mathbb{Z}$. This defines a functor $\underline{\mathbf{BS}} \longrightarrow \overline{\mathbf{BS}}$ (resp., $\overline{\mathbf{BS}} \to \underline{\mathbf{BS}}$) called the *conjugation functor*. Let $Y \in \mathbf{BS}$. Using the functors $\mathcal{L}(Y, \circ)$, $\mathcal{L}(\circ, Y)$, $\circ \widehat{\otimes} Y$ one can associate new Banach space complexes from the original complex $(\mathfrak{X}, \mathfrak{d})$. Note that differentials of $\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d}))$, $\mathcal{L}((\mathfrak{X}, \mathfrak{d}), Y)$ and $(\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y$ are given by the operators $L_{d_{k-1}}$, R_{d_k} and $d_{k-1} \otimes 1_Y$, respectively.

2.2. Ultrapowers

Let S be an infinite set and $\mathfrak{U}$ be a nontrivial (i.e. $\bigcap_{M\in\mathfrak{U}} M = \emptyset$) ultrafilter in S. The ultrafilter $\mathfrak{U}$ is said to be *countably incomplete* (see [48], [7]) if there exists a countable partition $\{S_n : n \in \mathbb{N}\}$ of S such that $S_n \notin \mathfrak{U}$ for each $n \in \mathbb{N}$. The filter of complements of finite subsets in $\mathbb{N}$ is called Fréchet filter. Any nontrivial ultrafilter in $\mathbb{N}$ is countably incomplete, as it dominates Fréchet filter. There exist countably incomplete ultrafilters in any infinite set S [7]. In the sequel, by an ultrafilter we mean a nontrivial countably incomplete ultrafilter, if it is not specially be indicated. Let $X \in \mathbf{BS}$ and let $\ell_\infty(S,X)$ be a Banach space of all bounded families $(x_s)_{s\in S}$ from X furnished with sup-norm. For an ultrafilter $\mathfrak{U}$ in I, let $N_{\mathfrak{U}}(X)$ be a closed subspace in $\ell_\infty(S,X)$ comprising those $(x_s)_{s\in S}$ with $\lim_{\mathfrak{U}} x_s = 0$. The ultrapower of X following $\mathfrak{U}$ is called the quotient space $X_{\mathfrak{U}} = \ell_\infty(S,X)/N_{\mathfrak{U}}(X)$. The element of $X_{\mathfrak{U}}$ which includes a representative of the family $(x_s)_{s\in S} \in \ell_\infty(S,X)$ is denoted by $[x_s]$. One can easily check that the norm $\|[x_s]\|$ is $\lim_{\mathfrak{U}} \|x_s\|$. The space X is contained in $X_{\mathfrak{U}}$ as a subspace generated by constant families in $\ell_\infty(S,X)$, and $X_{\mathfrak{U}} = X$ iff X is finite-dimensional space (see [7, Proposition 7]). For a subset $C \subseteq X$, the ultrapower of C following $\mathfrak{U}$ is $C_{\mathfrak{U}} = \{[c_s] \in X_{\mathfrak{U}} : c_s \in C\}$. Consider ultrafilters $\mathfrak{U}$ and $\mathfrak{V}$ in S and T respectively. Let $A_t = \{s \in S : (s,t) \in A\}$, $t \in T$, and $T_A = \{t \in T : A_t \in \mathfrak{U}\}$ for $A \subseteq S \times T$. The production $\mathfrak{U} \times \mathfrak{V}$ is defined as a family of subsets $A \subseteq S \times T$ for which $T_A \in \mathfrak{V}$. Then $\mathfrak{U} \times \mathfrak{V}$ is an ultrafilter [7]. The following assertion was proved in [7].

Lemma 2.1. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be nontrivial ultrafilters in S and T, respectively. If one of them is countably incomplete, then so is $\mathfrak{U} \times \mathfrak{V}$. Moreover, the canonical operator*

$$X_{\mathfrak{U}\times\mathfrak{V}} \to (X_{\mathfrak{U}})_{\mathfrak{V}}, \quad \left[x_{(s,t)}\right]_{(s,t)\in S\times T} \mapsto \left[\left[x_{(s,t)}\right]_{s\in S}\right]_{t\in T},$$

is an isometric isomorphism.

Let $X, Y \in \mathbf{BS}$. An operator $T \in \mathcal{L}(X,Y)$ is extended up to $T_{\mathfrak{U}} \in \mathcal{L}(X_{\mathfrak{U}}, Y_{\mathfrak{U}})$, $T_{\mathfrak{U}}[x_s] = [Tx_s]$ and $\|T_{\mathfrak{U}}\| = \|T\|$. Thus the assignment $X \mapsto X_{\mathfrak{U}}$, $T \mapsto T_{\mathfrak{U}}$, is a functor $\circ_{\mathfrak{U}} : \mathbf{BS} \to \mathbf{BS}$ called an ultrapower functor. The following assertion was proved in [7, Propositions 15,16,20,22].

Lemma 2.2. *Let $T \in \mathcal{L}(X,Y)$. Then $\ker(T)_{\mathfrak{U}} \subseteq \ker(T_{\mathfrak{U}})$, $\operatorname{im}(T_{\mathfrak{U}}) \subseteq \operatorname{im}(T)_{\mathfrak{U}}$ and $\overline{\operatorname{im}(T)} = Y \cap \overline{\operatorname{im}(T_{\mathfrak{U}})}$. Moreover, the following statements are equivalent:*
(i) $\operatorname{im}(T)$ is closed;
(ii) $\operatorname{im}(T_{\mathfrak{U}})$ is closed;
(iii) $\ker(T)_{\mathfrak{U}} = \ker(T_{\mathfrak{U}})$;
(iv) $\operatorname{im}(T_{\mathfrak{U}}) = \operatorname{im}(T)_{\mathfrak{U}}$.

Finally, let us recall that a Banach space X is said to be a super-reflexive if its each ultrapower $X_{\mathfrak{U}}$ is reflexive. For more detailed properties of super-reflexive spaces we refer the reader to [48]. We just mention the following result from [48] that will be used later.

Proposition 2.1. *If $\mathfrak{U}$ is countably incomplete, then $(X^*)_{\mathfrak{U}} = (X_{\mathfrak{U}})^*$ iff X is super-reflexive.*

If $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$ (or $(\mathfrak{X}, \mathfrak{d}) \in \overline{\mathbf{BS}}$) then its image by the ultrapower functor $\circ_{\mathfrak{U}}$ is called ultrapower of the complex and it is denoted by $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$.

2.3. The dual of a Banach space complex

Let X, $Y \in \mathbf{BS}$. If $T \in \mathcal{L}(X,Y)$ then $T^* \in \mathcal{L}(Y^*, X^*)$ denotes the dual operator. Let $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$ and let $(\mathfrak{X}^*, \mathfrak{d}^*)$ be its dual complex:

$$\cdots \longrightarrow X_{n-1}^* \xrightarrow{d_{n-1}^*} X_n^* \xrightarrow{d_n^*} X_{n+1}^* \longrightarrow \cdots.$$

Undoubtedly, $(\mathfrak{X}^*, \mathfrak{d}^*) = \mathcal{L}((\mathfrak{X}, \mathfrak{d}), \mathbb{C}) \in \overline{\mathbf{BS}}$.

The following classical result is well known [49, Ch. 0, item 5.2].

Theorem 2.1. *Let $(\mathfrak{X}, \mathfrak{d})$ be a (co)chain Banach space complex. Then following assertions are true.*

(i) If $(\mathfrak{X}, \mathfrak{d})$ is exact at the terms X_{n-1} and X_n (resp., X^n and X^{n+1}), then $(\mathfrak{X}^*, \mathfrak{d}^*)$ is exact at X_n^* (resp., X^{n*});

(ii) If $(\mathfrak{X}^*, \mathfrak{d}^*)$ is exact X_{n+1}^* (resp., X^{n-1*}), then $\operatorname{im}(d_n)$ (resp., $\operatorname{im}(d^{n-1})$) is closed in X_n (resp., X^n);

(iii) if $(\mathfrak{X}^*, \mathfrak{d}^*)$ is exact at X_n^* (resp., X^{n*}), then $\operatorname{im}(d_n)$ (resp., $\operatorname{im}(d^{n-1})$) is a dense subset of $\ker(d_{n-1})$ (resp., $\ker(d^n)$).

Now let $T : X \to Y$ be a Banach space operator with closed range and let $T^\sim : X/\ker(T) \to \operatorname{im}(T)$, $T^\sim(x^\sim \bmod \ker(T)) = Tx$, be the induced operator. The latter has a bounded inverse and the norm of this inverse operator is called the *inversion constant of* T and it is denoted by $\operatorname{ic}(T)$. Further, let $P = (S, T)$, $S \in \mathcal{L}(X, Y)$, $T \in \mathcal{L}(Y, Z)$ be a Banach space operators. We say that $P = (S, T)$ is *an operator pair* if $TS = 0$, and an operator pair P is said to be *exact* if $\operatorname{im}(S) = \ker(T)$ and $\operatorname{im}(T)$ is closed.

The following lemma was proved by A.S. Fainshtein [45, Lemma 1.2].

Lemma 2.3. *Let $X, Y, Z \in \mathbf{BS}$ and let $P = (S, T)$, $S \in \mathcal{L}(X, Y)$, $T \in \mathcal{L}(Y, Z)$, be an operator pair. Then P is not exact iff there exist bounded sequences $\{y_n\} \subset Y$ and $\{f_n\} \subset Y^*$ such that*

$$\lim_n Ty_n = 0, \quad \lim_n S^* f_n = 0, \quad f_n(y_n) = 1.$$

Proof. Let us assume that $\operatorname{im}(S) \neq \ker(T)$, that is, the operator $S : X \to \ker(T)$ does not epimorphism. There exists a sequence $\{g_n\} \subset \ker(T)^*$ such that $\|g_n\| = 1$ and $\lim_n S^* g_n = 0$ (see Theorem 2.1). For each g_n one can find $y_n \in \ker(T)$ such that $g_n(y_n) = 1$ and $\|y_n\| \le 2$. We may extend g_n up to a functional $f_n \in Y^*$ such that $\|f_n\| = f_n(y_n) = 1$. Then $Ty_n = 0$ and $\lim_n S^* f_n = 0$.

Now let us assume that $\operatorname{im}(T)$ is not closed, or the induced operator

$$T^\sim : Y/\ker(T) \to Z, \quad T^\sim(y^\sim \bmod \ker(T)) = Ty,$$

has unclosed image. There exists a sequence $\{y_n^\sim\} \subset Y/\ker(T)$ such that $\|y_n^\sim\| = 1$ and $\lim_n T^\sim y_n^\sim = 0$. Take functionals $\{g_n\}$ from $(Y/\ker(T))^*$ such that $\|g_n\| = g_n(y_n^\sim) = 1$. Let $f_n \in Y^*$, $f_n(y) = g_n(y^\sim)$ and $y_n \in y_n^\sim, \|y_n\| \le 2$. It is clear that $\|f_n\| = \|g_n\| = 1$, $f_n(y_n) = g_n(y_n^\sim) = 1$, $\lim_n Ty_n = 0$ and $(S^* f_n)(x) = g_n(Sx)^\sim = 0$ for all $x \in X$. Hence sequences $\{y_n\}$ and $\{f_n\}$ satisfy required conditions.

Conversely, let $\{y_n\}$ and $\{f_n\}$ be given sequences, but $\ker(T) = \operatorname{im}(S)$ and $\operatorname{im}(T)$ is closed. Since $\lim_n Ty_n = 0$, there exists a sequence $\{u_n\} \subset Y$, such that $\lim_n u_n = 0$ and $Tu_n = Ty_n$. Then the sequence $\{y_n - u_n\}$ is bounded and belongs to $\operatorname{im}(S)$. There exists a bounded sequence $\{x_n\} \subset X$ such that $Sx_n = y_n - u_n$. But $\lim_n f_n(Sx_n) = \lim_n (S^* f_n) x_n = 0$ and $\lim_n f_n(y_n - u_n) = 1$, a contradiction.◄

The following assertion belongs to B. E. Johnson [52].

Lemma 2.4. *Let $X, Y, Z \in \mathbf{BS}$ and let $S = (S_1, S_2)$, $S_1 \in \mathcal{L}(X, Y)$, $S_2 \in \mathcal{L}(Y, Z)$ be an exact operator pair. Then so is an operator pair sufficiently close to S. Namely, if $T = (T_1, T_2)$, $T_1 \in \mathcal{L}(X, Y)$, $T_2 \in \mathcal{L}(Y, Z)$ is an operator pair and $k_T < 1$, then T is exact, where $k_T = c_1\|S_1 - T_1\| + c_2\|S_2 - T_2\| + c_1 c_2\|S_1 - T_1\|\|S_2 - T_2\|$, $c_i > \operatorname{ic}(S_i)$, $i = 1, 2$. Moreover,*

$$\operatorname{ic}(T_1) \le (1 - k_T)^{-1} c_1 (1 + c_2\|S_2 - T_2\|), \quad \operatorname{ic}(T_2) \le (1 - k_T)^{-1} c_2 (1 + c_1\|S_1 - T_1\|).$$

Proof. We set $\varepsilon_i = \|S_i - T_i\|$, $i = 1, 2$. Take $y \in Y$. Then $\|S_2 y\| \leq \|T_2 y\| + \varepsilon_2 \|y\|$. By very definition of $\mathrm{ic}(S_2)$, there exists $y' \in Y$ such that $S_2 y' = S_2 y$ and $\|y'\| \leq c_2 \|S_2 y\| \leq c_2 (\|T_2 y\| + \varepsilon_2 \|y\|)$. Moreover, $\|y - y'\| \leq c_2 \|T_2 y\| + (1 + c_2\varepsilon_2) \|y\|$ and $y - y' \in \ker(S_2) = \mathrm{im}(S_1)$. Again, by definition of $\mathrm{ic}(S_1)$, we conclude that $y - y' = S_1 x$ and $\|x\| \leq c_1 \|y - y'\| \leq c_1 c_2 \|T_2 y\| + c_1 (1 + c_2\varepsilon_2) \|y\|$. Further,

$$\|y - T_1 x\| \leq \|y - S_1 x\| + \|S_1 x - T_1 x\| \leq \|y'\| + \varepsilon_1 \|x\| \leq$$
$$\leq c_2 (\|T_2 y\| + \varepsilon_2 \|y\|) + \varepsilon_1 c_1 c_2 \|T_2 y\| + \varepsilon_1 c_1 (1 + c_2\varepsilon_2) \|y\| =$$
$$= (1 + \varepsilon_1 c_1) c_2 \|T_2 y\| + k_T \|y\| .$$

Thus for arbitrary taken $y \in Y$ we find $x \in X$ such that

$$\|y - T_1 x\| \leq (1 + \varepsilon_1 c_1) c_2 \|T_2 y\| + k_T \|y\| ,$$
$$\|x\| \leq c_1 c_2 \|T_2 y\| + c_1 (1 + c_2\varepsilon_2) \|y\| .$$

If $y \in \ker(T_2)$ then $\|y - T_1 x_0\| \leq k_T \|y\|$ for some $x_0 \in X$, $\|x_0\| \leq c_1 (1 + c_2\varepsilon_2) \|y\|$, and $y_1 = y - T_1 x_0 \in \ker(T_2)$. On the same ground, $\|y_1 - T_1 x_1\| \leq k_T \|y_1\|$ for some $x_1 \in X$, $\|x_1\| \leq c_1 (1 + c_2\varepsilon_2) \|y_1\|$. Thus we define sequences x_i, y_i inductively by setting $y_0 = y$, $y_i = y_{i-1} - T_1 x_{i-1}$, $\|y_i\| \leq k_T \|y_{i-1}\|$, and $\|x_{i-1}\| \leq c_1 (1 + c_2\varepsilon_2) \|y_{i-1}\|$. It follows that the series $x' = \sum_i x_i$ converges in X and $T_1 x' = y$. Indeed, taking into account that $\|y_i\| \leq k_T^i \|y\|$ and $k_T < 1$, we infer $\sum_i \|x_i\| \leq c_1 (1 + c_2\varepsilon_2) \sum_i k_T^i \|y\| \leq (1 - k_T)^{-1} c_1 (1 + c_2\varepsilon_2) \|y\| < \infty$, that is

$$\|x'\| \leq (1 - k_T)^{-1} c_1 (1 + c_2\varepsilon_2) \|y\| . \tag{2.1}$$

Moreover, $\left\| y - T_1 \sum_{s=0}^{i} x_s \right\| = \|y_i - T_1 x_i\| = \|y_{i+1}\| \leq k_T^{i+1} \|y\|$, thereby $y = T_1 x'$. Consequently, we have proven that $\ker(T_2) = \mathrm{im}(T_1)$. In particular, $\mathrm{im}(T_1)$ is closed and $\mathrm{ic}(T_1) \leq (1 - k_T)^{-1} c_1 (1 + c_2\varepsilon_2)$ due to (2.1).

It remains to prove that T_2 has the closed range. Using (2.3), we obtain that

$$\|y^{\sim} \bmod \ker(T_2)\| = \|y^{\sim} \bmod \mathrm{im}(T_1)\| = \inf\{\|y - T_1 x\| : x \in X\} \leq$$
$$\leq (1 + \varepsilon_1 c_1) c_2 \|T_2 y\| + k_T \|y\| ,$$

whence $\|y^{\sim} \bmod \ker(T_2)\| = \|(y + z)^{\sim} \bmod \ker(T_2)\| \leq (1 + \varepsilon_1 c_1) c_2 \|T_2 y\| + k_T \|y + z\|$ for all $z \in \ker(T_2)$. It follows that

$$\|y^{\sim} \bmod \ker(T_2)\| \leq (1 - k_T)^{-1} (1 + \varepsilon_1 c_1) c_2 \|T_2 y\| ,$$

thereby $\mathrm{im}(T_2)$ is closed and $\mathrm{ic}(T_2) \leq (1 - k_T)^{-1} (1 + \varepsilon_1 c_1) c_2$.◄

2.4. The cone of an endomorphism

By the direct sum $X \oplus Y$ of Banach spaces X, Y, we mean the ℓ_1-norm sum, that is, the norm on the algebraic direct sum $X \oplus Y$ is given by the rule: $\|(x, y)\| = \|x\|_X + \|y\|_Y$, $(x, y) \in X \oplus Y$. Let $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$ and let $\beta = \{\beta_n \in \mathcal{B}(X_n)\}$ be a bounded endomorphism of this complex, thus $d_n \beta_n = \beta_{n-1} d_n$, $n \in \mathbb{Z}$. The cone $\mathrm{Con}((\mathfrak{X}, \mathfrak{d}), \beta)$ of the endomorphism β is the chain Banach space complex $\cdots \leftarrow Z_{n-1} \xleftarrow{\gamma_{n-1}} Z_n \leftarrow \cdots$, where $Z_n = X_{n+1} \oplus X_n$, $\gamma_{n-1}(x, y) = (d_n x + \beta_n y, -d_{n-1} y)$, $(x, y) \in Z_n$. If $(\mathfrak{X}, \mathfrak{d}) \in \overline{\mathbf{BS}}$ then $\mathrm{Con}((\mathfrak{X}, \mathfrak{d}), \beta)$ is the cochain complex $\cdots \to Z^n \xrightarrow{\gamma^n} Z^{n+1} \to \cdots$, where $Z^n = X^n \oplus X^{n-1}$, $\gamma^n(x, y) = \left(d^n x, -d^{n-1} y + \beta^n x\right)$, $(x, y) \in Z^n$.

Exercise 1. *Prove that* $\mathrm{Con}\left(\left(\mathfrak{X},\mathfrak{d}\right),\beta\right)^{*}=\mathrm{Con}\left(\left(\mathfrak{X}^{*},\mathfrak{d}^{*}\right),\beta^{*}\right)$ *to within an isomorphism in* $\underline{\mathbf{BS}}$ *(or* $\overline{\mathbf{BS}}$ *if* $\left(\mathfrak{X},\mathfrak{d}\right)\in\overline{\mathbf{BS}}$*).*

The following assertion demonstrates slight difference between chain and cochain versions of the cone.

Proposition 2.2. *Let* $\left(\mathfrak{X},\mathfrak{d}\right)\in\overline{\mathbf{BS}}$ *and* β *be a bounded endomorphism of* $\left(\mathfrak{X},\mathfrak{d}\right)$*. Then*

$$\underline{\mathrm{Con}\left(\left(\mathfrak{X},\mathfrak{d}\right),\beta\right)}=\mathrm{Con}\left(\left(\underline{\mathfrak{X}},\underline{\mathfrak{d}}\right),\beta\right)$$

to within an isomorphism in $\underline{\mathbf{BS}}$.

Proof. The relevant isomorphism of complexes can be implemented by the family of isomorphisms $\iota^{n}:X^{n}\oplus X^{n-1}\rightarrow X^{n-1}\oplus X^{n}$, $(x,y)\mapsto(-1)^{n}(-y,x)$, $n\in\mathbb{Z}$, in $\mathbf{BS}$. Indeed, let us verify that the following diagram

$$\begin{array}{ccc}
X^{n+1}\oplus X^{n} & \overset{\gamma^{n}}{\longleftarrow} & X^{n}\oplus X^{n-1}\\
\downarrow\iota^{n+1} & & \downarrow\iota^{n}\\
X^{n}\oplus X^{n+1} & \overset{\gamma_{n-1}}{\longleftarrow} & X^{n-1}\oplus X^{n},
\end{array}$$

is commutative, where γ^{n} (γ_{n-1}) is the differential of the complex $\underline{\mathrm{Con}\left(\left(\mathfrak{X},\mathfrak{d}\right),\beta\right)}$ $\left(\mathrm{Con}\left(\left(\underline{\mathfrak{X}},\underline{\mathfrak{d}}\right),\beta\right)\right)$. Take $(x,y)\in X^{n}\oplus X^{n-1}$. Then

$$\gamma_{n-1}\iota^{n}(x,y)=\gamma_{n-1}\Big((-1)^{n+1}y,(-1)^{n}x\Big)=\Big((-1)^{n+1}d^{n-1}y+(-1)^{n}\beta^{n}x,(-1)^{n+1}d^{n}x\Big)=$$
$$=\Big((-1)^{n}\left(-d^{n-1}y+\beta^{n}x\right),(-1)^{n-1}d^{n}x\Big)=\iota^{n+1}\left(d^{n}x,-d^{n-1}y+\beta^{n}x\right)=$$
$$=\iota^{n+1}\gamma^{n}(x,y),$$

that is, $\gamma_{n-1}\iota^{n}=\iota^{n+1}\gamma^{n}$ for all $n\in\mathbb{Z}$.◀

Lemma 2.5. *Let* $\left(\mathfrak{X},\mathfrak{d}\right)\in\underline{\mathbf{BS}}$ *(resp.,* $\left(\mathfrak{X},\mathfrak{d}\right)\in\overline{\mathbf{BS}}$*) and let* β *be a bounded endomorphism of* $\left(\mathfrak{X},\mathfrak{d}\right)$*. If* $H_{n}\left(\mathfrak{X},\mathfrak{d}\right)=\{0\}$ *and* $H_{n+1}\left(\mathfrak{X},\mathfrak{d}\right)=\{0\}$ *then* $H_{n}\left(\mathrm{Con}\left(\left(\mathfrak{X},\mathfrak{d}\right),\beta\right)\right)=\{0\}$ *(resp., if* $H^{n}\left(\mathfrak{X},\mathfrak{d}\right)=\{0\}$ *and* $H^{n-1}\left(\mathfrak{X},\mathfrak{d}\right)=\{0\}$ *then* $H^{n}\left(\mathrm{Con}\left(\left(\mathfrak{X},\mathfrak{d}\right),\beta\right)\right)=\{0\}$*).*

Proof. Let $\left(\mathfrak{X},\mathfrak{d}\right)\in\underline{\mathbf{BS}}$. Take $(x,y)\in\ker(\gamma_{n-1})\subseteq X_{n+1}\oplus X_{n}$. Then $d_{n-1}y=0$ and $d_{n}x=-\beta_{n}y$. Since $H_{n}\left(\mathfrak{X},\mathfrak{d}\right)=\{0\}$, it follows that $y=d_{n}z$ for some $z\in X_{n+1}$. Moreover, $-d_{n}x=\beta_{n}y=\beta_{n}d_{n}z=d_{n}\beta_{n+1}z$, which in turn implies that $x+\beta_{n+1}z=d_{n+1}w$, $w\in X_{n+2}$, owing to $H_{n+1}\left(\mathfrak{X},\mathfrak{d}\right)=\{0\}$. One can easily check that $(x,y)=\gamma_{n}(w,-z)$. Finally, the assertion for the cochain case can be reduced to the chain one on the ground of Proposition 2.2.◀

Lemma 2.6. *Let* $\left(\mathfrak{X},\mathfrak{d}\right)\in\underline{\mathbf{BS}}$*,* β *a bounded endomorphism of* $\left(\mathfrak{X},\mathfrak{d}\right)$*, and let* $\mathfrak{U}$ *be an ultrafilter. Then* $\mathrm{Con}\left(\left(\mathfrak{X},\mathfrak{d}\right),\beta\right)_{\mathfrak{U}}=\mathrm{Con}\left(\left(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}\right),\beta_{\mathfrak{U}}\right)$ *to within an isomorphism in* $\underline{\mathbf{BS}}$.

Proof. One can easily verify that the linear operator

$$f_{n}:\left(X_{n+1}\oplus X_{n}\right)_{\mathfrak{U}}\rightarrow X_{n+1\mathfrak{U}}\oplus X_{n\mathfrak{U}},\quad f_{n}\left[(x_{s},y_{s})\right]=\left([x_{s}],[y_{s}]\right),$$

is an isometric isomorphism. Note that

$$f_{n-1}\gamma_{n-1\mathfrak{U}}\left[(x_{s},y_{s})\right]=f_{n-1}\left[(d_{n}x_{s}+\beta_{n}y_{s},-d_{n-1}y_{s})\right]=\left([d_{n}x_{s}+\beta_{n}y_{s}],[-d_{n-1}y_{s}]\right)=$$
$$=\left(d_{n\mathfrak{U}}[x_{s}]+\beta_{n\mathfrak{U}}[y_{s}],-d_{n-1\mathfrak{U}}[y_{s}]\right)=\gamma_{\mathfrak{U}n-1}\left([x_{s}],[y_{s}]\right)=\gamma_{\mathfrak{U}n-1}f_{n}\left[(x_{s},y_{s})\right],$$

where $\gamma_{\mathfrak{U}n-1}$ is the differential of $\mathrm{Con}\left(\left(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}\right),\beta_{\mathfrak{U}}\right)$. Thus $f_{n-1}\gamma_{n-1\mathfrak{U}}=\gamma_{\mathfrak{U}n-1}f_{n}$, that is, the family $\{f_{n}\}$ implements relevant isomorphism of complexes.◀

2.5. The projective and flat Banach spaces

The projective (resp., flat) Banach spaces are important from infinite-dimensional spectral mapping properties view. Let us remind relevant definitions and simple properties of these spaces.

Let $X \in \mathbf{BS}$, $X^{\widehat{\otimes} n} = X \widehat{\otimes} \cdots \widehat{\otimes} X$ (n-times) the projective tensor product, S_n the group of all permutations over the finite set $\{1, \dots, n\}$, $\varepsilon(\tau)$ the sign of a permutation $\tau \in S_n$, and let

$$\delta_\tau \in \mathcal{B}\left(X^{\widehat{\otimes} n}\right), \quad \delta_\tau(x_1 \otimes \cdots \otimes x_n) = x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}, \quad \tau \in S_n, \quad n \in \mathbb{N}.$$

We define *the exterior power* $\wedge^n X$ *of* X as the image of the projection

$$A_n \in \mathcal{B}\left(X^{\widehat{\otimes} n}\right), \quad A_n = (n!)^{-1} \sum_{\tau \in S_n} \varepsilon(\tau)\, \delta_\tau.$$

We set $x_1 \wedge \cdots \wedge x_n = A_n(x_1 \otimes \cdots \otimes x_n)$ and $X^{\widehat{\otimes} 0} = \wedge^0 X = \mathbb{C}$. Take $\underline{x} = x_1 \wedge \cdots \wedge x_n \in \wedge^n X$. The following denotation is very convenient: $\underline{x}_i = x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n \in \wedge^{n-1} X$, where $\widehat{x_i}$ means the omission of x_i. If we omit two elements x_i and x_j from the expression of $\underline{x}$ then we write $\underline{x}_{i,j}$. Note that $\mathcal{L}(\wedge^n X, Y) = C^n(X, Y)$ to within an isometric isomorphism, where $Y \in \mathbf{BS}$, $C^n(X, Y)$ is the Banach space of all continuous skewsymmetric n-linear forms on X with values in Y.

Let $Y \in \mathbf{BS}$ and let $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$. The functor $\mathcal{L}(Y, \circ) : \mathbf{BS} \to \mathbf{BS}$ transforms the complex $(\mathfrak{X}, \mathfrak{d})$ into a new complex $\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d}))$:

$$\cdots \leftarrow \mathcal{L}(Y, X_{n-1}) \xleftarrow{\beta_{n-1}} \mathcal{L}(Y, X_n) \xleftarrow{\beta_n} \mathcal{L}(Y, X_{n+1}) \leftarrow \cdots,$$

where $\beta_n(T) = d_n \cdot T$, $T \in \mathcal{L}(Y, X_n)$. Respectively, it is defined a cochain Banach space complex $\mathcal{L}((\mathfrak{X}, \mathfrak{d}), Y)$. A Banach space Y is said to be *projective* (resp., *injective*) if the complex $\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d}))$ (resp., $\mathcal{L}((\mathfrak{X}, \mathfrak{d}), Y)$) is exact for each exact Banach space complex $(\mathfrak{X}, \mathfrak{d})$. A Banach space Y is said to be *flat* if its dual space Y^* is injective. The class of all projective (resp., injective, flat) Banach spaces is denoted by Proj (resp., Inj, Flat). It is easy to prove that $Y \in \mathrm{Proj}$ if and only if for an epimorphism of Banach spaces $T : X \to Z$ and an operator $\varphi \in \mathcal{L}(Y, Z)$ there exist $\psi \in \mathcal{L}(Y, X)$ such that $T \cdot \psi = \varphi$. By analogy, $Y \in \mathrm{Inj}$ iff for a topologically injective operator of Banach spaces $T : X \to Z$ and an operator $\varphi \in \mathcal{L}(X, Y)$ there exist $\psi \in B(Z, Y)$ such that $\psi \cdot T = \varphi$. In particular, if $Y_1, Y_2 \in \mathrm{Proj}$ (resp., $Y_1, Y_2 \in \mathrm{Inj}$), then $Y_1 \oplus Y_2 \in \mathrm{Proj}$ (resp., $Y_1 \oplus Y_2 \in \mathrm{Inj}$), moreover, $Y_1 \widehat{\otimes} Y_2 \in \mathrm{Proj}$.

For instance, the Banach space $\ell_1(S)$ (in particular, $\ell_1 = \ell_1(\mathbb{N})$) of all absolutely summable complex functions on a set S is projective [67, Proposition 4.3]. Obviously, each finite-dimensional normed space is projective, injective and flat simultaneously.

Lemma 2.7. *Let* $Y \in \mathbf{BS}$*. Then* $Y \in \mathrm{Flat}$ *iff* $(\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y$ *is exact whenever* $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$ *is an exact complex.*

Proof. Note that $(\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y$ is exact iff its dual $\left((\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y\right)^*$ is exact due to Theorem 2.1. But, $\left((\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y\right)^* = \mathcal{L}((\mathfrak{X}, \mathfrak{d}), Y^*)$ to within an isomorphism in $\overline{\mathbf{BS}}$. Thereby, $Y^* \in \mathrm{Inj}$ iff $(\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y$ is exact whenever $(\mathfrak{X}, \mathfrak{d})$ is exact.◀

In particular, the space of the form $L^1(\mu)$ is flat by virtue of Lemma 2.7, [67, Proposition 4.2].

Lemma 2.8. *Let* $Y \in \mathbf{BS}$*. Then* $\wedge^n Y \in \mathrm{Proj}$ *(resp.,* $\wedge^n Y \in \mathrm{Flat}$*) whenever* $Y \in \mathrm{Proj}$ *(resp.,* $Y \in \mathrm{Flat}$*),* $n \in \mathbb{N}$.

Proof. We have already noted that $Y^{\widehat{\otimes}n} \in \mathrm{Proj}$ if $Y \in \mathrm{Proj}$. Now take a Banach epimorphism $T : X \to Z$ and $\varphi \in \mathcal{L}(\wedge^n Y, Z)$. Then $\varphi \cdot A_n \in \mathcal{L}\left(Y^{\widehat{\otimes}n}, Z\right)$, where A_n is the projection onto $\wedge^n Y$. Thereby, there exists $\psi \in \mathcal{L}\left(Y^{\widehat{\otimes}n}, X\right)$ such that $T \cdot \psi = \varphi \cdot A_n$. Then $T \cdot \psi \cdot A_n = \varphi \cdot A_n^2 = \varphi \cdot A_n$. The latter means that $T \cdot \psi|_{\wedge^n Y} = \varphi$. Thus $\wedge^n Y \in \mathrm{Proj}$.

Now let us assume that $Y \in \mathrm{Flat}$. To prove that $\wedge^n Y \in \mathrm{Flat}$, we use Lemma 2.7. Take an exact complex $(\mathfrak{X}, \mathfrak{d}) \in \underline{\mathbf{BS}}$. By using induction on n, and using Lemma 2.7, we infer that the complex $(\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y^{\widehat{\otimes}n}$ remains exact. On the other hand, $\wedge^n Y$ is a complemented subspace in $Y^{\widehat{\otimes}n}$. Therefore $(\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} \wedge^n Y$ is also exact.◀

2.6. Banach space bicomplexes

One of the main role in our consideration will play Banach space bicomplexes, their total complexes, "diagonal chase" method and so on. Here we remind necessary definitions and results.

By a Banach space bicomplex we mean the triple $(\mathfrak{X}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ with $\mathfrak{X} = \{X^{n,m} : n, m \in \mathbb{Z}\}$, $X^{n,m} \in \mathbf{BS}$, $\mathfrak{d}_{\prime} = \left\{d_{\prime}^{n,m} \in \mathcal{L}\left(X^{n,m}, X^{n+1,m}\right)\right\}$, $\mathfrak{d}_{\prime\prime} = \left\{d_{\prime\prime}^{n,m} \in \mathcal{L}\left(X^{n,m}, X^{n,m+1}\right)\right\}$, such that the following diagram

$$
\begin{array}{ccccccc}
 & & \uparrow & & & & \\
\cdots & \longrightarrow & X^{n+1,m} & \longrightarrow & \cdots & & \\
 & & \uparrow d_{\prime}^{n,m} & & \uparrow & & \\
\cdots & \longrightarrow & X^{n,m} & \overset{d_{\prime\prime}^{n,m}}{\longrightarrow} & X^{n,m+1} & \longrightarrow & \cdots , \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

is commutative, and all columns $\left(\mathfrak{X}^{\bullet,m}, \mathfrak{d}_{\prime}^{\bullet,m}\right)$ and all rows $\left(\mathfrak{X}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}\right)$ are Banach space complexes, where $\mathfrak{X}^{\bullet,m} = \left\{X^{k,m} : k \in \mathbb{Z}\right\}$, $\mathfrak{X}^{n,\bullet} = \left\{X^{n,k} : k \in \mathbb{Z}\right\}$ and $\mathfrak{d}_{\prime}^{\bullet,m} = \left\{d_{\prime}^{k,m} : k \in \mathbb{Z}\right\}$, $\mathfrak{d}_{\prime\prime}^{n,\bullet} = \left\{d_{\prime\prime}^{n,k} : k \in \mathbb{Z}\right\}$. If we reverse the horizontal and (or) vertical arrows one occurs other versions of bicomplexes as well as chain and cochain complexes were for usual complexes. For us main interest will present suggested "double-cochain" and also "double-chain" versions of a bicomplex. The latter is dictated by our future applications to the spectral theory. We also say chain (resp., cochain) bicomplex instead of "double-chain" (resp., "double-cochain").

The operators $d_{\prime}^{n,m}$ and $d_{\prime\prime}^{n,m}$ are called the horizontal and vertical differentials, respectively. Note that $d_{\prime}^{n,m}\left(\ker\left(d_{\prime\prime}^{n,m}\right)\right) \subseteq \ker\left(d_{\prime\prime}^{n+1,m}\right)$ and $d_{\prime}^{n,m}\left(\mathrm{im}\left(d_{\prime\prime}^{n,m-1}\right)\right) \subseteq \mathrm{im}\left(d_{\prime\prime}^{n+1,m-1}\right)$, therefore the operator

$$D_{\prime}^{n,m} : H^m\left(\mathfrak{X}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}\right) \to H^m\left(\mathfrak{X}^{n+1,\bullet}, \mathfrak{d}_{\prime\prime}^{n+1,\bullet}\right), \quad D_{\prime}^{n,m}\left(x^{\sim}\right) = d_{\prime}^{n,m}(x)^{\sim},$$

is defined soundly. Moreover, the sequence

$$\cdots \longrightarrow H^m\left(\mathfrak{X}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}\right) \overset{D_{\prime}^{n,m}}{\longrightarrow} H^m\left(\mathfrak{X}^{n+1,\bullet}, \mathfrak{d}_{\prime\prime}^{n+1,\bullet}\right) \longrightarrow \cdots,$$

is a complex called m-th vertical cohomology complex of the bicomplex. By analogy, one defines nth horizontal cohomology complex

$$\cdots \longrightarrow H^n\left(\mathfrak{X}^{\bullet,m}, \mathfrak{d}_{\prime}^{\bullet,m}\right) \overset{D_{\prime\prime}^{n,m}}{\longrightarrow} H^n\left(\mathfrak{X}^{\bullet,m+1}, \mathfrak{d}_{\prime}^{\bullet,m+1}\right) \longrightarrow \cdots,$$

of the bicomplex.

We say that a bicomplex $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ is bounded (below) if one can find $N \in \mathbb{Z}$ such that $X^{n,m} = \{0\}$ whenever $n < N$ or $m < N$. The space $X^{N,N}$ is called the base space of the bicomplex. If $N = 0$ then we say that $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ is a nonnegative bicomplex with the base space $X^{0,0}$.

Now let $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ be a Banach space bicomplex with bounded diagonals and let

$$X^n = \bigoplus_{k+s=n} X^{k,s} \in \mathbf{BS},$$

be a sum of (bounded) diagonals of the bicomplex. One defines a Banach space complex

$$\cdots \longrightarrow X^n \xrightarrow{\delta^n} X^{n+1} \longrightarrow \cdots,$$

where $\delta^n(x) = d_{\prime\prime}^{k,s}(x) + (-1)^s d_{\prime}^{k,s}(x)$ whenever $x \in X^{k,s}$, $k+s=n$, $n \in \mathbb{Z}$. The latter is called the total complex of $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ and it is denoted by $\mathrm{Tot}(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$. Note that the cone of an endomorphism of a Banach space complex is a particular case of the total complex of a certain bicomplex. Namely, take $(\mathfrak{X}, \mathfrak{d}) \in \overline{\mathbf{BS}}$ and let β be an endomorphism of $(\mathfrak{X}, \mathfrak{d})$. The following diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X^n & \xrightarrow{-d^n} & X^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow \alpha^n & & \uparrow \alpha^{n+1} & & \\
\cdots & \longrightarrow & X^n & \xrightarrow{d^n} & X^{n+1} & \longrightarrow & \cdots,
\end{array}$$

is commutative, where $\alpha^n = (-1)^n \beta^n$, $n \in \mathbb{Z}$. Thereby, the latter defines a bicomplex $(\mathfrak{X}, \alpha, \mathfrak{d})$ and $\mathrm{Tot}(\mathfrak{X}, \alpha, \mathfrak{d}) = \mathrm{Con}((\mathfrak{X}, \mathfrak{d}), \beta)$.

Finally, if $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ is a Banach space bicomplex and $\mathfrak{U}$ is an ultrafilter then $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}'_{\mathfrak{U}}, \mathfrak{d}''_{\mathfrak{U}})$ is a bicomplex called an ultrapower of $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$, and $\mathrm{Tot}((\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}'_{\mathfrak{U}}, \mathfrak{d}''_{\mathfrak{U}})) = \mathrm{Tot}(\mathfrak{X}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})_{\mathfrak{U}}$ by the same argument carried out in the proof of Lemma 2.6, where $\mathfrak{X}_{\mathfrak{U}} = \{X_{\mathfrak{U}}^{n,m}\}$, $\mathfrak{d}'_{\mathfrak{U}} = \{d_{\prime\mathfrak{U}}^{n,m}\}$, $\mathfrak{d}''_{\mathfrak{U}} = \{d_{\prime\prime\mathfrak{U}}^{n,m}\}$.

2.7. The diagonal chase

The diagonal chase phenomena is appeared in the following lemmas and belong to so-called mathematical folklore.

Lemma 2.9. *Let $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ be a nonnegative cochain bicomplex such that all its rows are exact at first $i-1$ terms, $H^i\left(\mathfrak{X}^{0,\bullet}, \mathfrak{d}_{\prime\prime}^{0,\bullet}\right) \neq \{0\}$ and the differential $D_{\prime}^{0,i} : H^i\left(\mathfrak{X}^{0,\bullet}, \mathfrak{d}_{\prime\prime}^{0,\bullet}\right) \to H^i\left(\mathfrak{X}^{1,\bullet}, \mathfrak{d}_{\prime\prime}^{1,\bullet}\right)$ of the ith vertical cohomology complex is trivial, where $i \geq 1$. Then*

$$H^k\left(\mathfrak{X}^{\bullet,m}, \mathfrak{d}_{\prime}^{\bullet,m}\right) \neq \{0\},$$

for some $m, k \leq i$.

Proof. Let us assume that first i columns are exact at first i terms. By assumption $H^i\left(\mathfrak{X}^{0,\bullet}, \mathfrak{d}_{\prime\prime}^{0,\bullet}\right) \neq \{0\}$, therefore there is an element $x \in \ker\left(d_{\prime\prime}^{0,i}\right) \setminus \mathrm{im}\left(d_{\prime\prime}^{0,i-1}\right)$. Since $D_{\prime}^{0,i} = 0$, $d_{\prime}^{0,i}(x) \in \mathrm{im}\left(d_{\prime\prime}^{1,i-1}\right)$, that is, $d_{\prime}^{0,i}(x) = d_{\prime\prime}^{1,i-1}(x_{1,i-1})$ for some $x_{1,i-1} \in X^{1,i-1}$. Note that

$${}_{\prime\prime}^{2,i-1} d_{\prime}^{1,i-1}(x_{1,i-1}) = d_{\prime}^{1,i} d_{\prime\prime}^{1,i-1}(x_{1,i-1}) = d_{\prime}^{1,i} d_{\prime}^{0,i}(x) = 0,$$

$d$that is, $d_{\prime}^{1,i-1}(x_{1,i-1}) \in \ker\left(d_{\prime\prime}^{2,i-1}\right)$. But, the row $\left(\mathfrak{X}^{2,\bullet}, \mathfrak{d}_{\prime\prime}^{2,\bullet}\right)$ is exact at first $i-1$ terms by assumption, therefore $\ker\left(d_{\prime\prime}^{2,i-1}\right) = \operatorname{im}\left(d_{\prime\prime}^{2,i-2}\right)$ and $d_{\prime}^{1,i-1}(x_{1,i-1}) = d_{\prime\prime}^{2,i-2}(x_{2,i-2})$ for some $x_{2,i-2} \in X^{2,i-2}$. By induction, we could find some diagonal elements $x_{s,i-s} \in X^{s,i-s}$ such that

$$d_{\prime}^{s,i-s}(x_{s,i-s}) = d_{\prime\prime}^{s+1,i-s-1}(x_{s+1,i-s-1}),$$

and $x_{0,i} = x$. Roughly speaking, we rise by the ith diagonal of the bicomplex obtaining these elements. To finish the proof we just need to go down by the $i-1$th diagonal. Namely, note that

$$d_{\prime\prime}^{i+1,0} d_{\prime}^{i,0}(x_{i,0}) = d_{\prime}^{i,1} d_{\prime\prime}^{i,0}(x_{i,0}) = d_{\prime}^{i,1} d_{\prime}^{i-1,1}(x_{i-1,1}) = 0,$$

which implies that $d_{\prime}^{i,0}(x_{i,0}) \in \ker\left(d_{\prime\prime}^{i+1,0}\right) = \{0\}$ and $x_{i,0} \in \ker\left(d_{\prime}^{i,0}\right)$. But, first i columns are exact at first i terms as we have assumed. Therefore $x_{i,0} = d_{\prime}^{i-1,0}(y_{i-1,0})$ for some $y_{i-1,0} \in X^{i-1,0}$. Further,

$$d_{\prime}^{i-1,1}\left(x_{i-1,1} - d_{\prime\prime}^{i-1,0}(y_{i-1,0})\right) = d_{\prime\prime}^{i,0}(x_{i,0}) - d_{\prime\prime}^{i,0}(x_{i,0}) = 0,$$

which implies that $x_{i-1,1} - d_{\prime\prime}^{i-1,0}(y_{i-1,0}) = d_{\prime}^{i-2,1}(y_{i-2,1})$ for some $y_{i-2,1} \in X^{i-2,1}$. By induction, we find elements $y_{i-k,k-1} \in X^{i-k,k-1}$ such that $x_{i-k,k} - d_{\prime\prime}^{i-k,k-1}(y_{i-k,k-1}) = d_{\prime}^{i-k-1,k}(y_{i-k-1,k})$. Then $y = y_{0,i-1} \in X^{0,i-1}$, and

$$\begin{aligned} d_{\prime}^{0,i}\left(x - d_{\prime\prime}^{0,i-1}(y)\right) &= d_{\prime\prime}^{1,i-1}\left(x_{1,i-1} - d_{\prime}^{0,i-1}(y_{0,i-1})\right) = \\ &= d_{\prime\prime}^{1,i-1}\left(x_{1,i-1} - x_{1,i-1} + d_{\prime\prime}^{1,i-2}(y_{1,i-2})\right) = 0. \end{aligned}$$

Since $\ker\left(d_{\prime}^{0,i}\right) = \{0\}$, it follows that $x = d_{\prime\prime}^{0,i-1}(y) \in \operatorname{im}\left(d_{\prime\prime}^{0,i-1}\right)$, a contradiction.◀

Exercise 2. *Prove the chain version of the assertion stated in Lemma 2.9. Namely, let* $\left(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}''\right)$ *be a nonnegative chain bicomplex such that all its rows are exact at first* $i-1$ *terms,* $H_i\left(\mathfrak{X}_{0,\bullet}, \mathfrak{d}''_{0,\bullet}\right) \neq \{0\}$ *and the differential* $D'_{0,i} : H_i\left(\mathfrak{X}_{1,\bullet}, \mathfrak{d}''_{1,\bullet}\right) \to H_i\left(\mathfrak{X}_{0,\bullet}, \mathfrak{d}''_{0,\bullet}\right)$ *of the* i*th vertical homology complex is trivial, where* $i \geq 1$*. Then* $H_k\left(\mathfrak{X}_{\bullet,m}, \mathfrak{d}'_{\bullet,m}\right) \neq \{0\}$ *for some* $m, k \leq i$*.*

Lemma 2.10. *Let* $(\mathfrak{X}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ *be a nonnegative cochain bicomplex such that all its rows (or columns) are exact at first* n *terms. Then* $\operatorname{Tot}(\mathfrak{X}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ *is exact at first* n *terms.*

Proof. Observing that the assertion is trivial for $n = 0$, we proceed by induction on n. Take $x = (x_{k,s}) \in X^n = \bigoplus_{k+s=n} X^{k,s}$ such that $\delta^n x = 0$. Then $d_{\prime\prime}^{0,n}(x_{0,n}) = 0$ and $d_{\prime\prime}^{1,n-1}(x_{1,n-1}) + (-1)^n d_{\prime}^{0,n}(x_{0,n}) = 0$. Since the first row is exact at $X^{0,n}$, $x_{0,n} = d_{\prime\prime}^{0,n-1}(y_{0,n-1})$ for a certain $y_{0,n-1} \in X^{0,n-1}$. Moreover,

$$d_{\prime\prime}^{1,n-1}\left(x_{1,n-1} - (-1)^{n-1} d_{\prime}^{0,n-1}(y_{0,n})\right) = (-1)^{n-1} d_{\prime}^{0,n}(x_{0,n}) - (-1)^{n-1} d_{\prime}^{0,n} d_{\prime\prime}^{0,n-1}(y_{0,n-1}) = 0.$$

Using the exactness of the second row, we infer $x_{1,n-1} - (-1)^{n-1} d_{\prime}^{0,n-1}(y_{0,n-1}) = d_{\prime\prime}^{1,n-2}(y_{1,n-2})$ for some $y_{1,n-2} \in X^{1,n-2}$. By induction, we find some $y_{s,t} \in X^{s,t}$, $s+t = n-1$, such that

$$x_{k,n-k} = (-1)^{n-k} d_{\prime}^{k-1,n-k}(y_{k-1,n-k}) + d_{\prime\prime}^{k,n-k-1}(y_{k,n-k-1}),$$

for all k, $1 \leq k \leq n$. But, the latter equalities mean that $x = \delta^{n-1}(y)$, where $y = (y_{s,t}) \in X^{n-1}$. Thus the total complex is exact at X^n, and by induction hypothesis, $\operatorname{Tot}(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ is exact at first $n-1$ terms.◀

Exercise 3. *Formulate and prove the chain version of the assertion from Lemma 2.10.*

Using the same idea on diagonal chase carried out in lemmas 2.9, 2.10, prove the following more general fact on bicomplexes.

Exercise 4. *Let $(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ be a nonnegative (co)chain bicomplex such that first n vertical (or horizontal) (co)homology complexes are exact. Then* $\operatorname{Tot}(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}'')$ *is exact at first n terms.*

2.8. The inverse closed subalgebras

Let A be a unital associative algebra. The spectrum (in A) of an element $a \in A$ is denoted by $\mathrm{sp}_A(a)$. For $A = \mathcal{L}(X)$ and an operator $T \in \mathcal{L}(X)$, we write $\mathrm{sp}(T)$ instead of $\mathrm{sp}_A(T)$. A subalgebra $B \subseteq A$ is said to be *an inverse closed subalgebra* [5, 1.1.4], if any invertible in A element of B is invertible in B. Thus $\mathrm{sp}_A(b) = \mathrm{sp}_B(b)$ for all $b \in B$. One can easily verify that the inverse closed subalgebras are stable with respect to any intersections, so it makes sense to define the inverse closed subalgebra $\mathcal{R}(M)$ in A generated by a subset $M \subseteq A$. The elements of the subalgebra $\mathcal{R}(M)$ can be interpreted as a set of values of all formal "rational functions" in variables M in the algebra A (see [69], [25, Section 2]). Namely, let S be a set with a mapping $\pi : S \to A$ into the algebra A and let $M = \operatorname{im}(\pi)$. One can define the "rational functions" with the set S of variables and their actions in A as a collection of formal expressions from $\mathcal{R}_{S,\pi} = \bigcup_{n\in\mathbb{Z}_+} \mathcal{R}^n_{S,\pi}$ with the canonical mapping $\widehat{\pi} : \mathcal{R}_{S,\pi} \to A$, $\widehat{\pi}(f(S)) = f(M)$, extending π, which is inductively defined by the following way. Let $\mathcal{R}^0_{S,\pi}$ be the free algebra (of all polynomials) generated by the set S, and let $\mathcal{R}^0(M) = \left\{f(M) : f(S) \in \mathcal{R}^0_{S,\pi}\right\}$ be their values in A by means of π. If the collection $\mathcal{R}^{n-1}_{S,\pi}$ and their images $\mathcal{R}^{n-1}(M)$ in A have been defined, then $\mathcal{R}^n_{S,\pi}$ is defined as the free algebra (of all polynomials) generated by $\mathcal{R}^{n-1}_{S,\pi}$ and all formal expressions $f^{-1}(S)$, $f(S) \in \mathcal{R}^{n-1}_{S,\pi}$, for which $f(M)$ is invertible in A. We set $\widehat{\pi}\left(f^{-1}(S)\right) = f^{-1}(M) = f(M)^{-1}$. Thus $\mathcal{R}(M) = \bigcup_{n\in\mathbb{Z}_+} \mathcal{R}^n(M)$. If $f(S) \in \mathcal{R}^n_{S,\pi}$ then we say that $f(S)$ has an order n. Note also that if $\varepsilon : A \to B$ is a unital algebra homomorphism and $N = \varepsilon(M)$ then $\mathcal{R}_{S,\pi} \subseteq \mathcal{R}_{S,\varepsilon\cdot\pi}$ and $\varepsilon(f(M)) = f(N)$, $f(S) \in \mathcal{R}_{S,\pi}$.

Lemma 2.11. *Let S be a set and let $\pi : S \to A$, $\varsigma : S \to B$ be mappings into algebras A and B, respectively. If $\mathcal{R}_{S,\pi} \subseteq \mathcal{R}_{S,\varsigma}$ and $\mathrm{sp}_A(f(\pi(S))) = \mathrm{sp}_B(f(\varsigma(S)))$ for all $f(S) \in \mathcal{R}_{S,\pi}$, then $\mathcal{R}_{S,\pi} = \mathcal{R}_{S,\varsigma}$.*

Proof. We proceed by induction on the order of rational functions taken from $\mathcal{R}_{S,\varsigma}$. It is beyond a doubt $\mathcal{R}^0_{S,\varsigma} \subseteq \mathcal{R}_{S,\pi}$. Take $f(S) \in \mathcal{R}^n_{S,\varsigma}$. By its very definition, $f(S) = p(\Phi)$ is a (free) polynomial taken by a finite set $\Phi = \left\{g_\iota(S), g_\kappa^{-1}(S) : g_\iota(S), g_\kappa(S) \in \mathcal{R}^{n-1}_{S,\varsigma}\right\}$. By induction hypothesis, $\bigcup_{k=0}^{n-1} \mathcal{R}^k_{S,\varsigma} \subseteq \mathcal{R}_{S,\pi}$. Therefore, one suffices to set that $f(S) = g^{-1}(S)$ for some $g(S) \in \mathcal{R}^{n-1}_{S,\varsigma}$. Then $g(S) \in \mathcal{R}_{S,\pi}$ and $g(\varsigma(S))$ is invertible in B. With $\mathrm{sp}_A(g(\pi(S))) = \mathrm{sp}_B(g(\varsigma(S)))$ in mind, infer that $g(\pi(S))$ is invertible in A, too. The latter in turn implies that $g^{-1}(S) \in \mathcal{R}_{S,\pi}$, that is, $f(S) \in \mathcal{R}_{S,\pi}$.◀

Now let S and W be sets with a surjective mapping $\tau : S \to W$ and let $\varsigma : W \to A$ be a mapping into an algebra. We have the mappings $\widehat{\varsigma} : \mathcal{R}_{W,\varsigma} \to A$ and $\widehat{\pi} : \mathcal{R}_{S,\pi} \to A$ extending ς and π, respectively, where $\pi = \varsigma \cdot \tau$.

Lemma 2.12. *There exists a unique mapping $\widetilde{\tau}:\mathcal{R}_{S,\pi}\to\mathcal{R}_{W,\varsigma}$ extending τ such that $\widehat{\varsigma}\cdot\widetilde{\tau}=\widehat{\pi}$.*

Proof. We proceed by induction on the order of rational functions. Note that τ is uniquely extended up to an algebra homomorphism $\tau^{0}:\mathcal{R}_{S,\pi}^{0}\to\mathcal{R}_{W,\varsigma}^{0}$, $\tau^{0}(f(S))=f(W)$. Evidently, $\widehat{\varsigma}\cdot\tau^{0}=\widehat{\pi}$.

By induction hypothesis, it uniquely defines a mapping $\tau^{n-1}:\mathcal{R}_{S,\pi}^{n-1}\to\mathcal{R}_{W,\varsigma}^{n-1}$ such that $\widehat{\varsigma}\cdot\tau^{n-1}=\widehat{\pi}$. Take $f^{-1}(S)\in\mathcal{R}_{S,\pi}$ such that $f(S)\in\mathcal{R}_{S,\pi}^{n-1}$. By its very definition, $f(\pi(S))$ is invertible in A. But, $f(\pi(S))=\widehat{\pi}(f(S))=\widehat{\varsigma}\tau^{n-1}(f(S))$, whence $g^{-1}(W)\in\mathcal{R}_{W,\varsigma}^{n}$, where $g(W)=\tau^{n-1}(f(S))$. We set $\tau^{n}\left(f^{-1}(S)\right)=g^{-1}(W)$. Then

$$\widehat{\varsigma}\left(\tau^{n}\left(f^{-1}(S)\right)\right)=\widehat{\varsigma}\left(g^{-1}(W)\right)=\widehat{\varsigma}(g(W))^{-1}=\widehat{\pi}(f(S))^{-1}=\widehat{\pi}\left(f^{-1}(S)\right).$$

We have the mapping $\tau^{n}:\mathcal{R}_{S,\pi}^{n-1}\cup\overline{\mathcal{R}}_{S,\pi}^{n-1}\to\mathcal{R}_{W,\varsigma}^{n}$, where $\overline{\mathcal{R}}_{S,\pi}^{n-1}=\left\{f^{-1}(S):f(S)\in\mathcal{R}_{S,\pi}^{n-1}\right\}$. The latter is uniquely extended up to an algebra homomorphism $\tau^{n}:\mathcal{R}_{S,\pi}^{n}\to\mathcal{R}_{W,\varsigma}^{n}$. Obviously, $\widehat{\varsigma}\cdot\tau^{n}=\widehat{\pi}$.◄

Now let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The lower central series of $\mathfrak{g}$ is defined as a sequences of the Lie ideals $\mathfrak{g}^{(i)}$, $i\in\mathbb{N}$, where $\mathfrak{g}^{(1)}=\mathfrak{g}$, $\mathfrak{g}^{(i)}=\left[\mathfrak{g},\mathfrak{g}^{(i-1)}\right]$ for $i>1$. A Lie algebra $\mathfrak{g}$ is said to be a nilpotent if $\mathfrak{g}^{(k)}=\{0\}$ for a certain k. If $\mathfrak{g}^{(k+1)}=\{0\}$ and $\mathfrak{g}^{(k)}\neq\{0\}$ then k is called the *nilpotent step* of $\mathfrak{g}$. In particular, a Lie algebra with the nilpotent step 1 is a commutative Lie algebra and a non-commutative Lie algebra $\mathfrak{g}$ has the nilpotent step 2, iff $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=\{0\}$. A nilpotent Lie algebra $\mathfrak{g}$ is said to be a *Heisenberg algebra* if $\dim([\mathfrak{g},\mathfrak{g}])=1$, in particular, $\mathfrak{g}$ has the nilpotent step 2. A simple example of a Heisenberg algebra is the Lie algebra with the basis e_1,e_2,e_3 and relations $[e_1,e_2]=e_3$ and $[e_i,e_3]=0$ for all i. Finally, a Lie algebra $\mathfrak{g}$ is said to be *a solvable* if $\mathcal{D}^{k}(\mathfrak{g})=\{0\}$ for a certain k, where $\mathcal{D}^{1}(\mathfrak{g})=\mathfrak{g}$ and $\mathcal{D}^{i}(\mathfrak{g})=\left[\mathcal{D}^{i-1}(\mathfrak{g}),\mathcal{D}^{i-1}(\mathfrak{g})\right]$ for $i>1$. Note that a nilpotent Lie algebra is automatically solvable one, for $\mathcal{D}^{i}(\mathfrak{g})\subseteq\mathfrak{g}^{(i)}$, $i\in\mathbb{N}$. A simple example of a solvable Lie algebra which is not nilpotent is 2-dimensional Lie algebra $\mathfrak{g}=\mathbb{C}e_1\oplus\mathbb{C}e_2$ such that $[e_1,e_2]=e_2$.

Let $\mathfrak{g}$ be a finite-dimensional solvable (resp., nilpotent) Lie algebra. There exists a basis $e=(e_1,\ldots,e_n)$ in $\mathfrak{g}$ such that the adjoint representation of $\mathfrak{g}$ is reduced to the (resp., strictly) upper triangular form with respect to e due to Engel and Lie theorems [6, 1.4.2, 1.5.3] In particular, $c_{ij}^{k}=0$ if $k<\max\{i,j\}$ (resp., $k\leq\max\{i,j\}$), where c_{ij}^{k} are structure constants of $\mathfrak{g}$ calculated by e. We call e a (resp., *strongly*) *triangular basis* in $\mathfrak{g}$. The space of all Lie characters of a Lie algebra $\mathfrak{g}$ is denoted by $\Delta(\mathfrak{g})$, that is,

$$\Delta(\mathfrak{g})=\left\{\lambda\in\mathfrak{g}^{*}:\lambda\left(\mathfrak{g}^{(2)}\right)=\{0\}\right\}.$$

The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is denoted by $\mathcal{U}(\mathfrak{g})$, which is a locally convex algebra equipped with the finest locally convex topology. The character space $\operatorname{Spec}\mathcal{U}(\mathfrak{g})$ is identified with $\Delta(\mathfrak{g})$, that is, each Lie character $\lambda\in\Delta(\mathfrak{g})$ has unique extension up to a character on $\mathcal{U}(\mathfrak{g})$ denoted by λ also. By Poincare-Birkhoff-Witt theorem (see [6, 1.2.7], [8, Ch. 2, item 2.1]), the subset $\mathfrak{M}_e=\left\{e^{J}:J\in\mathbb{Z}_{+}^{n}\right\}\subseteq\mathcal{U}(\mathfrak{g})$ (here $e^{J}=e_1^{j_1}\cdots e_n^{j_n}$) of all ordered monomials taken by any basis $e=(e_1,\ldots,e_n)$ in $\mathfrak{g}$, is an algebraic basis in $\mathcal{U}(\mathfrak{g})$.

The following key lemma was proved by Yu. V. Turovskii [68], [70].

Lemma 2.13. *Let $B\in\mathbf{BA}$ and let $\mathfrak{g}$ be its finite-dimensional nilpotent (resp., solvable) Lie subalgebra such that the inverse closed (resp., associative) subalgebra $\mathcal{R}(\mathfrak{g})\subseteq B$ (resp., $\mathcal{R}^{0}(\mathfrak{g})\subseteq B$) generated by $\mathfrak{g}$ is dense in B. Then B is commutative modulo its Jacobson radical* $\operatorname{Rad}B$*.*

Now let $A \in \mathbf{BA}$ and let $\mathfrak{g}$ be its finite-dimensional nilpotent Lie subalgebra. The closed associative hull $B(= \overline{\mathcal{R}^0(\mathfrak{g})})$ of $\mathfrak{g}$ in A is a commutative algebra modulo its Jacobson radical $\operatorname{Rad} B$ by virtue of by Turovskii lemma 2.13. Therefore $\operatorname{Rad} B$ is the set of all quasinilpotent elements in B, and it is the (left or right) closed ideal generated by the Lie ideal $[\mathfrak{g}, \mathfrak{g}]$.

2.9. The joint spectral radius

One of the main tools of our investigations is the joint spectral radius technique. More deep exposition of this technique has been presented in the paper [61], we refer the reader to this paper for details.

Let $A \in \mathbf{BA}$ and let $M \subseteq A$. We set $M^n = \{a_1 \cdots a_n : a_i \in M\}$, $n \in \mathbb{N}$. The union $\bigcup_{n \in \mathbb{N}} M^n$ is the multiplicative semigroup generated by M in A, denoted by $\mathrm{SG}(M)$. For a bounded set M we set $\|M\| = \sup\{\|a\| : a \in M\}$ and $\rho(M) = \lim_n \|M^n\|^{1/n}$. The number $\rho(M)$ is called the (joint) spectral radius of a bounded set M [60]. Note that the limit $\lim_n \|M^n\|^{1/n}$ exists and it equals to $\inf\left\{\|M^n\|^{1/n} : n \in \mathbb{N}\right\}$.

Exercise 5. *Let $A \in \mathbf{BA}$ and let $M \subseteq A$ be a bounded subset. If $\rho(M) < 1$ then $\mathrm{SG}(M)$ is bounded in A, that is, $\|\mathrm{SG}(M)\| < \infty$.*

The following properties of the joint spectral radius can be easily verified:
(1) $\rho(M) \leq \|M\|$;
(2) $\rho(\lambda M) = |\lambda| \rho(M)$, $\lambda \in \mathbb{C}$;
(3) $\rho(N) \leq \rho(M)$ for $N \subseteq M$;
(4) $\rho(M) = \rho\left(\overline{M}\right)$, where $\overline{M}$ is the norm-closure;
(5) $\rho(MN) = \rho(NM)$ for any bounded $M, N \subseteq A$.

Let $\mathcal{N}(A)$ be a set of all algebraic norms on A which are equivalent to the original norm on A. It is proved [60] that $\rho(M) = \inf\{q(M) : q \in \mathcal{N}(A)\}$. The absolutely convex hull in A of a subset $M \subseteq A$ is denoted by $\operatorname{abc}(M)$, by definition $\operatorname{abc}(M)$ consists of all finite absolutely convex linear combinations $\sum_{i=1}^n \lambda_i a_i \in A$, $\sum_i |\lambda_i| \leq 1$.

Exercise 6. *Let $q \in \mathcal{N}(A)$. Prove that $q(\operatorname{abc}(M)) = q(M)$ for a bounded subset $M \subseteq A$. In particular, $\rho(\operatorname{abc}(M)) = \rho(M)$.*

Now let $A \in \mathbf{BA}$. We write $[a, b]$ instead of $ab - ba$ for all $a, b \in A$. If M and N are subsets in A then we write $[M, N]$ instead of $\{[a, b] : a \in M, b \in N\}$. The following lemma was suggested in [61].

Lemma 2.14. *Let M and N be bounded subsets in a Banach algebra A. If $[M, N] = \{0\}$ then $\rho(M + N) \leq \rho(M) + \rho(N)$.*

Proof. Take $\lambda_1 > \rho(M)$ and $\lambda_2 > \rho(N)$. Then $\|M^n\| < \mu\lambda_1^n$ and $\|N^n\| < \mu\lambda_2^n$, $n \in \mathbb{N}$, for a certain $\mu \in \mathbb{R}_+$. Note that $(M+N)^n \subseteq \sum_{k=0}^n \left(\sum_{i=1}^{\binom{n}{k}} M^k N^{n=k}\right)$, for $[M, N] = \{0\}$. Therefore,

$$\|(M+N)^n\| \leq \sum_{k=0}^n \sum_{i=1}^{\binom{n}{k}} \left\|M^k\right\| \left\|N^{n=k}\right\| < \sum_{k=0}^n \binom{n}{k} \mu^2 \lambda_1^k \lambda_2^{n-k} = \mu^2 (\lambda_1 + \lambda_2)^n,$$

whence $\rho(M+N) \leq \lambda_1 + \lambda_2$ and therefore $\rho(M+N) \leq \rho(M) + \rho(N)$.◄

Let again $A \in \mathbf{BA}$. Consider the left (resp., right) representation $L : A \to \mathcal{B}(A)$, $a \mapsto L_a$ (resp., $R : A \to \mathcal{B}(A)$, $a \mapsto R_a$). One can easily verify that $\|L_a\| \leq \|a\|$ (resp., $\|R_a\| \leq \|a\|$). For a subset $M \subseteq A$ we set $L_M = \{L_a : a \in M\}$ (resp., $R_M = \{R_a : a \in M\}$). Then $\|L_M\| \leq \|M\|$ (resp., $\|R_M\| \leq \|M\|$). Moreover, $L_M L_N = L_{MN}$ and $R_M R_N = R_{NM}$. In particular, $L_M^n = L_{M^n}$ and $R_M^n = R_{M^n}$, $n \in \mathbb{N}$. The latter in turn implies that $\rho(L_M) \leq \rho(M)$ for a bounded subset $M \subseteq A$.

3. Slodkowski spectra

In this section we introduce Slodkowski spectra of parametrized Banach space complexes. We prove that these spectra are stable under taking functors $\mathcal{L}(Y, \circ)$ ($Y \in \mathrm{Proj}$), $\mathcal{L}(\circ, \mathbb{C})$, $\circ \widehat{\otimes} Y$ ($Y \in \mathrm{Flat}$) and ultrapowers $\circ_{\mathfrak{U}}$.

Let Ω be a topological space and let $\mathfrak{X} = \{X_n : n \in \mathbb{Z}\}$ be a family of Banach spaces. Assume that there exists a family of continuous mappings $\mathfrak{d} = \{d_n : n \in \mathbb{Z}\}$, $d_n : \Omega \to \mathcal{L}(X_{n+1}, X_n)$, such that $(\mathfrak{X}, \mathfrak{d}(\lambda))$ (with $\mathfrak{d}(\lambda) = \{d_n(\lambda)\}$) is a chain Banach space complex

$$\cdots \longleftarrow X_{n-1} \overset{d_{n-1}(\lambda)}{\longleftarrow} X_n \overset{d_n(\lambda)}{\longleftarrow} X_{n+1} \longleftarrow \cdots, \tag{3.1}$$

for each $\lambda \in \Omega$. The family of Banach space complexes $(\mathfrak{X}, \mathfrak{d}(\lambda))$, $\lambda \in \Omega$, is called *a parametrized chain Banach space complex* or briefly *chain Ω-Banach complex* and it is denoted by $(\mathfrak{X}, \mathfrak{d})$. If $(\mathfrak{X}, \mathfrak{d}(\lambda))$ is a cochain complex for each $\lambda \in \Omega$ then $(\mathfrak{X}, \mathfrak{d})$ is said to be *a parametrized cochain Banach space complex* or *cochain Ω-Banach complex*. A morphism $\mathbf{f} : (\mathfrak{X}, \mathfrak{d}) \to (\mathfrak{Y}, \mathfrak{d}')$ of (co)chain Ω-Banach complexes is defined as a family of continuous mappings $\mathbf{f} = \{f_n : n \in \mathbb{Z}\}$, $f_n : \Omega \to \mathcal{L}(X_n, Y_n)$, such that $\mathbf{f}(\lambda) = \{f_n(\lambda)\} : (\mathfrak{X}, \mathfrak{d}(\lambda)) \to (\mathfrak{Y}, \mathfrak{d}'(\lambda))$ is a morphism in $\underline{\mathbf{BS}}$ ($\overline{\mathbf{BS}}$) for each $\lambda \in \Omega$. Using functors $\mathcal{L}(Y, \circ)$, $\mathcal{L}(\circ, Y)$, $\circ \widehat{\otimes} Y$, and $\circ_{\mathfrak{U}}$, we can associate the new Ω-Banach complexes from the original Ω-Banach complex $(\mathfrak{X}, \mathfrak{d})$. In particular, $\mathcal{L}((\mathfrak{X}, \mathfrak{d}), \mathbb{C}) = (\mathfrak{X}^*, \mathfrak{d}^*)$, where $(\mathfrak{X}^*, \mathfrak{d}^*) = \left\{ \left(\mathfrak{X}^*, \mathfrak{d}(\lambda)^*\right) : \lambda \in \Omega \right\}$ is the dual parametrized complex of $(\mathfrak{X}, \mathfrak{d})$.

A parametrized (co)chain Banach space bicomplex is defined as a bicomplex $\left(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}''\right)$ (see Subsection 2.6) such that all its rows $\left(\mathfrak{X}_{n,\bullet}, \mathfrak{d}''_{n,\bullet}\right)$ are Ω-Banach complexes, columns $\left(\mathfrak{X}_{\bullet,m}, \mathfrak{d}'_{\bullet,m}\right)$ are Λ-Banach complexes, and $\left(\mathfrak{X}, \mathfrak{d}'(\lambda), \mathfrak{d}''(\mu)\right)$ is a Banach space bicomplex for all $\lambda \in \Omega$ and $\mu \in \Lambda$. In this case we say that $\left(\mathfrak{X}, \mathfrak{d}', \mathfrak{d}''\right)$ is a $\Omega \times \Lambda$-Banach bicomplex.

Let $(\mathfrak{X}, \mathfrak{d})$ be a (co)chain parametrized Banach space complex,

$$\Sigma_n(\mathfrak{X}, \mathfrak{d}) = \{\lambda \in \Omega : H_n(\mathfrak{X}, \mathfrak{d}) \neq \{0\}\},$$

and $\Sigma^n(\mathfrak{X}, \mathfrak{d}) = \{\lambda \in \Omega : H^n(\mathfrak{X}, \mathfrak{d}) \neq \{0\}\}$ if $(\mathfrak{X}, \mathfrak{d})$ is a cochain complex, $n \in \mathbb{Z}$. We set

$$\sigma_{\delta,n}(\mathfrak{X}, \mathfrak{d}) = \bigcup_{k \leq n} \Sigma_k(\mathfrak{X}, \mathfrak{d}),$$

$$\sigma_{\pi,n}(\mathfrak{X}, \mathfrak{d}) = \left\{ \lambda \in \Omega : \lambda \in \bigcup_{k \geq n} \Sigma_k(\mathfrak{X}, \mathfrak{d}) \text{ or } \operatorname{im}(d_{n-1}(\lambda)) \text{ is not closed} \right\}.$$

Similarly, we have

$$\sigma^{\delta,n}(\mathfrak{X}, \mathfrak{d}) = \bigcup_{k \geq n} \Sigma^k(\mathfrak{X}, \mathfrak{d}),$$

$$\sigma^{\pi,n}(\mathfrak{X}, \mathfrak{d}) = \left\{ \lambda \in \Omega : \lambda \in \bigcup_{k \leq n} \Sigma^k(\mathfrak{X}, \mathfrak{d}) \text{ or } \operatorname{im}(d^n(\lambda)) \text{ is not closed} \right\},$$

whenever $(\mathfrak{X},\mathfrak{d})$ is a cochain complex. One can easily verify that $\sigma^{\delta,n}(\mathfrak{X},\mathfrak{d})=\sigma_{\delta,n}(\underline{\mathfrak{X}},\underline{\mathfrak{d}})$ and $\sigma_{\pi,n}(\mathfrak{X},\mathfrak{d})=\sigma^{\pi,n}\left(\overline{\mathfrak{X}},\overline{\mathfrak{d}}\right)$, $n\in\mathbb{Z}$.

Definition 3.1. *The set-valued functions $\sigma_{\delta,n}$, $\sigma_{\pi,n}$ ($\sigma^{\delta,n}$, $\sigma^{\pi,n}$), $n\in\mathbb{Z}$, defined on the class of all parametrized (co)chain Banach space complexes are called Slodkowski spectra. The set*

$$\sigma_{\mathrm{t}}(\mathfrak{X},\mathfrak{d})=\sigma_{\delta,\infty}(\mathfrak{X},\mathfrak{d})=\sigma_{\pi,-\infty}(\mathfrak{X},\mathfrak{d})=\bigcup_{n\in\mathbb{Z}}\Sigma_n(\mathfrak{X},\mathfrak{d}),$$

(or $\sigma_{\mathrm{t}}(\mathfrak{X},\mathfrak{d})=\sigma^{\pi,\infty}(\mathfrak{X},\mathfrak{d})=\sigma^{\delta,-\infty}(\mathfrak{X},\mathfrak{d})=\bigcup_{n\in\mathbb{Z}}\Sigma^n(\mathfrak{X},\mathfrak{d})$ for the cochain complex $(\mathfrak{X},\mathfrak{d})$) is called Taylor spectrum of $(\mathfrak{X},\mathfrak{d})$. We set $\mathfrak{S}=\mathfrak{S}_{\cdot}\cup\mathfrak{S}^{\cdot}$, where $\mathfrak{S}_{\cdot}=\mathfrak{S}_\delta\cup\mathfrak{S}_\pi$ ($\mathfrak{S}^{\cdot}=\mathfrak{S}^\delta\cup\mathfrak{S}^\pi$), $\mathfrak{S}_\delta=\{\sigma_{\delta,n}:n\in\mathbb{Z}\cup\{\infty\}\}$, $\mathfrak{S}_\pi=\{\sigma_{\pi,n}:n\in\{-\infty\}\cup\mathbb{Z}\}$ ($\mathfrak{S}^\delta=\left\{\sigma^{\delta,n}:n\in\{-\infty\}\cup\mathbb{Z}\right\}$, $\mathfrak{S}^\pi=\{\sigma^{\pi,n}:n\in\mathbb{Z}\cup\{\infty\}\}$).

In the sequel, the implication $\sigma\in\mathfrak{S}$ indicates that we have taken in a Slodkowski spectrum if the latter will not specially be indicated. Moreover, it is convenient to introduce the *conjugate to σ spectrum* $\overline{\sigma}$ by setting $\overline{\sigma}=\sigma^{\delta,k}$ (resp., $\overline{\sigma}=\sigma^{\pi,k}$) if $\sigma=\sigma_{\delta,k}$ (resp., $\sigma=\sigma_{\pi,k}$) and vice-verse, and also *the dual to σ spectrum* σ^* by setting $\sigma^*=\sigma^{\delta,k}$ (resp., $\sigma^*=\sigma^{\pi,k}$) if $\sigma=\sigma_{\pi,k}$ (resp., $\sigma=\sigma_{\delta,k}$) and vice-verse. For a subset $S\subseteq\mathfrak{S}$ we write $\overline{S}=\{\overline{\sigma}:\sigma\in S\}$ and $S^*=\{\sigma^*:\sigma\in S\}$. Thus $\overline{\mathfrak{S}_\delta}=\mathfrak{S}^\delta$, $\overline{\mathfrak{S}_\pi}=\mathfrak{S}^\pi$, $\overline{\mathfrak{S}^\delta}=\mathfrak{S}_\delta$, $\overline{\mathfrak{S}^\pi}=\mathfrak{S}_\pi$, and $\mathfrak{S}_\delta^*=\mathfrak{S}^\pi$, $\mathfrak{S}_\pi^*=\mathfrak{S}^\delta$, $\mathfrak{S}^{\delta*}=\mathfrak{S}_\pi$, $\mathfrak{S}^{\pi*}=\mathfrak{S}_\delta$. In particular, we have the following identities

$$\sigma(\underline{\mathfrak{X}},\underline{\mathfrak{d}})=\overline{\sigma}(\mathfrak{X},\mathfrak{d}),\quad\sigma\left(\overline{\mathfrak{X}},\overline{\mathfrak{d}}\right)=\overline{\sigma}(\mathfrak{X},\mathfrak{d}),$$

for all $\sigma\in\mathfrak{S}$. Note also that $\overline{\sigma}^*=\overline{\sigma^*}$ for all $\sigma\in\mathfrak{S}$.

Lemma 3.1. *Let $(\mathfrak{X},\mathfrak{d})$ be a (co)chain Ω-Banach complex. Then*
(i) $\Sigma^n(\mathfrak{X}^*,\mathfrak{d}^*)\subseteq\Sigma_{n-1}(\mathfrak{X},\mathfrak{d})\cup\Sigma_n(\mathfrak{X},\mathfrak{d})$ (resp., $\Sigma_n(\mathfrak{X}^*,\mathfrak{d}^*)\subseteq\Sigma^n(\mathfrak{X},\mathfrak{d})\cup\Sigma^{n+1}(\mathfrak{X},\mathfrak{d})$);
(ii) $\Sigma_n(\mathfrak{X},\mathfrak{d})\subseteq\Sigma^n(\mathfrak{X}^*,\mathfrak{d}^*)\cup\Sigma^{n+1}(\mathfrak{X}^*,\mathfrak{d}^*)$ (resp., $\Sigma^n(\mathfrak{X},\mathfrak{d})\subseteq\Sigma_{n-1}(\mathfrak{X}^*,\mathfrak{d}^*)\cup\Sigma_n(\mathfrak{X}^*,\mathfrak{d}^*)$).

Proof. If $\lambda\notin\Sigma_{n-1}(\mathfrak{X},\mathfrak{d})\cup\Sigma_n(\mathfrak{X},\mathfrak{d})$, then the complex $(\mathfrak{X},\mathfrak{d}(\lambda))$ is exact at terms X_{n-1} and X_n. By Theorem 2.1 (i), the dual complex $\left(\mathfrak{X}^*,\mathfrak{d}(\lambda)^*\right)$ is exact at the term X_n^*, that is, $\lambda\notin\Sigma^n(\mathfrak{X}^*,\mathfrak{d}^*)$.

Let us prove (ii). Assume that $\lambda\notin\Sigma^n(\mathfrak{X}^*,\mathfrak{d}^*)\cup\Sigma^{n+1}(\mathfrak{X}^*,\mathfrak{d}^*)$. Then the dual complex $\left(\mathfrak{X}^*,\mathfrak{d}(\lambda)^*\right)$ is exact at terms X_n^* and X_{n+1}^*. By Theorem 2.1 (ii), (iii), $(\mathfrak{X},\mathfrak{d}(\lambda))$ is exact at the term X_n, that is, $\lambda\notin\Sigma_n(\mathfrak{X},\mathfrak{d})$.◄

Theorem 3.1. *Let $(\mathfrak{X},\mathfrak{d})$ be a Ω-Banach complex. Then $\sigma(\mathfrak{X}^*,\mathfrak{d}^*)=\sigma^*(\mathfrak{X},\mathfrak{d})$ for all $\sigma\in\mathfrak{S}$.*

Proof. It suffices to prove the equality for $\sigma\in\mathfrak{S}_\delta$ and $\sigma\in\mathfrak{S}^\pi$. Indeed, being proved that we use the conjugate functor to calculate other Slodkowski spectra of the dual complex. For instance, if $(\mathfrak{X},\mathfrak{d})$ is a chain Ω-Banach complex and $\sigma\in\mathfrak{S}_\pi$ then $\overline{\sigma}\in\mathfrak{S}^\pi$ and

$$\sigma(\mathfrak{X},\mathfrak{d})=\overline{\sigma}\left(\overline{\mathfrak{X}},\overline{\mathfrak{d}}\right)=\overline{\sigma}^*\left(\overline{\mathfrak{X}}^*,\overline{\mathfrak{d}}^*\right)=\overline{\sigma}^*\left(\overline{\mathfrak{X}^*},\overline{\mathfrak{d}^*}\right)=\overline{\sigma^*}\left(\overline{\mathfrak{X}^*},\overline{\mathfrak{d}^*}\right)=\sigma^*(\mathfrak{X}^*,\mathfrak{d}^*).$$

By analogy, the same is valid for a spectrum $\sigma\in\mathfrak{S}^\delta$ and a cochain complex $(\mathfrak{X},\mathfrak{d})$.

Now let $(\mathfrak{X},\mathfrak{d})$ be a chain Ω-Banach complex and let $\sigma=\sigma_{\delta,n}\in\mathfrak{S}_\delta$. Then $\bigcup_{k\le n}\Sigma^k(\mathfrak{X}^*,\mathfrak{d}^*)\subseteq\bigcup_{k\le n}\Sigma_k(\mathfrak{X},\mathfrak{d})$ by Lemma 3.1 (i). Moreover, if $\operatorname{im}\left(d_n(\lambda)^*\right)$ is not closed then so is $\operatorname{im}(d_n(\lambda))$, and therefore $\lambda\in\Sigma_n(\mathfrak{X},\mathfrak{d})$. Thus $\sigma^{\pi,n}(\mathfrak{X}^*,\mathfrak{d}^*)\subseteq\sigma_{\delta,n}(\mathfrak{X},\mathfrak{d})$. Conversely, if $\lambda\notin\sigma^{\pi,n}(\mathfrak{X}^*,\mathfrak{d}^*)$, then $\operatorname{im}\left(d_n(\lambda)^*\right)$ is closed and $\lambda\notin\bigcup_{k\le n}\Sigma^k(\mathfrak{X}^*,\mathfrak{d}^*)$. By Theorem 2.1 (iii), $\lambda\notin\Sigma_n(\mathfrak{X},\mathfrak{d})$. Using Lemma 3.1 (ii), we obtain that $\Sigma_k(\mathfrak{X},\mathfrak{d})\subseteq\Sigma^k(\mathfrak{X}^*,\mathfrak{d}^*)\cup\Sigma^{k+1}(\mathfrak{X}^*,\mathfrak{d}^*)$ for all k, $k\le n$. Thereupon, $\lambda\notin\bigcup_{k\le n}\Sigma_k(\mathfrak{X},\mathfrak{d})=\sigma_{\delta,n}(\mathfrak{X},\mathfrak{d})$. Thus $\sigma_{\delta,n}(\mathfrak{X},\mathfrak{d})=\sigma^{\pi,n}(\mathfrak{X}^*,\mathfrak{d}^*)$.

Now we assume that $(\mathfrak{X}, \mathfrak{d})$ is a cochain Ω-Banach complex and $\sigma = \sigma^{\pi,n} \in \mathfrak{S}^{\pi}$. Take $\lambda \notin \sigma^{\pi,n}(\mathfrak{X}, \mathfrak{d})$. Then $\lambda \notin \bigcup_{k\leq n} \Sigma^k(\mathfrak{X}, \mathfrak{d})$ and $\operatorname{im}(d^n(\lambda))$ is closed. By Lemma 3.1 (i),

$$\bigcup_{k\leq n-1} \Sigma_k(\mathfrak{X}^*, \mathfrak{d}^*) \subseteq \bigcup_{k\leq n} \Sigma^k(\mathfrak{X}, \mathfrak{d}),$$

whence $\lambda \notin \bigcup_{k\leq n-1} \Sigma_k(\mathfrak{X}^*, \mathfrak{d}^*)$. Taking into account that $\sigma_{\delta,n}(\mathfrak{X}^*, \mathfrak{d}^*) = \bigcup_{k\leq n} \Sigma_k(\mathfrak{X}^*, \mathfrak{d}^*)$, we need only to prove that $\lambda \notin \Sigma_n(\mathfrak{X}^*, \mathfrak{d}^*)$. But, bearing in mind that $\lambda \notin \Sigma^n(\mathfrak{X}, \mathfrak{d})$ and $\operatorname{im}(d^n(\lambda))$ is closed, we conclude that $\lambda \notin \Sigma_n(\mathfrak{X}^*, \mathfrak{d}^*)$ by virtue of Theorem 2.1 (i). Thus $\sigma_{\delta,n}(\mathfrak{X}^*, \mathfrak{d}^*) \subseteq \sigma^{\pi,n}(\mathfrak{X}, \mathfrak{d})$. Conversely, let $\lambda \notin \sigma_{\delta,n}(\mathfrak{X}^*, \mathfrak{d}^*) = \bigcup_{k\leq n} \Sigma_k(\mathfrak{X}^*, \mathfrak{d}^*)$. By Lemma 3.1 (ii),

$$\bigcup_{k\leq n} \Sigma^k(\mathfrak{X}, \mathfrak{d}) \subseteq \bigcup_{k\leq n} \Sigma_k(\mathfrak{X}^*, \mathfrak{d}^*),$$

whence $\lambda \notin \bigcup_{k\leq n} \Sigma^k(\mathfrak{X}, \mathfrak{d})$. Moreover, $\operatorname{im}(d^n(\lambda))$ is closed out of that $\lambda \notin \Sigma_n(\mathfrak{X}^*, \mathfrak{d}^*)$ (see Theorem 2.1 (ii)). Thus $\lambda \notin \sigma_{\pi,n}(\mathfrak{X}^*, \mathfrak{d}^*)$.◀

Theorem 3.2. *Let $(\mathfrak{X}, \mathfrak{d})$ be a chain Ω-Banach complex and let $Y \in$ **BS**. Then $\sigma(\mathfrak{X}, \mathfrak{d}) \subseteq \sigma(\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d})))$ for all $\sigma \in \mathfrak{S}$.. Moreover, $\sigma(\mathfrak{X}, \mathfrak{d}) = \sigma(\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d})))$ whenever $Y \in$* Proj.

Proof. Take $\lambda \in \sigma(\mathfrak{X}, \mathfrak{d})$. By Lemma 2.3, there exist bounded sequences $\{x_n\} \subset X_k$ and $\{x'_n\} \subset X_k^*$ such that $\lim_n d_{k-1}(\lambda) x_n = 0$, $\lim_n d_k(\lambda)^* x'_n = 0$, $x'_n(x_n) = 1$, for some k. Take $x' \in Y^*$, $\|x'\| = 1$, and $x \in Y$ such that $x'(x) = 1$. Consider bounded sequences $\{x' \otimes x_n\} \subset \mathcal{L}(Y, X_k)$ and $\{F_n\} \subset \mathcal{L}(Y, X_k)^*$, where $F_n(u) = u^*(x'_n) x$. Note that $\lim_n L_{d_{k-1}(\lambda)} x' \otimes x_n = \lim_n x' \otimes d_{k-1}(\lambda) x_n = 0$ and $L^*_{d_k(\lambda)} F_n(u) = F_n \cdot L_{d_k(\lambda)}(u) = u^*\left(d_k(\lambda)^* x'_n\right)(y)$ for all $u \in \mathcal{L}(Y, X_{k+1})$. Then $\lim_n L^*_{d_k(\lambda)} F_n = 0$. Appealing Lemma 2.3 again, we infer that $\lambda \in \sigma(\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d})))$.

Now let $Y \in$ Proj and let $\sigma = \sigma_{\pi,n} \in \mathfrak{S}_{\pi}$ for a certain n. Take $\lambda \notin \sigma(\mathfrak{X}, \mathfrak{d})$. Thus the complex $(\mathfrak{X}, \mathfrak{d})$ is exact at all members k, $k \geq n$, and $\operatorname{im}(d_{n-1}(\lambda))$ is closed. Then $\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d}))$ is also exact at all members k, $k \geq n$ and $\operatorname{im}\left(L_{d_{n-1}(\lambda)}\right) \subseteq \mathcal{L}(Y, \operatorname{im}(d_{n-1}(\lambda)))$. Moreover, if $u \in \mathcal{L}(Y, \operatorname{im}(d_{n-1}(\lambda)))$ then there exists $v \in \mathcal{L}(Y, X_n)$ such that $d_{n-1}(\lambda) \cdot v = u$, owing to $Y \in$ Proj (see Subsection 2.5). The latter means that $L_{d_{n-1}(\lambda)} v = u$, that is, $\operatorname{im}\left(L_{d_{n-1}(\lambda)}\right) = \mathcal{L}(Y, \operatorname{im}(d_{n-1}(\lambda)))$ and $\operatorname{im}\left(L_{d_{n-1}(\lambda)}\right)$ is closed. Therefore, $\lambda \notin \sigma_{\pi,n}(\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d})))$. Thus $\sigma(\mathfrak{X}, \mathfrak{d}) = \sigma(\mathcal{L}(Y, (\mathfrak{X}, \mathfrak{d})))$ for all $\sigma \in \mathfrak{S}_{\pi}$.

The same argument could be applied for the spectra $\sigma \in \mathfrak{S}_{\delta}$.◀

Exercise 7. *Let $(\mathfrak{X}, \mathfrak{d})$ be a chain Ω-Banach complex and let $Y \in$ **BS**. Using the same idea carried out in the proof of Theorem 3.2, prove that $\sigma(\mathfrak{X}, \mathfrak{d}) \subseteq \sigma\left((\mathfrak{X}, \mathfrak{d}) \widehat{\otimes} Y\right)$ for all $\sigma \in \mathfrak{S}$., moreover, the latter inclusion becomes the equality whenever $Y \in$* Flat *(see Subsection 2.5). What about the same type relations with respect to the functor $\mathcal{L}(\circ, Y)$? When $Y = \mathbb{C}$ the assertion was proved in Theorem 3.1.*

Proposition 3.1. *Let $(\mathfrak{X}, \mathfrak{d})$ be a nonnegative chain Ω-Banach complex and let $\sigma^{\pi,n}(\mathfrak{X}, \mathfrak{d})$ be its π-type Slodkowski spectrum. Then $\sigma^{\pi,n}(\mathfrak{X}, \mathfrak{d})$ is closed for all $n \in \mathbb{Z}_+$.*

Proof. Take $\lambda_0 \notin \sigma^{\pi,n}(\mathfrak{X}, \mathfrak{d})$. Then $\operatorname{im}\left(d^{s-1}(\lambda_0)\right) = \ker(d^s(\lambda_0))$ for all s, $1 \leq s \leq n$, and the image $\operatorname{im}(d^n(\lambda_0))$ is closed. Fix constants c_s such that $c_s > \operatorname{ic}(d^s(\lambda_0))$, $1 \leq s \leq n$. Using continuity of $\mathfrak{d}$, we infer that there exists an open neighborhood U in Ω of λ_0 such that $\varepsilon_s = \|d^s(\lambda) - d^s(\lambda_0)\| \leq 2^{-s-2} c_s^{-1}$ for all $\lambda \in U$. Demonstrate that $U \cap \sigma^{\pi,n}(\mathfrak{X}, \mathfrak{d}) = \oslash$. Take $\lambda \in U$. Note that

$$\sum_{s=0}^{n-1}\left(c_s\varepsilon_s+c_{s+1}\varepsilon_{s+1}+c_sc_{s+1}\varepsilon_s\varepsilon_{s+1}\right)\leq\sum_{s=0}^{n-1}\left(2^{-s-2}+2^{-s-3}+2^{-s-2}2^{-s-3}\right)\leq$$
$$\leq\sum_{s=0}^{n-1}2^{-s-1}<1.$$

By Lemma 2.4, $\operatorname{im}\left(d^{s-1}(\lambda)\right)=\ker\left(d^s(\lambda)\right)$ for all s, $1\leq s\leq n$, and $\operatorname{im}\left(d^n(\lambda)\right)$ is closed. Thereby, $\lambda\notin\sigma^{\pi,n}(\mathfrak{X},\mathfrak{d})$.◀

Now we investigate stability property of spectra with respect to the ultrapower functor (see Subsection 2.2).

Lemma 3.2. *Let $(\mathfrak{X},\mathfrak{d})$ be a (co)chain Ω-Banach complex and let $\mathfrak{U}$ be an ultrafilter. Then $\sigma_{\pi,n}(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})=\bigcup_{k\geq n}\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$ for all $\sigma_{\pi,n}\in\mathfrak{S}_\pi$.*

Proof. By Definition 3.1, $\bigcup_{k\geq n}\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})\subseteq\sigma_{\pi,n}(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$. Let $\lambda\notin\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$ for all k, $k\geq n$. To prove that $\lambda\notin\sigma_{\pi,n}(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$ one needs to establish the closedness of the image $\operatorname{im}(d_{n-1}(\lambda)_\mathfrak{U})$. Note that $\ker(d_{n-1}(\lambda)_\mathfrak{U})=\operatorname{im}(d_n(\lambda)_\mathfrak{U})$, therefore $\operatorname{im}(d_n(\lambda)_\mathfrak{U})$ is closed. Then $\operatorname{im}(d_n(\lambda))$ is closed too and

$$\operatorname{im}(d_n(\lambda))=X_n\cap\operatorname{im}(d_n(\lambda)_\mathfrak{U})=X_n\cap\ker(d_{n-1}(\lambda)_\mathfrak{U})=\ker(d_{n-1}(\lambda)),$$

by virtue of Lemma 2.2. It follows that

$$\ker(d_{n-1}(\lambda))_\mathfrak{U}=\operatorname{im}(d_n(\lambda))_\mathfrak{U}=\operatorname{im}(d_n(\lambda)_\mathfrak{U})=\ker(d_{n-1}(\lambda)_\mathfrak{U}).$$

Appealing Lemma 2.2 once more, we conclude that $\operatorname{im}(d_{n-1}(\lambda)_\mathfrak{U})$ is closed.

The cochain version of the assertion we left to the reader.◀

Theorem 3.3. *Let $(\mathfrak{X},\mathfrak{d})$ be a (co)chain Ω-Banach complex and let $\mathfrak{U}$ be an ultrafilter in a set S. Then $\sigma(\mathfrak{X},\mathfrak{d})=\sigma(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$ for all $\sigma\in\mathfrak{S}$.*

Proof. First, assume that $(\mathfrak{X},\mathfrak{d})$ is a chain complex. Let us prove that $\Sigma_k(\mathfrak{X},\mathfrak{d})\subseteq\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$. Take $\lambda\in\Sigma_k(\mathfrak{X},\mathfrak{d})$. If $\operatorname{im}(d_k(\lambda))$ is not closed then so is $\operatorname{im}(d_k(\lambda)_\mathfrak{U})$ by Lemma 2.2, therefore $\lambda\in\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$. Let us assume that $\operatorname{im}(d_k(\lambda))$ is closed. Take $x\in\ker(d_{k-1}(\lambda))\setminus\operatorname{im}(d_k(\lambda))$. By Lemma 2.2, $X_k\cap\operatorname{im}(d_k(\lambda)_\mathfrak{U})=\operatorname{im}(d_k(\lambda))$, therefore $x\notin\operatorname{im}(d_k(\lambda)_\mathfrak{U})$. But, $x\in\ker(d_{k-1}(\lambda))_\mathfrak{U}$ and $\ker(d_{k-1}(\lambda))_\mathfrak{U}\subseteq\ker(d_{k-1}(\lambda)_\mathfrak{U})$. It follows that $\lambda\in\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$. Thus we deduce that $\sigma_{\delta,n}(\mathfrak{X},\mathfrak{d})=\bigcup_{k\leq n}\Sigma_k(\mathfrak{X},\mathfrak{d})\subseteq\sigma_{\delta,n}(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$ and $\sigma_{\pi,n}(\mathfrak{X},\mathfrak{d})\subseteq\sigma_{\pi,n}(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$.

Conversely, let us prove that

$$\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})\subseteq\{\lambda\in\Omega:\operatorname{im}(d_{k-1}(\lambda))\text{ is not closed}\}\cup\Sigma_k(\mathfrak{X},\mathfrak{d})\subseteq\Sigma_{k-1}(\mathfrak{X},\mathfrak{d})\cup\Sigma_k(\mathfrak{X},\mathfrak{d}).\quad(3.2)$$

Take $\lambda\in\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})$. By Lemma 2.2, we could assume that the images $\operatorname{im}(d_{k-1}(\lambda)_\mathfrak{U})$, $\operatorname{im}(d_k(\lambda)_\mathfrak{U})$ are closed. Further, $\ker(d_{k-1}(\lambda))_\mathfrak{U}=\ker(d_{k-1}(\lambda)_\mathfrak{U})$, $\operatorname{im}(d_k(\lambda))_\mathfrak{U}=\operatorname{im}(d_k(\lambda)_\mathfrak{U})$ by virtue Lemma 2.2. Take $[x_s]\in\ker(d_{k-1}(\lambda)_\mathfrak{U})\setminus\operatorname{im}(d_k(\lambda)_\mathfrak{U})$, where $x_s\in\ker(d_{k-1}(\lambda))$. Then, $x_{s_0}\notin\operatorname{im}(d_k(\lambda))$ for some $s_0\in S$, whence $x_{s_0}\in\ker(d_{k-1}(\lambda))\setminus\operatorname{im}(d_k(\lambda))$ and therefore $\lambda\in\Sigma_k(\mathfrak{X},\mathfrak{d})$.

In particular,

$$\sigma_{\delta,n}(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})=\bigcup_{k\leq n}\Sigma_k(\mathfrak{X}_\mathfrak{U},\mathfrak{d}_\mathfrak{U})\subseteq\bigcup_{k\leq n}\Sigma_k(\mathfrak{X},\mathfrak{d})=\sigma_{\delta,n}(\mathfrak{X},\mathfrak{d}),$$

that is, $\sigma_{\delta,n}(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}) = \sigma_{\delta,n}(\mathfrak{X},\mathfrak{d})$.

Finally, take $\lambda \in \sigma_{\pi,n}(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$. Using Lemma 3.2, we infer that $\lambda \in \Sigma_k(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$ for some k, $k \geq n$. It follows that $\lambda \in \Sigma_k(\mathfrak{X},\mathfrak{d})$ or $\operatorname{im}(d_{k-1}(\lambda))$ is not closed due to (3.2). The latter merely means that $\lambda \in \sigma_{\pi,n}(\mathfrak{X},\mathfrak{d})$, that is, $\sigma_{\delta,n}(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}) = \sigma_{\delta,n}(\mathfrak{X},\mathfrak{d})$.

For a cochain complex $(\mathfrak{X},\mathfrak{d})$ we apply the conjugate functor. Namely, if $\sigma \in \mathfrak{S}^{\delta} \cup \mathfrak{S}^{\pi}$ then $\sigma(\mathfrak{X},\mathfrak{d}) = \overline{\sigma}(\underline{\mathfrak{X}},\underline{\mathfrak{d}}) = \overline{\sigma}(\underline{\mathfrak{X}}_{\mathfrak{U}},\underline{\mathfrak{d}}_{\mathfrak{U}}) = \overline{\sigma}\left(\underline{\mathfrak{X}_{\mathfrak{U}}},\underline{\mathfrak{d}_{\mathfrak{U}}}\right) = \sigma(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$.◄

Corollary 3.1. *If $0 \leftarrow (\mathfrak{X},\mathfrak{d}) \xleftarrow{\mathbf{f}} (\mathfrak{Y},\mathfrak{d}') \xleftarrow{\mathbf{g}} (\mathfrak{Z},\mathfrak{d}'') \leftarrow 0$ is a short exact sequence of (co)chain Ω-Banach complexes then $\sigma(\mathfrak{Y},\mathfrak{d}') \subseteq \sigma(\mathfrak{X},\mathfrak{d}) \cup \sigma(\mathfrak{Z},\mathfrak{d}'')$ for all $\sigma \in \mathfrak{S}$.*

Proof. Take $\lambda \in \sigma(\mathfrak{Y},\mathfrak{d}')$. Using Theorem 3.3 and Lemma 3.2, one can assume that $\lambda \in \Sigma_i(\mathfrak{Y}_{\mathfrak{U}},\mathfrak{d}'_{\mathfrak{U}})$ for some i, $i \in \mathbb{Z}$. Then $H_i(\mathfrak{Y}_{\mathfrak{U}},\mathfrak{d}'_{\mathfrak{U}}(\lambda)) \neq \{0\}$. Note that $0 \leftarrow X_n \xleftarrow{f_n(\mu)} Y_n \xleftarrow{g_n(\mu)} Z_n \leftarrow 0$, $\mu \in \Omega$, is a Ω-Banach complex with the empty Taylor spectrum for each $n \in \mathbb{Z}$. It follows that so its ultrapower on the ground of Theorem 3.3. The latter merely means that the following short sequence of complexes

$$0 \leftarrow (\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}(\lambda)) \xleftarrow{\mathbf{f}(\mu)_{\mathfrak{U}}} (\mathfrak{Y}_{\mathfrak{U}},\mathfrak{d}'_{\mathfrak{U}}(\lambda)) \xleftarrow{\mathbf{g}(\mu)_{\mathfrak{U}}} (\mathfrak{Z}_{\mathfrak{U}},\mathfrak{d}''_{\mathfrak{U}}(\lambda)) \leftarrow 0,$$

is exact for all $\mu \in \Omega$. It remains to use the long exact sequence of (co)homologies

$$\cdots \leftarrow H_i(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}(\lambda)) \leftarrow H_i(\mathfrak{Y}_{\mathfrak{U}},\mathfrak{d}'_{\mathfrak{U}}(\lambda)) \leftarrow H_i(\mathfrak{Z}_{\mathfrak{U}},\mathfrak{d}''_{\mathfrak{U}}(\lambda)) \leftarrow H_{i+1}(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}(\lambda)) \leftarrow \cdots,$$

induced by the short exact sequence for $\mu = \lambda$. Then $H_i(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}(\lambda)) \neq \{0\}$ or $H_i(\mathfrak{Z}_{\mathfrak{U}},\mathfrak{d}''_{\mathfrak{U}}(\lambda)) \neq \{0\}$.◄

4. Projection property

In this section, we investigate the projection property of Slodkowski spectra.

First, let us introduce the following definition played important role for Slodkowski spectra of infinite-parametrized Banach space complexes.

Definition 4.1. *Let $(\mathfrak{X},\mathfrak{d})$ be a chain Ω-Banach complex. We say that $(\mathfrak{X},\mathfrak{d})$ is π-stable (resp., δ-stable) if $\bigcap_{n\in\mathbb{Z}} \bigcup_{k\geq n} \Sigma_k(\mathfrak{X},\mathfrak{d}) = \oslash$ (resp., $\bigcap_{n\in\mathbb{Z}} \bigcup_{k\leq n} \Sigma_k(\mathfrak{X},\mathfrak{d}) = \oslash$). If $(\mathfrak{X},\mathfrak{d})$ is a cochain Ω-Banach complex then it is said to be π-stable (resp., δ-stable) if so is the chain complex $(\underline{\mathfrak{X}},\underline{\mathfrak{d}})$.*

It is clear that a chain complex $(\mathfrak{X},\mathfrak{d})$ is π-stable (resp., δ-stable) iff $\bigcap_{n\in\mathbb{Z}} \sigma_{\pi,n}(\mathfrak{X},\mathfrak{d}) = \oslash$ (resp., $\bigcap_{n\in\mathbb{Z}} \sigma_{\delta,n}(\mathfrak{X},\mathfrak{d}) = \oslash$). By analogy, a cochain complex $(\mathfrak{X},\mathfrak{d})$ is π-stable (resp., δ-stable) iff $\bigcap_{n\in\mathbb{Z}} \sigma^{\pi,n}(\mathfrak{X},\mathfrak{d}) = \oslash$ (resp., $\bigcap_{n\in\mathbb{Z}} \sigma^{\delta,n}(\mathfrak{X},\mathfrak{d}) = \oslash$).

Lemma 4.1. *Let $(\mathfrak{X},\mathfrak{d})$ be a chain Ω-Banach complex. Then*
(i) $(\mathfrak{X},\mathfrak{d})$ is π-stable (resp., δ-stable) iff $(\mathfrak{X}^*,\mathfrak{d}^*)$ is δ-stable (resp., π-stable);
(ii) $(\mathfrak{X},\mathfrak{d})$ is π-stable (resp., δ-stable) iff so is its ultrapower $(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$.

Proof. The first assertion follows from Theorem 3.1, and the second one from Theorem 3.3.◄

Note that if a chain Ω-Banach complex $(\mathfrak{X},\mathfrak{d})$ is vanishing to the right (resp., left), that is, $X_k = \{0\}$, $k \geq n$ (resp., $k \leq n$) for some n, then $(\mathfrak{X},\mathfrak{d})$ is automatically π-stable (resp., δ-stable). In particular, a finite parametrized Banach space complex is π-stable and δ-stable, simultaneously.

Now let $(\mathfrak{X},\mathfrak{d})$ be a chain Ω-Banach complex and let $\beta = \{\beta_n \in \mathcal{B}(X_n)\}$ be a bounded endomorphism of $(\mathfrak{X},\mathfrak{d})$ ($d_n(\lambda)\beta_n = \beta_{n-1}d_n(\lambda)$, $\lambda \in \Omega$, $n \in \mathbb{Z}$). We set $\beta - \mu = \{\beta_n - \mu \in \mathcal{B}(X_n)\}$

whenever $\mu \in \mathbb{C}$, and the spectrum $\operatorname{sp}(\beta)$ of β is assumed to equal to the union of the ordinary spectra $\operatorname{sp}(\beta_n)$. It beyond a doubt $\beta-\mu$ is an endomorphism of $(\mathfrak{X}, \mathfrak{d})$ for all μ. The Ω-Banach complex

$$\operatorname{Con}((\mathfrak{X}, \mathfrak{d}), \beta-\mu)=\{\operatorname{Con}((\mathfrak{X}, \mathfrak{d}), \beta-\mu), \lambda \in \Omega\},$$

is called *a cone of the endomorphism* $\beta-\mu$. If Ω is a sole point then the latter is none other than the cone of a Banach space complex (see Subsection 2.4). The family of Banach space complexes $\operatorname{Con}((\mathfrak{X}, \mathfrak{d}), \beta-\mu)$ (for the differential we use notation $\gamma_n(\lambda, \mu)$), $(\lambda, \mu) \in \Omega \times \mathbb{C}$, is a $\Omega \times \mathbb{C}$-Banach complex (where $\Omega \times \mathbb{C}$ furnished with the product topology) and it is denoted by $\operatorname{Con}_\beta(\mathfrak{X}, \mathfrak{d})$.

Lemma 4.2. *Let us assume that* $\lambda \in \Sigma_n(\mathfrak{X}, \mathfrak{d})$ *and* $\operatorname{im}(d_n(\lambda))$ *is closed. There exists* $\mu \in \mathbb{C}$ *such that* $\lambda \in \Sigma_n(\operatorname{Con}((\mathfrak{X}, \mathfrak{d}), \beta-\mu))$ *or* $\operatorname{im}(\gamma_{n-1}(\lambda, \mu))$ *is not closed.*

Proof. Let $Y_n=X_n / \operatorname{im}(d_n(\lambda))$ and $T_n \in \mathcal{L}(Y_n, X_{n-1})$, $T_n x^{\sim}=d_{n-1}(\lambda) x$. Then $\ker(T_n) \neq 0$ and $\beta_{n-1} T_n=T_n \beta_n^{\sim}$, where $\beta_n^{\sim} \in \mathcal{B}(Y_n)$, $\beta_n^{\sim} x^{\sim}=(\beta_n x)^{\sim}$. Thus the kernel $\ker(T_n)$ is invariant under the operator $\beta_n^{\sim}$. By using nonvoidness of the approximate point spectrum $\sigma^{\mathrm{ap}}(\beta_n^{\sim})$ of the operator $\beta_n^{\sim}$, we see that there exist $\mu \in \mathbb{C}$ and a sequence $\{x_k^{\sim}\} \subset \ker(T_n)$, $\|x_k^{\sim}\|=1$, such that $\lim_k(\beta_n^{\sim}-\mu) x_k^{\sim}=0$. One can find a sequence $\{y_k\} \subset X_{n+1}$ such that $\lim_k(\beta_n-\mu) x_k+d_n(\lambda) y_k=0$. A direct computation shows that

$$\gamma_{n-1}(\lambda, \mu)(y_k, x_k)=(d_n(\lambda) y_k+(\beta_n-\mu) x_k,-d_{n-1}(\lambda) x_k)=(d_n(\lambda) y_k+(\beta_n-\mu) x_k, 0) \rightarrow 0,$$

whenever $k \rightarrow \infty$. If $\lambda \notin \Sigma_n(\operatorname{Con}((\mathfrak{X}, \mathfrak{d}), \beta-\mu))$ then $\ker(\gamma_{n-1}(\lambda, \mu))=\operatorname{im}(\gamma_n(\lambda, \mu))$ and the norm $r_k=\|(y_k, x_k)^{\sim}\|$ of $(y_k, x_k)^{\sim} \in X_{n+1} \oplus X_n / \operatorname{im}(\gamma_n(\lambda, \mu))$ is estimated below:

$$r_k=\inf_{(z, w) \in X_{n+2} \oplus X_{n+1}}\|(y_k, x_k)+(d_{n+1}(\lambda) z+(\beta_{n+1}-\mu) w,-d_n(\lambda) w)\| \geq$$

$$\geq \inf_{w \in X_{n+1}}\|x_k-d_n(\lambda) w\|=\|x_k^{\sim}\|=1.$$

Thus, $\inf\{r_k: k \in \mathbb{N}\} \geq 1$ and $\lim_k \gamma_{n-1}(\lambda, \mu)(y_k, x_k)=0$, whence the image of the operator

$$X_{n+1} \oplus X_n / \ker(\gamma_{n-1}(\lambda, \mu)) \rightarrow X_n \oplus X_{n-1},$$

induced by $\gamma_{n-1}(\lambda, \mu)$ is not closed, thereby so is the image of $\gamma_{n-1}(\lambda, \mu)$.◀

Theorem 4.1. *Let* $(\mathfrak{X}, \mathfrak{d})$ *be a (co)chain* Ω*-Banach complex,* β *a bounded endomorphism of* $(\mathfrak{X}, \mathfrak{d})$ *and let* $\Pi: \Omega \times \mathbb{C} \rightarrow \nleq$ *is the canonical projection. If* $(\mathfrak{X}, \mathfrak{d})$ *is* π*-stable (resp.,* δ*-stable) then* $\sigma(\mathfrak{X}, \mathfrak{d})=\Pi(\sigma(\operatorname{Con}_\beta(\mathfrak{X}, \mathfrak{d})))$ *for all* $\sigma \in \mathfrak{S}_\pi$ *(resp.,* $\sigma \in \mathfrak{S}_\delta$*).*

Proof. First, let us assume that $(\mathfrak{X}, \mathfrak{d})$ is a chain Ω-Banach complex, $\sigma=\sigma_{\pi, n} \in \mathfrak{S}_\pi$ and let $\mathfrak{U}$ be an ultrafilter. By Lemma 2.6, $\operatorname{Con}_\beta(\mathfrak{X}, \mathfrak{d})_{\mathfrak{U}}=\operatorname{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ to within an isomorphism of $\Omega \times \mathbb{C}$-Banach complexes. Using Theorem 3.3, infer $\sigma(\operatorname{Con}_\beta(\mathfrak{X}, \mathfrak{d}))=\sigma(\operatorname{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}))$. Moreover,

$$\sigma(\operatorname{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}))=\bigcup_{k \geq n} \Sigma_k(\operatorname{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})),$$

by virtue of Lemma 3.2. Take $(\lambda, \mu) \in \Sigma_k(\operatorname{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}))$ for some k, $k \geq n$. Then $\lambda \in \Sigma_k(\operatorname{Con}((\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}), \beta_{\mathfrak{U}}-\mu))$. By Lemma 2.5

$$\Sigma_k(\operatorname{Con}((\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}), \beta_{\mathfrak{U}}-\mu)) \subseteq \Sigma_k(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}) \cup \Sigma_{k+1}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}).$$

Therefore $\lambda \in \bigcup_{k \geq n} \Sigma_k(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}) \subseteq \sigma(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$. By using Theorem 3.3 again, we obtain that $\lambda \in \sigma(\mathfrak{X}, \mathfrak{d})$. Thus $\Pi(\sigma(\operatorname{Con}_\beta(\mathfrak{X}, \mathfrak{d}))) \subseteq \sigma(\mathfrak{X}, \mathfrak{d})$.

Conversely, take $\lambda \in \sigma(\mathfrak{X}, \mathfrak{d})$. By Lemma 4.1 (ii), and Definition 4.1, $\lambda \notin \sigma_{\pi,t}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ for some t, $t > n$. Moreover, by Theorem 3.3 and Lemma 3.2

$$\sigma(\mathfrak{X}, \mathfrak{d}) = \sigma(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}) = \bigcup_{k \geq n} \Sigma_k(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}),$$

and $\sigma_{\pi,t}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}) = \bigcup_{k \geq t} \Sigma_k(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$. Let s, $n \leq s < t$, be the greatest number such that $\lambda \in \Sigma_s(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$. Then $\operatorname{im}(d_s(\lambda)_{\mathfrak{U}})$ is closed. Indeed, if $\operatorname{im}(d_s(\lambda)_{\mathfrak{U}})$ is not closed then by Definition 3.1, $\lambda \in \sigma_{\pi,s+1}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ and, by Lemma 3.2, $\lambda \in \bigcup_{k \geq s+1} \Sigma_k(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$, which contradicts to the choice of the number s. Now, we use Lemma 4.2, there exists $\mu \in \mathbb{C}$ such that

$$(\lambda, \mu) \in \sigma_{\pi,s}(\mathrm{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})) \subseteq \sigma(\mathrm{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})).$$

By Lemma 2.6 and Theorem 3.3, $\sigma(\mathrm{Con}_{\beta_{\mathfrak{U}}}(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})) = \sigma(\mathrm{Con}_{\beta}(\mathfrak{X}, \mathfrak{d}))$, so $(\lambda, \mu) \in \sigma(\mathrm{Con}_{\beta}(\mathfrak{X}, \mathfrak{d}))$. Thus the assertion has been proven for all spectra $\sigma \in \mathfrak{S}_{\pi}$ if $(\mathfrak{X}, \mathfrak{d})$ is a chain π-stable. If $(\mathfrak{X}, \mathfrak{d})$ is a cochain Ω-Banach complex then

$$\sigma(\mathfrak{X}, \mathfrak{d}) = \overline{\sigma}(\underline{\mathfrak{X}}, \underline{\mathfrak{d}}) = \Pi(\overline{\sigma}(\mathrm{Con}_{\beta}(\underline{\mathfrak{X}}, \underline{\mathfrak{d}}))) = \Pi\left(\overline{\sigma}\left(\underline{\mathrm{Con}_{\beta}(\mathfrak{X}, \mathfrak{d})}\right)\right) = \Pi(\sigma(\mathrm{Con}_{\beta}(\mathfrak{X}, \mathfrak{d}))),$$

by virtue of Proposition 2.2.

Now let us assume that $(\mathfrak{X}, \mathfrak{d})$ is a chain δ-stable complex and $\sigma = \sigma_{\delta,n} \in \mathfrak{S}_{\delta}$. Using Lemma 4.1 (i), Theorem 3.1 and the assertion from Exercise 1, we deduce that

$$\sigma(\mathfrak{X}, \mathfrak{d}) = \sigma^*(\mathfrak{X}^*, \mathfrak{d}^*) = \Pi(\sigma^*(\mathrm{Con}_{\beta^*}(\mathfrak{X}^*, \mathfrak{d}^*))) = \Pi\left(\sigma^*(\mathrm{Con}_{\beta}(\mathfrak{X}, \mathfrak{d}))^*\right) = \Pi(\sigma(\mathrm{Con}_{\beta}(\mathfrak{X}, \mathfrak{d}))),$$

thus the assertion has been proven.◀

5. Spectral mapping properties

In this section we present cochain version of the spectral mapping properties for π-type Slodkowski spectra. As we will see below our approach strongly depends on the ultrapower functor. We have noted above (Theorem 3.1) that δ-type Slodkowski spectra of chain complexes are reduced to the π-type spectra of its dual complex. That would allow us to formulate the relevant assertions for δ-type spectra. But, it is well known that ultrapower functor and the dual functor are not compatible (see Proposition 2.1). Therefore we present δ-type spectra in the applications of suggested in this section scheme to the spectral theory of Banach Lie algebra representations.

Let $(\mathfrak{X}, \mathfrak{d})$ and $(\mathfrak{Y}, \overline{\mathfrak{d}})$ be nonnegative (co)chain parametrized Banach space complexes such that both complexes have the same first term $X = X_0 = Y_0$ ($X = X^0 = Y^0$) and let Ω and Λ be their space of parameters, respectively. We say that these complexes are δ-*spectrally connected* if there exists nonnegative cochain $\Omega \times \Lambda$-Banach bicomplex $(\mathfrak{Z}, \mathfrak{d}', \mathfrak{d}'')$ such $\left(\mathfrak{Z}_{0,\bullet}, \mathfrak{d}''_{0,\bullet}\right) = (\mathfrak{X}, \mathfrak{d})$, $\left(\mathfrak{Z}_{\bullet,0}, \mathfrak{d}'_{\bullet,0}\right) = (\mathfrak{Y}, \overline{\mathfrak{d}})$ and $\sigma\left(\mathfrak{Z}_{n,\bullet}, \mathfrak{d}''_{n,\bullet}\right) \subseteq \sigma(\mathfrak{X}, \mathfrak{d})$, $\sigma\left(\mathfrak{Z}_{\bullet,m}, \mathfrak{d}'_{\bullet,m}\right) \subseteq \sigma(\mathfrak{Y}, \overline{\mathfrak{d}})$ for all $\sigma \in \mathfrak{S}_{\delta}$, and n, $m \in \mathbb{N}$. By analogy, we say that these complexes are π-*spectrally connected* if $\left(\mathfrak{Z}^{0,\bullet}, \mathfrak{d}_{''}^{0,\bullet}\right) = (\mathfrak{X}, \mathfrak{d})$, $\left(\mathfrak{Z}^{\bullet,0}, \mathfrak{d}_{'}^{\bullet,0}\right) = (\mathfrak{Y}, \overline{\mathfrak{d}})$ and

$$\sigma\left(\mathfrak{Z}^{n,\bullet}, \mathfrak{d}_{''}^{n,\bullet}\right) \subseteq \sigma(\mathfrak{X}, \mathfrak{d}), \quad \sigma\left(\mathfrak{Z}^{\bullet,m}, \mathfrak{d}_{'}^{\bullet,m}\right) \subseteq \sigma(\mathfrak{Y}, \overline{\mathfrak{d}}), \quad \sigma \in \mathfrak{S}^{\pi},$$

for a nonnegative cochain $\Omega \times \Lambda$-Banach bicomplex $(\mathfrak{Z}, \mathfrak{d}_{'}, \mathfrak{d}_{''})$. Thus $(\mathfrak{Z}, \mathfrak{d}'(\lambda), \mathfrak{d}''(\mu))$ is a nonnegative Banach space bicomplex with the base space X for each $(\lambda, \mu) \in \Omega \times \Lambda$. Their total complexes $\mathrm{Tot}(\mathfrak{Z}, \mathfrak{d}'(\lambda), \mathfrak{d}''(\mu))$, $(\lambda, \mu) \in \Omega \times \Lambda$, define $\Omega \times \Lambda$-Banach complex $\mathrm{Tot}(\mathfrak{Z}, \mathfrak{d}', \mathfrak{d}'')$ and let $\sigma(\mathfrak{Z}, \mathfrak{d}', \mathfrak{d}'')$ denotes Slodkowski spectrum of the latter complex.

Proposition 5.1. *Let* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *be a* π*-spectrally connected cochain complexes and let* $\sigma \in \mathfrak{S}^{\pi}$. *Then* $\sigma(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime}) \subseteq \sigma(\mathfrak{X}, \mathfrak{d}) \times \sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *for a parametrized Banach space bicomplex* $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ *connecting* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$.

Proof. Let $\mathfrak{U}$ be an ultrafilter. Undoubtedly, the ultrapower $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})_{\mathfrak{U}} (= (\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime\mathfrak{U}}, \mathfrak{d}_{\prime\prime\mathfrak{U}}))$ is a parametrized Banach space bicomplex connecting $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ and $\left(\mathfrak{Y}_{\mathfrak{U}}, \overline{\mathfrak{d}}_{\mathfrak{U}}\right)$. Using Theorem 3.3 and Lemma 3.2, we infer

$$\sigma(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime}) = \sigma(\mathrm{Tot}(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})) = \sigma(\mathrm{Tot}(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})_{\mathfrak{U}}) = \sigma(\mathrm{Tot}(\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime\mathfrak{U}}, \mathfrak{d}_{\prime\prime\mathfrak{U}})) = \sigma(\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime\mathfrak{U}}, \mathfrak{d}_{\prime\prime\mathfrak{U}}),$$

and $\sigma(\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime\mathfrak{U}}, \mathfrak{d}_{\prime\prime\mathfrak{U}}) = \bigcup_{k \leq n} \Sigma^k (\mathrm{Tot}(\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime\mathfrak{U}}, \mathfrak{d}_{\prime\prime\mathfrak{U}}))$. Now take $(\lambda, \mu) \in \sigma(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$. If $\lambda \notin \sigma(\mathfrak{X}, \mathfrak{d})$ then $\lambda \notin \bigcup_{k \leq n} \Sigma^k (\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$. Since complexes $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ and $\left(\mathfrak{Y}_{\mathfrak{U}}, \overline{\mathfrak{d}}_{\mathfrak{U}}\right)$ are π-spectrally connected, it follows that $\lambda \notin \bigcup_{k \leq n} \Sigma^k \left(\mathfrak{Z}_{\mathfrak{U}}^{m,\bullet}, \mathfrak{d}_{\prime\prime\mathfrak{U}}^{m,\bullet}\right)$ for all $m \in \mathbb{Z}_+$. Thus all rows of the bicomplex $(\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime}(\lambda)_{\mathfrak{U}}, \mathfrak{d}_{\prime\prime}(\mu)_{\mathfrak{U}})$ are exact at first n terms, whence $\mathrm{Tot}(\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime}(\lambda)_{\mathfrak{U}}, \mathfrak{d}_{\prime\prime}(\mu)_{\mathfrak{U}})$ is exact at first n terms by virtue of Lemma 2.10. The latter means that $(\lambda, \mu) \notin \bigcup_{k \leq n} \Sigma^k (\mathrm{Tot}(\mathfrak{Z}_{\mathfrak{U}}, \mathfrak{d}_{\prime\mathfrak{U}}, \mathfrak{d}_{\prime\prime\mathfrak{U}}))$, or $(\lambda, \mu) \notin \sigma(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$, a contradiction. The same argument with columns of the bicomplex amounts $\mu \in \sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$.◀

Exercise 8. *Prove the chain version of the assertion from above Proposition 5.1. Namely, let* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *be a* δ*-spectrally connected chain complexes. Then* $\sigma(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime}) \subseteq \sigma(\mathfrak{X}, \mathfrak{d}) \times \sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *for a parametrized bicomplex* $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ *connecting* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$, $\sigma \in \mathfrak{S}_{\delta}$ *(use Exercise 3).*

Let us introduce the following key notions of the π-spectral mapping properties.

Definition 5.1. *Let* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *be* π*-spectrally connected complexes parametrized on the topological spaces* Ω *and* Λ*, respectively, and let* $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ *be a* $\Omega \times \Lambda$*-bicomplex connecting these complexes. By* π*-spectral mapping with respect to* $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ *we mean a continuous map* $f : \Omega \to \Lambda$ *such that*

1_{π}*) all vertical cohomology complexes*

$$0 \to H^m(\mathfrak{X}, \mathfrak{d}(\lambda)) \overset{D_{\prime}^{0,m}(\mu)}{\longrightarrow} \cdots \longrightarrow H^m\left(\mathfrak{Z}^{n,\bullet}, \mathfrak{d}_{\prime\prime}^{n,\bullet}(\lambda)\right) \overset{D_{\prime}^{n,m}(\mu)}{\longrightarrow} H^m\left(\mathfrak{Z}^{n+1,\bullet}, \mathfrak{d}_{\prime\prime}^{n+1,\bullet}(\lambda)\right) \longrightarrow \cdots,$$

of the bicomplex $(\mathfrak{Z}, \mathfrak{d}_{\prime}(\lambda), \mathfrak{d}_{\prime\prime}(\mu))$ *are exact whenever* $\mu \neq f(\lambda)$*;*

2_{π}*)* $D_{\prime}^{0,m}(f(\lambda)) = 0$ *whenever the cohomology space* $H^m(\mathfrak{X}, \mathfrak{d}(\lambda))$ *is Hausdorff.*

If just the condition 2_{π}*) is satisfied then we say that* f *is* π*-prespectral mapping.*

Let us prove the forward and backward spectral mapping theorems of π-spectrally connected complexes.

Theorem 5.1. *Let* $(\mathfrak{X}, \mathfrak{d})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *be a cochain complexes parametrized on* Ω *and* Λ*, respectively,* $\mathfrak{U}$ *an ultrafilter, and let* $\sigma \in \mathfrak{S}^{\pi}$. *If* $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ *and* $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$ *are* π*-spectrally connected and* $f : \Omega \to \Lambda$ *is* π*-prespectral mapping then* $f(\sigma(\mathfrak{X}, \mathfrak{d})) \subseteq \sigma\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$.

Proof. Let $(\mathfrak{Z}, \mathfrak{d}_{\prime}, \mathfrak{d}_{\prime\prime})$ be a $\Omega \times \Lambda$-bicomplex connecting $(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$ and $\left(\mathfrak{Y}, \overline{\mathfrak{d}}\right)$, and let $\sigma = \sigma^{\pi,n}$. Take $\lambda \in \sigma(\mathfrak{X}, \mathfrak{d})$ and let $\mu = f(\lambda)$. By Theorem 3.3 and Lemma 3.2,

$$\sigma(\mathfrak{X}, \mathfrak{d}) = \sigma(\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}) = \bigcup_{k \leq n} \Sigma^k (\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}}).$$

Choose the lowest i, such that $\lambda \in \Sigma^i (\mathfrak{X}_{\mathfrak{U}}, \mathfrak{d}_{\mathfrak{U}})$. One should note that such possibility is allowed by the π-stability of the cochain complex $(\mathfrak{X}, \mathfrak{d})$ (see Definition 4.1 and Lemma 4.1). Actually, herein $\Sigma^k(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})=\oslash$ for all negative k. Further, $\lambda\notin\sigma^{\pi,i-1}(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$, in contrary case $\lambda\in\bigcup_{k\leq i-1}\Sigma^k(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$ by virtue of Lemma 3.2. Since $(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$ and $(\mathfrak{Y},\overline{\mathfrak{d}})$ are π-spectrally connected complexes (by means of $(\mathfrak{Z},\mathfrak{d}_{\prime},\mathfrak{d}_{\prime\prime})$), it follows that $\sigma^{\pi,i-1}\left(\mathfrak{Z}^{n,\bullet},\mathfrak{d}_{\prime\prime}^{n,\bullet}\right)\subseteq\sigma^{\pi,i-1}(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}})$ for all n. Thus all rows of $(\mathfrak{Z},\mathfrak{d}_{\prime},\mathfrak{d}_{\prime\prime})$ are exact at first $i-1$ terms and $H^i(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}(\lambda))$ is a nontrivial Banach space. Moreover, the differential

$$D_{\prime}^{0,i}(\mu):H^i(\mathfrak{X}_{\mathfrak{U}},\mathfrak{d}_{\mathfrak{U}}(\lambda))\to H^i\left(\mathfrak{Z}^{1,\bullet},\mathfrak{d}_{\prime\prime}^{1,\bullet}\right),$$

of the i-th vertical cohomology complex is trivial by Definition 5.1. Then $H^k\left(\mathfrak{Z}^{\bullet,m},\mathfrak{d}_{\prime}^{\bullet,m}\right)\neq\{0\}$ for some $m\leq i$ and $k\leq i$, by virtue of Lemma 2.9. The latter means that $\mu\in\sigma\left(\mathfrak{Z}^{\bullet,m},\mathfrak{d}_{\prime}^{\bullet,m}\right)$. But $\sigma\left(\mathfrak{Z}^{\bullet,m},\mathfrak{d}_{\prime}^{\bullet,m}\right)\subseteq\sigma\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)$, therefore $\mu\in\sigma\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)$.◄

Theorem 5.2. *Let $(\mathfrak{X},\mathfrak{d})$ and $\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)$ be a π-spectrally connected Banach space complexes parametrized on Ω and Λ, respectively, $f:\Omega\to\Lambda$ a π-spectral mapping with respect to a $\Omega\times\Lambda$-bicomplex $(\mathfrak{Z},\mathfrak{d}_{\prime},\mathfrak{d}_{\prime\prime})$ connecting $(\mathfrak{X},\mathfrak{d})$ and $\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)$, and let $\sigma\in\mathfrak{S}^{\pi}$. If $\sigma\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)=\Pi_{\Lambda}(\sigma(\mathfrak{Z},\mathfrak{d}_{\prime},\mathfrak{d}_{\prime\prime}))$ then $\sigma\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)\subseteq f(\sigma(\mathfrak{X},\mathfrak{d}))$, where $\Pi_{\Lambda}:\Omega\times\Lambda\to\Lambda$ is the canonical projection.*

Proof. Take $\mu\in\sigma\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)$. By assumption, $(\lambda,\mu)\in\sigma(\mathfrak{Z},\mathfrak{d}_{\prime},\mathfrak{d}_{\prime\prime})$ for some $\lambda\in\Omega$. Then $\lambda\in\sigma(\mathfrak{X},\mathfrak{d})$ by virtue of Proposition 5.1. If $f(\lambda)\neq\mu$ then all vertical cohomology complexes of the bicomplex $(\mathfrak{Z},\mathfrak{d}_{\prime}(\lambda),\mathfrak{d}_{\prime\prime}(\mu))$ are exact by Definition 5.1. Then $\operatorname{Tot}(\mathfrak{Z},\mathfrak{d}_{\prime}(\lambda),\mathfrak{d}_{\prime\prime}(\mu))$ is an exact complex (see Exercise 4). But the latter means that $(\lambda,\mu)\notin\sigma_{\mathrm{t}}(\mathfrak{Z},\mathfrak{d}_{\prime},\mathfrak{d}_{\prime\prime})$. In particular, $(\lambda,\mu)\notin\sigma(\mathfrak{Z},\mathfrak{d}_{\prime},\mathfrak{d}_{\prime\prime})$, a contradiction. Therefore, $\mu=f(\lambda)\in f(\sigma(\mathfrak{X},\mathfrak{d}))$. Thus $\sigma\left(\mathfrak{Y},\overline{\mathfrak{d}}\right)\subseteq f(\sigma(\mathfrak{X},\mathfrak{d}))$.◄

6. Ultraspectra of Banach Lie algebra representations

In this section, we consider a particular case of parametrized Banach space complexes. Namely, we focus on a fixed Banach module over a Banach Lie algebra, which generates Banach space complex parametrized at the character space of the Lie algebra. We investigate nonvoidness of spectra of this parametrized Banach complex.

A *normed Lie algebra* (resp., *Banach Lie* (shortly, B-L) *algebra*) E is a normed (resp., Banach) space and a Lie algebra with the continuous Lie brackets $[\cdot,\cdot]:E\times E\to E$, $(a,b)\mapsto[a,b]$. A *Banach module over a B-L algebra E* (shortly, *a Banach E-module*) is a Banach space X with a bounded Lie representation $\alpha:E\to\mathcal{B}(X)$. To indicate the Lie representation, we shortly say that the pair (X,α) is a Banach E-module. A functional $\lambda\in E^*$ is said to be a *Lie character of* E, if $\lambda([E,E])=0$. The space of all Lie characters (equipped with the $*$-weak topology) of a B-L algebra E is denoted by $\Delta(E)$ $(\subseteq E^*)$. The *dual module to* X is defined as the pair (X^*,α^*), where $\alpha^*:E^{\mathrm{op}}\to\mathcal{B}(X^*)$, $\alpha^*(a)=\alpha(a)^*$ is the dual Lie representation. A Banach E-module (X,α) generates the following chain Banach space complex

$$C_{\bullet}(\alpha):0\leftarrow X\xleftarrow{d_0}X\widehat{\otimes}E\xleftarrow{d^1}\cdots\xleftarrow{d^{n-1}}X\widehat{\otimes}\wedge^n E\xleftarrow{d^n}\cdots,$$

with the differential

$$d^n(x\otimes\underline{a})=\sum_{i=1}^{n+1}(-1)^{i+1}\alpha(a_i)x\otimes\underline{a}_i+\sum_{i<j}(-1)^{i+j-1}x\otimes[a_i,a_j]\wedge\underline{a}_{i,j},$$

where $\underline{a}=a_1\wedge\ldots\wedge a_{n+1}\in\wedge^{n+1}E$. If $\dim(E)<\infty$ then the latter complex is known as the Koszul complex of the E-module X and it denoted by $\operatorname{Kos}(X,\alpha)$. The E-module (X,α) generates also cochain complex

$$C^{\bullet}(\alpha): 0 \to X \overset{d^0}{\to} C(E,X) \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} C^n(E,X) \overset{d^n}{\to} \cdots,$$

with the differential

$$d^n\omega(\underline{a}) = \sum_{i=1}^{n+1} (-1)^{i+1} \alpha(a_i)\,\omega(\underline{a}_i) + \sum_{i<j} (-1)^{i+j} \omega\left([a_i, a_j] \wedge \underline{a}_{i,j}\right),$$

where $\omega \in C^n(E,X) = \mathcal{L}(\wedge^n E, X)$.

Exercise 9. *Prove that $C_{\bullet}(\alpha)^* = C^{\bullet}(\alpha^*)$ up to an (isometric) isomorphism in $\overline{\mathbf{BS}}$.*

The parametrized at the space $\Delta(E)$ (co)chain Banach space complex $C_{\bullet}(\alpha-\lambda)$ (resp., $C^{\bullet}(\alpha-\lambda)$), $\lambda \in \Delta(E)$, is denoted by $\mathcal{C}_{\bullet}(\alpha)$ (resp., $\mathcal{C}^{\bullet}(\alpha)$). It worth to note that $\mathcal{C}_{\bullet}(\alpha)$ (resp., $\mathcal{C}^{\bullet}(\alpha)$) is δ-stable (resp., π-stable) $\Delta(E)$-Banach complex (see Definition 4.1). If (X,α) and (Y,β) are Banach E-modules and $\varphi: (X,\alpha) \to (Y,\beta)$ is a bounded E-module morphism (that is, $\varphi(\alpha(a)x) = \beta(a)\varphi(x)$), then φ can be extended up to a morphism of Banach space complexes

$$\varphi_{\bullet}(\lambda): C_{\bullet}(\alpha-\lambda) \to C_{\bullet}(\beta-\lambda), \quad \varphi_{\bullet}(\lambda)\, x \otimes \underline{a} = \varphi(x) \otimes \underline{a},$$

(resp., $\varphi^{\bullet}(\lambda): C^{\bullet}(\alpha-\lambda) \to C^{\bullet}(\beta-\lambda)$, $\varphi^{\bullet}(\lambda)\omega = \varphi\omega$), for each $\lambda \in \Delta(E)$. Therefore the assignment

$$\varphi_{\bullet}: \mathcal{C}_{\bullet}(\alpha) \to \mathcal{C}_{\bullet}(\beta), \quad \varphi_{\bullet} = \{\varphi_{\bullet}(\lambda)\},$$

(resp., $\varphi^{\bullet}: \mathcal{C}^{\bullet}(\alpha) \to \mathcal{C}^{\bullet}(\beta)$, $\varphi^{\bullet} = \{\varphi^{\bullet}(\lambda)\}$) defines a morphism of $\Delta(E)$-Banach complexes.

Lemma 6.1. *Assume that*

$$0 \leftarrow (X,\alpha) \overset{\varphi}{\longleftarrow} (Y,\beta) \overset{\psi}{\longleftarrow} (Z,\gamma) \leftarrow 0,$$

is a complex of Banach E-modules, which is either admissible or exact and $E \in$ Flat *(resp., $E \in$* Proj*). The sequence of $\Delta(E)$-Banach complexes*

$$0 \leftarrow \mathcal{C}_{\bullet}(\alpha) \overset{\varphi_{\bullet}}{\longleftarrow} \mathcal{C}_{\bullet}(\beta) \overset{\psi_{\bullet}}{\longleftarrow} \mathcal{C}_{\bullet}(\gamma) \leftarrow 0,$$

(resp., $0 \leftarrow \mathcal{C}^{\bullet}(\alpha) \overset{\varphi^{\bullet}}{\longleftarrow} \mathcal{C}^{\bullet}(\beta) \overset{\psi^{\bullet}}{\longleftarrow} \mathcal{C}^{\bullet}(\gamma) \leftarrow 0$) is exact.

Proof. In both assumptions, the sequence of Banach space complexes

$$0 \leftarrow C_{\bullet}(\alpha-\lambda) \overset{\varphi_{\bullet}(\lambda)}{\longleftarrow} C_{\bullet}(\beta-\lambda) \overset{\psi_{\bullet}(\lambda)}{\longleftarrow} C_{\bullet}(\gamma-\lambda) \leftarrow 0,$$

remains exact for each $\lambda \in \Delta(E)$, by Lemmas 2.7, 2.8. But, the latter means that the required (co)chain complex of $\Delta(E)$-Banach complexes is exact.◀

Let us introduce spectra of Banach E-modules or bounded Lie representations.

Definition 6.1. *Let (X,α) be a Banach E-module, $\sigma(\mathcal{C}_{\bullet}(\alpha))$, $\sigma \in \mathfrak{S}_{\cdot}$, spectra of the chain $\Delta(E)$-Banach complex $\mathcal{C}_{\bullet}(\alpha)$, and let $\sigma(\mathcal{C}^{\bullet}(\alpha))$, $\sigma \in \mathfrak{S}^{\cdot}$, be spectra of the cochain $\Delta(E)$-Banach complex $\mathcal{C}^{\bullet}(\alpha)$. We call these sets Slodkowski spectra (resp., Taylor spectrum) of the E-module X or the Lie representation α and denote them by $\sigma(\alpha)$, $\sigma \in \mathfrak{S}$.*

Since $C_{\bullet}(\alpha-\lambda)^{*}=C^{\bullet}(\alpha^{*}-\lambda)$ (see Exercise 9) to within an isomorphism in $\overline{\mathbf{BS}}$, it follows using Theorem 3.1 that $\sigma(\alpha)=\sigma^{*}(\alpha^{*})$ for all $\sigma\in\mathfrak{S}_{\cdot}$.

For brevity, further we consider only cochain case. Actually, all suggested below assertions have their chain versions too, we left them to the reader.

The *point spectrum* $\sigma^{\mathrm{p}}(\alpha)$ (resp., *approximate point spectrum* $\sigma^{\mathrm{ap}}(\alpha)$) of a representation $\alpha:E\rightarrow\mathcal{L}(X)$ is defined (see [69], [56]) as a set of functions $\lambda:E\rightarrow\mathbb{C}$ such that there exists $x\in X$, $\alpha(a)x=\lambda(a)x$ (resp., there exists a net $(x_{\gamma})\subseteq X$, $\|x_{\gamma}\|=1$, $(\alpha(a)-\lambda(a))x_{\gamma}\rightarrow 0$) for all $a\in E$. It is clear that $\sigma^{\mathrm{p}}(\alpha)=\Sigma^{0}(\mathcal{C}^{\bullet}(\alpha))$, and $\lambda\in E^{*}$, $\lambda(a)\in\sigma(\alpha(a))$, $a\in E$, whenever $\lambda\in\sigma^{\mathrm{ap}}(\alpha)$. Moreover, $\alpha([a,b])x_{\gamma}\rightarrow 0$ for all $a,b\in E$, that is, $\lambda\in\Delta(E)$. For each $S\subseteq\mathcal{B}(X)$, we define $\sigma^{\mathrm{p}}(S)$ and $\sigma^{\mathrm{ap}}(S)$ as the relevant spectra of the identity representation of the closed Lie subalgebra in $\mathcal{B}(X)$ generated by S. Undoubtedly,

$$\sigma^{\mathrm{p}}(\alpha(E))\cdot\alpha=\sigma^{\mathrm{p}}(\alpha),\quad \sigma^{\mathrm{ap}}(\alpha(E))\cdot\alpha=\sigma^{\mathrm{ap}}(\alpha)\quad\text{and}\quad \sigma^{\mathrm{p}}(\alpha)\subseteq\sigma^{\pi,0}(\alpha)\subseteq\sigma^{\mathrm{ap}}(\alpha).$$

If E is finite-dimensional then $\sigma^{\pi,0}(\alpha)=\sigma^{\mathrm{ap}}(\alpha)$. But, in the general case $\sigma^{\pi,0}(\alpha)\neq\sigma^{\mathrm{ap}}(\alpha)$.

Example 6.1. *Let $E=\ell_1$ be a commutative B-L algebra and let $\alpha:E\rightarrow\mathcal{B}(X)$, $\alpha(f_n)=P_n$, be a bounded representation on a separable Hilbert space X, where $\{f_n\}$ is the canonical basis in E, P_n is the orthogonal projection onto the linear span of first n vectors with respect to a Hilbert basis $\{e_m\}_{m\in\mathbb{N}}$ in X. Then $\lim_m P_n e_m=0$ for all n, that is, $0\in\sigma^{\mathrm{ap}}(\alpha)$. Nevertheless, $0\notin\sigma^{\pi,0}(\alpha)$. Indeed, take $x=\sum_{m=1}^{\infty}a_m e_m\in X$. Then*

$$\left\|d^{0}x\right\|=\sup_{n\in\mathbb{N}}\left\|\left(d^{0}x\right)f_n\right\|_X=\sup_{n\in\mathbb{N}}\left\|P_n x\right\|_X=\sup_{n\in\mathbb{N}}\left(\sum_{m=1}^{n}\left|a_m\right|^2\right)^{1/2}\geq\|x\|_X,$$

where $d^0:X\rightarrow\mathcal{L}(E,X)$, $\left(d^0x\right)f_n=P_nx$, is the differential of the complex $C^{\bullet}(\alpha)$. Thus $\operatorname{im}\left(d^0\right)$ *is closed, therefore $0\notin\sigma^{\pi,0}(\alpha)$.*

Now, let $\mathfrak{U}$ be an ultrafilter, and let $X_{\mathfrak{U}}$ be the ultrapower of the Banach space X. A Lie representation $\alpha:E\rightarrow\mathcal{B}(X)$ induces the Lie representation

$$\alpha_{\mathfrak{U}}:E\rightarrow\mathcal{L}(X_{\mathfrak{U}}),\quad \alpha_{\mathfrak{U}}(a)=\alpha(a)_{\mathfrak{U}},$$

called *the ultrapower of* α. Thus $(X_{\mathfrak{U}},\alpha_{\mathfrak{U}})$ is a Banach E-module.

Definition 6.2. *Let (X,α) be a Banach E-module and $\sigma\in\mathfrak{S}^{\cdot}$. We define ultraspectrum $\sigma_{\mathrm{u}}(\alpha)$ of the module (X,α) (or the Lie representation α) as the union of spectra $\sigma(\alpha_{\mathfrak{U}})$ taken over all countably incomplete ultrafilters $\mathfrak{U}$, and we write $\sigma_{\mathrm{u}}(\alpha)=\sigma_{\mathrm{u}}^{\pi,n}(\alpha)$ (resp., $\sigma_{\mathrm{u}}^{\delta,n}(\alpha)$) whenever $\sigma=\sigma^{\pi,n}\in\mathfrak{S}^{\pi}$ ($\sigma=\sigma^{\delta,n}\in\mathfrak{S}^{\delta}$). The relevant union of all $\Sigma^{0}(\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}}))$ we call the ultrapoint spectrum of α and denote it by $\sigma^{\mathrm{up}}(\alpha)$.*

Lemma 6.2. *If $\dim(E)<\infty$ then $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}=\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\sigma_{\mathrm{u}}(\alpha)=\sigma(\alpha)$ for all $\sigma\in\mathfrak{S}^{\cdot}$.*

Proof. Consider the following linear operator

$$\varphi_n:\mathcal{L}(\wedge^n E,X)_{\mathfrak{U}}\rightarrow\mathcal{L}(\wedge^n E,X_{\mathfrak{U}}),\quad (\varphi_n[\omega_i])u=[\omega_i(u)],\quad u\in\wedge^n E.$$

Note that $\varphi_{n+1}(d^n)_{\mathfrak{U}}=d^n_{\mathfrak{U}}\varphi_n$, where $d^n_{\mathfrak{U}}$ is the differential of $C^{\bullet}(\alpha_{\mathfrak{U}})$. The assumption $\dim(E)<\infty$ implies that φ_n is an isometry for all n (see [48, Lemma 7.4]). Thus $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}=\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ to within an isomorphism in $\overline{\mathbf{BS}}$ and therefore $\sigma(\alpha_{\mathfrak{U}})=\sigma(\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}}))=\sigma(\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}})$. But, $\sigma(\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}})=\sigma(\mathcal{C}^{\bullet}(\alpha))=\sigma(\alpha)$ by virtue of Theorem 3.3, that is, $\sigma_{\mathrm{u}}(\alpha)=\sigma(\alpha)$.◀

Theorem 6.1. *Let E be a B-L algebra and let (X,α) be a Banach E-module. Then*

$$\sigma_{\mathrm{u}}^{\pi,0}(\alpha)=\sigma^{\mathrm{up}}(\alpha)=\sigma^{\mathrm{ap}}(\alpha).$$

Proof. It is clear that $\sigma^{\mathrm{up}}(\alpha)\subseteq\sigma_{\mathrm{u}}^{\pi,0}(\alpha)\cap\sigma^{\mathrm{ap}}(\alpha)$. Let us prove that $\sigma^{\mathrm{ap}}(\alpha)\subseteq\sigma^{\mathrm{up}}(\alpha)$. Take $\lambda\in\sigma^{\mathrm{ap}}(\alpha(E))$. We should prove that $\lambda\cdot\alpha\in\sigma^{\mathrm{up}}(\alpha)$. By definition, there exists a family $\{x_s\}_{s\in S}\subseteq X$, such that $\|x_s\|=1$ and $\lim_{\mathcal{F}}(T-\lambda(T))x_s=0$ for each $T\in\alpha(E)$, where $\mathcal{F}$ is a filter in the index set S. If $\mathfrak{U}$ is an ultrafilter in S majorized $\mathcal{F}$ then $\lim_{\mathfrak{U}}(T-\lambda(T))x_s=0$, $T\in\alpha(E)$.

If $\mathfrak{U}$ is a trivial filter then there exists a joint eigenvector $x\in X$, $\|x\|=1$, $(T-\lambda(T))x=0$, $T\in\alpha(E)$. Then $(\alpha(a)_{\mathfrak{V}}-\lambda(\alpha(a)))[x]=0$, $a\in E$, for each countably incomplete ultrafilter $\mathfrak{V}$, that is, $d_{\mathfrak{V}}^{0}(\lambda\cdot\alpha)[x]=0$, where $d_{\mathfrak{V}}^{0}(\lambda\cdot\alpha)$ is the differential of the complex $C^{\bullet}(\alpha_{\mathfrak{V}}-\lambda\cdot\alpha)$. Thus $\lambda\cdot\alpha\in\Sigma^{0}(\mathcal{C}(\alpha_{\mathfrak{V}}))\subseteq\sigma^{\mathrm{up}}(\alpha)$.

Now, let us assume that $\mathfrak{U}$ is a nontrivial (but not necessarily countably incomplete) ultrafilter. Then S is an infinite set. Let us replace $\mathfrak{U}$ by a countably incomplete ultrafilter. Take a countably incomplete ultrafilter $\mathfrak{V}$ in $\mathbb{N}$. By Lemma 2.1, $\mathfrak{U}\times\mathfrak{V}$ is countably incomplete. Now assume that $x_{(s,n)}=x_s$, $n\in\mathbb{N}$. Then $\lim_{\mathfrak{U}\times\mathfrak{V}}(T-\lambda(T))x_{(s,n)}=0$ for each $T\in\alpha(E)$, that is, $(T_{\mathfrak{U}\times\mathfrak{V}}-\lambda(T))\left[x_{(s,n)}\right]=0$ and $\lambda\cdot\alpha\in\sigma^{\mathrm{up}}(\alpha)$. Thus $\sigma^{\mathrm{ap}}(\alpha)\subseteq\sigma^{\mathrm{up}}(\alpha)$.

It remains to prove that $\sigma_{\mathrm{u}}^{\pi,0}(\alpha)\subseteq\sigma^{\mathrm{up}}(\alpha)$. Take $\lambda\in\sigma^{\pi,0}(\alpha_{\mathfrak{U}})\backslash\Sigma^{0}(\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}}))$, where $\mathfrak{U}$ is an ultrafilter in a certain set S. There exists a sequence $\{[x_s^n]\}_{n\in\mathbb{N}}\subset X_{\mathfrak{U}}$, $\|[x_s^n]\|=1$, such that $\lim_n[(\alpha(a)-\lambda(a))x_s^n]=0$, $a\in E$, for $\operatorname{im}\left(d_{\mathfrak{U}}^{0}(\lambda)\right)$ is not closed. Let $\mathfrak{V}$ be an ultrafilter in $\mathbb{N}$ majorized the Fréchet filter. It is clear that $\lim_{\mathfrak{V}}[(\alpha(a)-\lambda(a))x_s^n]=0$. Consider the Lie representation $(\alpha_{\mathfrak{U}})_{\mathfrak{V}}:E\to\mathcal{B}((X_{\mathfrak{U}})_{\mathfrak{V}})$. By Lemma 2.1, $X_{\mathfrak{U}\times\mathfrak{V}}=(X_{\mathfrak{U}})_{\mathfrak{V}}$ and $\alpha_{\mathfrak{U}\times\mathfrak{V}}=(\alpha_{\mathfrak{U}})_{\mathfrak{V}}$. Take $[[x_s^n]]\in X_{\mathfrak{U}\times\mathfrak{V}}$. Then $\|[[x_s^n]]\|=\lim_{\mathfrak{V}}\|[x_s^n]\|=1$ and

$$\|[[(\alpha(a)-\lambda(a))x_s^n]]\|=\lim_{\mathfrak{U}\times\mathfrak{V}}\|(\alpha(a)-\lambda(a))x_s^n\|=\lim_{\mathfrak{V}}\|[(\alpha(a)-\lambda(a))x_s^n]\|=0.$$

Thus $d_{\mathfrak{U}\times\mathfrak{V}}^{0}(\lambda)[[x_s^n]]=0$ or $\lambda\in\Sigma^{0}(\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}\times\mathfrak{V}}))$. It follows that $\lambda\in\sigma^{\mathrm{up}}(\alpha)$.◀

Corollary 6.1. *Let E be a solvable B-L algebra, (X,α) a Banach E-module and let $\sigma\in\mathfrak{S}^{\cdot}$. The ultraspectrum $\sigma^{\mathrm{u}}(\alpha)$ is nonvoid.*

Proof. Indeed, by assumption, $\alpha(E)$ is a solvable Lie algebra of operators. By [47], $\sigma^{\mathrm{ap}}(\alpha(E))\neq\emptyset$. Then, also, $\sigma^{\mathrm{ap}}(\alpha)\neq\emptyset$. Using Theorem 6.1, we infer that $\sigma_{\mathrm{u}}^{\pi,0}(\alpha)=\sigma^{\mathrm{ap}}(\alpha)$. It remains to note that $\sigma_{\mathrm{u}}^{\pi,0}(\alpha)\subseteq\sigma_{\mathrm{u}}(\alpha)$.◀

Theorem 6.2. *Let (X,α) be a Banach E-module. There exists an ultrafilter $\mathfrak{U}$ such that*

$$\sigma^{\mathrm{p}}(\alpha_{\mathfrak{U}})=\sigma^{\pi,0}(\alpha_{\mathfrak{U}})=\sigma^{\mathrm{ap}}(\alpha).$$

In particular, $\sigma_{\mathrm{u}}^{\pi,0}(\alpha)=\sigma^{\pi,0}(\alpha_{\mathfrak{U}})$.

Proof. Let S be the set of all pairs $s=\left(N,n^{-1}\right)$, where N is a finite subset in E and $n\in\mathbb{N}$. Assume that $s_1\leq s_2$ whenever $N_1\subseteq N_2$ and $n_1\leq n_2$, where $s_i=\left(N_i,n_i^{-1}\right)$. Then $(S,\leq)$ is a poset and for each pair $s_1,s_2\in S$ there exists $s_3\in S$, $\sup\{s_1,s_2\}\leq s_3$. Thus the set of all sections $\Gamma(s)$, $s\in S$ (here $\Gamma(s)=\{\gamma\in S:s\leq\gamma\}$) generates a filter base in S. Let $\mathfrak{U}$ be an ultrafilter majorized this filter base. Then $\mathfrak{U}$ is countably incomplete. Indeed, let $S_n=\left\{s\in S:s=\left(N,n^{-1}\right)\right\}$, $n\in\mathbb{N}$. Evidently, $S=\bigcup_n S_n$ and $S_n\cap\Gamma(s_n)=\emptyset$ for each $n\in\mathbb{N}$, where $s_n=\left(N,(n+1)^{-1}\right)$. Thereby $S_n\notin\mathfrak{U}$.

Now let us prove that $\sigma^{\mathrm{ap}}(\alpha) \subseteq \sigma^{\mathrm{p}}(\alpha_{\mathfrak{U}})$. Assume that $0 \in \sigma^{\mathrm{ap}}(\alpha)$. By definition, for each finite subset $N \subset E$ and $n \in \mathbb{N}$ one can find $x \in X$, $\|x\| = 1$, such that $\|\alpha(N)x\| < n^{-1}$. We set $x_s = x$ whenever $s = (N, n^{-1})$. Then $\alpha(a)x_s \to 0$ by the section filter in S, $a \in E$, which in turn implies that $\lim_{\mathfrak{U}} \alpha(a)x_s = 0$ or $\alpha(a)_{\mathfrak{U}}[x_s] = 0$, $a \in E$. With $\|[x_s]\| = 1$ in mind, infer $0 \in \sigma^{\mathrm{p}}(\alpha_{\mathfrak{U}})$. Thus $\sigma^{\mathrm{ap}}(\alpha) = \sigma^{\mathrm{p}}(\alpha_{\mathfrak{U}})$. By Theorem 6.1, $\sigma^{\pi,0}(\alpha_{\mathfrak{U}}) = \sigma_{\mathrm{u}}^{\pi,0}(\alpha)$.◀

7. Quasinilpotent B-L algebras

In this section, we investigate the projection property of spectra $\sigma(\alpha)$, $\sigma \in \mathfrak{S}$, of a quasinilpotent B-L algebra representation α. A Banach-Lie algebra E with quasinilpotent operators $\mathrm{ad}(a) \in \mathcal{B}(E)$, $\mathrm{ad}(a)b = [a,b]$ $(a \in E)$, of its adjoint representation is called a *quasinilpotent B-L algebra* (see [76]). In the sequel, we shall use B-L algebras for which $\mathrm{sp}(\mathrm{ad}(a)) = 0$, but only for elements $a \in S$ from a subset $S \subseteq E$ of topological Lie generators (that is, the Lie subalgebra generated by S is dense in E). In this case, we say that E is a *quasinilpotent B-L algebra generated by S*. We will especially be interested in finitely generated quasinilpotent B-L algebras (see examples in [33]).

7.1. The Lie representation θ

Let E be a B-L algebra and let I be its closed ideal. Then its exterior power $\wedge^n I$ (see Subsection 2.5) turnes into a Banach E-module by means of the Lie representation

$$T_{n,I} : E \to \mathcal{B}(\wedge^n I), \quad T_{n,I}(a)(\underline{u}) = \sum_{i=1}^{n} (-1)^{i-1} (\mathrm{ad}(a) u_i) \wedge \underline{u}_i,$$

where $\underline{u} = u_1 \wedge \ldots \wedge u_n \in \wedge^n I$. The latter extends the adjoint representation of E. If (X, α) is a Banach E-module, then $C^n(I, X)$ is a Banach E-module with the θ-representation

$$\theta_{n,I} : E \to \mathcal{B}(C^n(I,X)), \quad \theta_{n,I}(a) = L_{\alpha(a)} - R_{T_{n,I}(a)},$$

where $L_{\alpha(a)}$ (resp., $R_{T_{n,I}(a)}$) is the left (resp., right) multiplication operator. We set $T_n = T_{n,E}$ and $\theta_n = \theta_{n,E}$. Respectively, $X \widehat{\otimes} \wedge^n E$ is a E-module via the representation

$$\vartheta_n : E \to \mathcal{B}\left(X \widehat{\otimes} \wedge^n E\right), \quad \vartheta_n(a) = \alpha(a) \otimes 1 + 1 \otimes T_n(a).$$

Let us remind the following well known (see [6, Ch. 1]) cohomological formulae

$$d^n \theta_n(a) = \theta_{n+1}(a) d^n, \tag{7.1}$$

$$d^{n-1} i_n(a) + i_{n+1}(a) d^n = \theta_n(a), \tag{7.2}$$

$$\theta_{n-1}(a) i_n(b) - i_n(b) \theta_n(a) = i_n([a,b]), \tag{7.3}$$

where d^n is the differential of the complex $C^\bullet(\alpha)$ and

$$i_n(a) : C^n(E,X) \to C^{n-1}(E,X), \quad (i_n(a)\omega) b = \omega(a \wedge b),$$

is so called *homotopy operator*. The relevant homological formulae are also true, namely,

$$d_{n-1} \vartheta_n(a) = \vartheta_{n-1}(a) d_{n-1},$$

$$d_n \kappa_n(a) + \kappa_{n-1}(a) d_{n-1} = \vartheta_n(a),$$

$$\vartheta_{n+1}(a) \kappa_n(b) - \kappa_n(b) \vartheta_n(a) = \kappa_n([a,b]),$$

where $\kappa_n(a) : X \widehat{\otimes} \wedge^n E \to X \widehat{\otimes} \wedge^{n+1} E$, $\kappa_n(a)(x \otimes \underline{u}) = x \otimes a \wedge \underline{u}$.

Lemma 7.1. *Let E be a B-L algebra. If* $\operatorname{sp}(\operatorname{ad}(a)) = \{0\}$ *for some* $a \in E$, *then* $\operatorname{sp}(T_n(a)) = \{0\}$, $n \in \mathbb{Z}_+$. *Moreover,* $\operatorname{sp}(\theta_n(a)) = \operatorname{sp}(\alpha(a))$ *for a Banach E-module* (X, α).

Proof. Let $\operatorname{ad}_i(a) = 1 \otimes \ldots \otimes \operatorname{ad}(a) \otimes \ldots \otimes 1 \in \mathcal{B}\left(E^{\widehat{\otimes} n}\right)$, $1 \leq i \leq n$, where $\operatorname{ad}(a)$ stands at i-th place, and let $S_n(a) = \sum_{i=1}^{n} \operatorname{ad}_i(a)$. It is beyond a doubt $S_n(a)$ is a sum of mutually commuting operators. By assumption, $\operatorname{sp}(\operatorname{ad}_i(a)) = \{0\}$, therefore $\operatorname{sp}(S_n(a)) = \{0\}$. Evidently, $A_n S_n(a) = S_n(a) A_n$, where $A_n \in \mathcal{B}\left(E^{\widehat{\otimes} n}\right)$ is the projection onto $\wedge^n E$ defined in Subsection 2.5. Moreover, $T_n(a)$ is the restriction of the operator $S_n(a)$ to the invariant subspace $\wedge^n E$, whence $\operatorname{sp}(T_n(a)) = \{0\}$.

Finally, let (X, α) be a Banach E-module. Since $\left[L_{\alpha(a)}, R_{T_n(a)}\right] = 0$ and $R_{T_n(a)}$ is a quasinilpotent operator, it follows using spectral (for instance, Taylor spectrum of commuting families) mapping theorem that $\operatorname{sp}(\theta_n(a)) = \operatorname{sp}\left(L_{\alpha(a)}\right) = \operatorname{sp}(\alpha(a))$.◄

Exercise 10. *If* (X, α) *is a Banach E-module and* $\operatorname{sp}(\operatorname{ad}(a)) = \{0\}$ *for a certain* $a \in E$, *then* $\operatorname{sp}(\vartheta_n(a)) = \operatorname{sp}(\alpha(a))$.

Lemma 7.2. *Let E be a B-L algebra,* (X, α) *a Banach E-module and let* $\rho(\theta_n(\operatorname{ball} E))$ *be the joint spectral radius of the bounded set* $\theta_n(\operatorname{ball} E)$. *Then*

$$\rho(\theta_n(\operatorname{ball} E)) \leq \rho(\alpha(\operatorname{ball} E)) + \rho(T_n(\operatorname{ball} E)).$$

Moreover, $\rho(T_n(\operatorname{ball} E)) = 0$ *whenever E is a finite-dimensional nilpotent Lie algebra.*

Proof. Let $M = L_{\alpha(\operatorname{ball} E)}$ and let $N = -R_{T_n(\operatorname{ball} E)}$. By definition, $\theta_n(u) = L_{\alpha(u)} - R_{T_n(u)}$, $u \in E$. Therefore, $\theta_n(\operatorname{ball} E) \subseteq M + N$. Moreover, $[M, N] = \{0\}$. Using Lemma 2.14, infer that

$$\rho(\theta_n(\operatorname{ball} E)) \leq \rho(M + N) \leq \rho(M) + \rho(N) \leq \rho(\alpha(\operatorname{ball} E)) + \rho(T_n(\operatorname{ball} E)).$$

Now assume that $\dim(E) < \infty$ and E is nilpotent. By Lemma 7.1, $T_n(E)$ is a nilpotent Lie algebra comprising nilpotent operators acting on the finite-dimensional space $\wedge^n E$. Thereby, $T_n(E)$ generates a nilpotent associative subalgebra in $\mathcal{B}(\wedge^n E)$ by virtue of Engel theorem. Then $T_n(E)^k = \{0\}$ for sufficiently large k. It follows that $\rho(T_n(\operatorname{ball} E)) = 0$.◄

Exercise 11. *Prove that* $\rho(\vartheta_n(\operatorname{ball} E)) \leq \rho(\alpha(\operatorname{ball} E)) + \rho(T_n(\operatorname{ball} E))$.

Lemma 7.3. *Let E be a quasinilpotent B-L algebra and let* $\sigma \in \mathfrak{S}$. *If* $\lambda \in \sigma(\alpha)$ *then* $\lambda(a) \in \operatorname{sp}(\alpha(a))$ *for all* $a \in E$. *In particular, the spectrum* $\sigma(\alpha)$ *is precompact, and it is compact whenever* $\dim(E) < \infty$.

Proof. We prove cochain version leaving the chain version to the reader. By Lemma 7.1, $\operatorname{sp}(\theta_n(a)) = \operatorname{sp}(\alpha(a))$, $n \in \mathbb{Z}_+$. If $\lambda(a) \notin \operatorname{sp}(\alpha(a))$ for a certain $a \in E$, then $\lambda(a) \notin \bigcup_{n \in \mathbb{Z}_+} \operatorname{sp}(\theta_n(a))$. But

$$d^{n-1}(\lambda)\, i_n(a) + i_{n+1}(a)\, d^n(\lambda) = \theta_n(a) - \lambda(a),$$

by virtue of (7.2), where $d^n(\lambda)$ is the differential of the complex $C^\bullet(\alpha - \lambda)$. Taking into account that $\theta_n(a) - \lambda(a)$ is invertible and using (7.1), infer that

$$d^{n-1}(\lambda)\left(\theta_{n-1}(a) - \lambda(a)\right)^{-1} i_n(a) + \left(\theta_n(a) - \lambda(a)\right)^{-1} i_{n+1}(a)\, d^n(\lambda) = 1,$$

which in turn follows that the complex $C^\bullet(\alpha - \lambda)$ is admissible. In particular, $\lambda \notin \Sigma^n(\mathcal{C}^\bullet(\alpha))$ for all n. Therefore $\lambda \notin \sigma_{\mathrm{t}}(\alpha)$, a contradiction.

Thus $\lambda(a) \in \mathrm{sp}(\alpha(a))$ for every $a \in E$. In particular, $\sigma(\alpha)$ is embedded into the topological direct product $\prod_{a\in E} \mathrm{sp}(\alpha(a))$. Bearing in mind that $\sigma(\alpha)$ furnished with the $*$-weak topology (inherited from $\Delta(E)$), we conclude that $\sigma(\alpha)$ is a $*$-weak precompact subset in E^*.

Now let $\dim(E) < \infty$. Then $\mathcal{C}^\bullet(\alpha)$ is a finite parametrized $\Delta(E)$-complex, whence its spectrum $\sigma(\mathcal{C}^\bullet(\alpha))$ is closed due to Proposition 3.1. Consequently, $\sigma(\alpha)$ is a compact set.◄

Proposition 7.1. *Let* $\iota_n : \left(X\widehat{\otimes}\wedge^n E\right)^* \to \mathcal{L}(\wedge^n E, X^*)$ *be the canonical isomorphism in* **BS** *given by the rule* $\iota_n(f)(\underline{u})(x) = f(x\otimes\underline{u})$, $x \in X$, $\underline{u} \in \wedge^n E$. *Then* $\theta_n'(a)\iota_n = \iota_n\vartheta_n(a)^*$, *where* $\theta_n' : E^{\mathrm{op}} \to \mathcal{B}(C^n(E, X^*))$, $\theta_n'(a) = L_{\alpha^*(a)} - R_{T_n^{\mathrm{op}}(a)}$, *is the* θ*-representation induced by the dual representation* $\alpha^* : E^{\mathrm{op}} \to \mathcal{B}(X^*)$.

Proof. Let $T_n^{\mathrm{op}} : E^{\mathrm{op}} \to \mathcal{B}(\wedge^n E)$ be the extension of the adjoint representation of E^{op}. Note that $T_n^{\mathrm{op}}(a) = -T_n(a)$, $a \in E$. Then

$$R_{T_n^{\mathrm{op}}(a)}\iota_n(f)(\underline{u})(x) = -\iota_n(f)(T_n(a)\underline{u})(x) = -f(x\otimes T_n(a)\underline{u}) =$$

$$= -(1\otimes T_n(a))^*(f)(x\otimes\underline{u}) = -\iota_n\left((1\otimes T_n(a))^*(f)\right)(\underline{u})(x),$$

that is, $-R_{T_n^{\mathrm{op}}(a)}\iota_n = \iota_n(1\otimes T_n(a))^*$. Further

$$L_{\alpha^*(a)}\iota_n(f)(\underline{u})(x) = \alpha^*(a)(\iota_n(f)(\underline{u}))(x) = \iota_n(f)(\underline{u})(\alpha(a)x) = f(\alpha(a)x\otimes\underline{u}) =$$
$$= (\alpha(a)\otimes 1)^*(f)(x\otimes\underline{u}) = \iota_n\left((\alpha(a)\otimes 1)^*(f)\right)(\underline{u})(x),$$

that is, $L_{\alpha^*(a)}\iota_n(f) = \iota_n\left((\alpha(a)\otimes 1)^*(f)\right)$. It follows that

$$\theta_n'(a)\iota_n = \left(L_{\alpha^*(a)} - R_{T_n^{\mathrm{op}}(a)}\right)\iota_n = \iota_n\left((\alpha(a)\otimes 1)^* + (1\otimes T_n(a))^*\right) = \iota_n\vartheta_n(a)^*,$$

that is, $\theta_n'(a)\iota_n = \iota_n\vartheta_n(a)^*$ ◄.

Note that all ι_n are isomorphisms and $\left(X\widehat{\otimes}\wedge^n E\right)^* = X^*\widehat{\otimes}(\wedge^n E)^*$ whenever $\dim(E) < \infty$, and in this case, the dual representation $\vartheta^* : E^{\mathrm{op}} \to \mathcal{B}\left(\left(X\widehat{\otimes}\wedge E\right)^*\right)$, $\vartheta^*(a) = \vartheta(a)^*$ ($\vartheta(a) = \sum_n \vartheta_n(a)$), is reduced (to within an isomorphism) to the Lie representation $\theta' : E^{\mathrm{op}} \to \mathcal{B}(\mathcal{L}(\wedge E, X^*))$, $\theta'(a) = \sum_n \theta_n'(a)$, by Proposition 7.1.

Corollary 7.1. *Let E be a finite-dimensional nilpotent Lie algebra and let (X, α) be a Banach E-module. The dual representation* $\theta^* : E^{\mathrm{op}} \to \mathcal{B}\left(\mathcal{L}(\wedge E, X)^*\right)$, $\theta^*(a) = \theta(a)^*$, *is reduced (to within an isomorphism) to the* θ*-representation* $\theta' : E^{\mathrm{op}} \to \mathcal{B}(\mathcal{L}(\wedge E, X^*))$, $\theta'(a) = L_{\alpha^*(a)} - R_{T^{\mathrm{op}}(a)}$.

Proof. Let $n = \dim(E)$. Note that $\wedge^k E^* = \left(\wedge^k E\right)^*$ and the map $\gamma_k : X\otimes\wedge^k E^* \to \mathcal{L}\left(\wedge^k E, X\right)$, $\gamma_k(x\otimes f)(\underline{u}) = f(\underline{u})x$, $f \in \wedge^k E^*$, $\underline{u} \in \wedge^k E$, is an isomorphism in **BS**. Moreover, $\theta_k(a)\gamma_k = \gamma_k\left(\alpha(a)\otimes 1 - 1\otimes T_k(a)^*\right)$. Now let $\tau_{\underline{w}}^{(k)} : \wedge^k E^* \to \wedge^{n-k} E$ be an isomorphism depending on the choice of some fixed $\underline{w} \in \wedge^n E$ (see [2, Ch.1, Section 11]). Taking into account that E is a nilpotent Lie algebra, we conclude $\tau_{\underline{w}}^{(k)} T_k(a)^* = -T_{n-k}(a)\tau_{\underline{w}}^{(k)}$ by virtue of Corollary 1 from [2, Ch. 1, Section 11]. It follows that

$$\left(1_X\otimes\tau_{\underline{w}}^{(k)}\right)\left(\alpha(a)\otimes 1 - 1\otimes T_k(a)^*\right) = \left(\alpha(a)\otimes 1 + 1\otimes T_{n-k}(a)\right)\left(1_X\otimes\tau_{\underline{w}}^{(k)}\right).$$

Thus the linear map $\varepsilon = \sum_k \left(1_X\otimes\tau_{\underline{w}}^{(k)}\right)\gamma_k^{-1}$ implements a topological isomorphism $\mathcal{L}(\wedge E, X) \to X\otimes\wedge E$ such that $\varepsilon\theta(a) = \vartheta(a)\varepsilon$ for all $a \in E$, that is, $\theta = \vartheta$ to within an isomorphism. Using Proposition 7.1, we infer that $\theta^* = \vartheta^* = \theta'$ to within an isomorphism.◄

7.2. Projection property

Now we suggest the forward projection property onto closed ideals of a quasinilpotent B-L algebra. As a corollary we obtain the projection property onto Lie subalgebras of a nilpotent Lie algebra.

Lemma 7.4. *Let E be a B-L algebra, F a closed ideal in E of codimension one, $e \in E \backslash F$ and let (X, α) be a Banach E-module. Then $C^n(E,X) = C^n(F,X) \oplus C^{n-1}(F,X)$, $n \in \mathbb{Z}_+$. The operator $\theta_n(e)$ leaves invariant the subspace $C^n(F,X)$, and*

$$C^{\bullet}(\alpha) = \operatorname{Con}\left(C^{\bullet}(\alpha|_F), \theta(e)\right),$$

where $\theta(e) = \{\theta_n(e)\}$.

Proof. By assumption, $E = \mathbb{C}e \oplus F$. We define the following bounded linear operator

$$f_n : C^n(E,X) \to C^n(F,X) \oplus C^{n-1}(F,X), \quad f_n(\varpi) = \left(\varpi|_F, (i_n(e)\varpi)|_F\right),$$

where $\varpi|_F$ and $(i_n(e)\varpi)|_F$ are restrictions of the relevant forms onto F. It is clear that $\ker(f_n) = \{0\}$. Take $(\omega, \upsilon) \in C^n(F,X) \oplus C^{n-1}(F,X)$ and we set

$$\varpi(c_1 e + u_1, \ldots, c_n e + u_n) = \omega(u_1, \ldots, u_n) + \sum_{i=1}^{n} (-1)^{i+1} c_i \upsilon(u_1, \ldots \widehat{u_i}, \ldots, u_n),$$

where $u_i \in F, c_i \in \mathbb{C}$. Undoubtedly, $\varpi \in C^n(E,X)$ and $\varpi|_F = \omega$, $(i_n(e)\varpi)|_F = \upsilon$. Thus $C^n(F,X)$ is identified with a complemented subspace in $C^n(E,X)$. Let us prove that $C^n(F,X)$ is invariant under the operator $\theta_n(e)$. Take $\omega \in C^n(F,X)$ and let

$$\xi(u_1, \ldots, u_n) = \alpha(e)\omega(u_1, \ldots, u_n) + \sum_{i=1}^{n} (-1)^{i+1} \omega([e, u_i], u_1, \ldots \widehat{u_i}, \ldots, u_n).$$

Then $\xi \in C^n(F,X)$ and $(\theta_n(e)\varpi)|_F = \xi$, where $\varpi = f_n^{-1}(\omega, 0)$. We have $\theta_n(e)\omega = \xi$.

Now let d, d' be the differentials of complexes $C^{\bullet}(\alpha)$ and $C^{\bullet}(\alpha|_F)$, respectively. It is clear that $(d\varpi)|_F = d'(\varpi|_F)$, and by (7.1), $d'\theta_n(e)(\varpi|_F) = \theta_n(e)d'(\varpi|_F), \varpi \in C^n(E,X)$, that is, $\theta(e) = \{\theta_n(e)\}$ is an endomorphism of $C^{\bullet}(\alpha|_F)$. By (7.2), $(i_{n+1}(e)d\varpi)|_F = (\theta_n(e)\varpi)|_F - (di_n(e)\varpi)|_F = \theta_n(e)(\varpi|_F) - d'(i_n(e)\varpi|_F)$. Finally, $f_{n+1}d\varpi = (d'(\varpi|_F), -d'(i_n(e)\varpi|_F) + \theta_n(e)(\varpi|_F)) = \gamma f_n \varpi$, where γ is the differential of the cone $\operatorname{Con}(C^{\bullet}(\alpha|_F), \theta(e))$. Thus the family $\{f_n\}$ implements required isomorphism of complexes.◄

Remark 7.1. *For each n, we can identify $C^n(E,X) = C^n(F,X) \oplus C^{n-1}(F,X)$. We have just proven that $C^n(F,X)$ is invariant with respect to the operators $\theta_n(a)$, $a \in E$, that means $C^n(F,X)$ is a closed E-submodule in $C^n(E,X)$. It worth to note that the second summand $C^{n-1}(F,X)$ is also E-submodule. Indeed, if $\upsilon \in C^{n-1}(F,X)$ then there exists $\varpi \in C^n(E,X)$ such that $\varpi|_F = 0$ and $\upsilon = (i_n(e)\varpi)|_F$. By using (7.3), we obtain that*

$$\theta_{n-1}(a)\upsilon = (\theta_{n-1}(a)i_n(e)\varpi)|_F = (i_n(e)\theta_n(a)\varpi)|_F + (i_n([a,e])\varpi)|_F = (i_n(e)\theta_n(a)\varpi)|_F,$$

where $a \in E$, $[a,e] \in F$.

Exercise 12. *Find the matrix of the operator $\theta_n(a) \in \mathcal{B}(C^n(E,X))$ $(a \in E)$ with respect to the decomposition from Lemma 7.4.*

Exercise 13. *Prove that* $C_\bullet(\alpha) = \operatorname{Con}(C_\bullet(\alpha|_F), \vartheta(e))$ *(the chain version of the decomposition from Lemma 7.4).*

Corollary 7.2. *Let* E *be a finite-dimensional nilpotent Lie algebra and let* (X, α) *be a Banach* E*-module. The topological isomorphism* $\varepsilon : \mathcal{L}(\wedge E, X) \to X \otimes \wedge E$ *suggested in Corollary 7.1 implements an isomorphism* $\Delta(E)$*-Banach complexes* $\underline{\mathcal{C}^\bullet(\alpha)} \to \mathcal{C}_\bullet(\alpha)$*. In particular,* $\sigma(\alpha) = \overline{\sigma}(\alpha)$ *for all* $\sigma \in \mathfrak{S}$*.*

The proof of this assertion is based on Lemma 7.4, Exercise 13, Corollary 7.1, and the inductive argument on $\dim(E)$. For details we refer the reader to [4], [2, 1.11], [46].

Theorem 7.1. *Let* E *be a quasinilpotent B-L algebra,* F *a closed ideal in* E *of finite codimension,* $\sigma \in \mathfrak{S}_\delta \cup \mathfrak{S}^\pi$*, and let* (X, α) *be a Banach* E*-module. Then* $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$*. Moreover, if* $\alpha([E, E])$ *consists of quasinilpotent operators then* $\sigma(\alpha)|_F = \sigma(\alpha|_F)$*. In particular, the projection property onto all Lie subalgebras of a finite-dimensional nilpotent Lie algebra is valid for all Slodkowski spectra, that is,* $\sigma(\alpha)|_F = \sigma(\alpha|_F)$ *whenever* F *is a Lie subalgebra of a finite-dimensional nilpotent Lie algebra* E *and* $\sigma \in \mathfrak{S}$*.*

Proof. The chain case is reduced to the cochain case by using the dual representation. Namely, $\sigma(\alpha)|_F = \sigma^*(\alpha^*)|_F$ (see Section 6) for all $\sigma \in \mathfrak{S}$, and $\alpha^*|_F = (\alpha|_F)^*$.

Fix $\sigma \in \mathfrak{S}^\pi$. Evidently, E/F is a quasinilpotent B-L algebra. Then E/F is a finite-dimensional nilpotent Lie algebra due to Engel theorem. Using the central lower series of E/F, we could reduce the situation to the case when F has codimension one. So, assume $\dim(E/F) = 1$. Take $\lambda \in \sigma(\alpha) = \sigma(\mathcal{C}^\cdot(\alpha))$ and $e \in E \backslash F$. By Lemma 7.4,

$$C^\bullet(\alpha - \lambda) = \operatorname{Con}(C^\bullet((\alpha - \lambda)|_F), \theta(e) - \lambda(e)), \quad \lambda \in \Delta(E),$$

that is, $\mathcal{C}^\cdot(\alpha) = \operatorname{Con}_{\theta(e)}(\mathcal{C}^\bullet(\alpha|_F))$. Then, $\lambda|_F \in \Delta(F)$ and $(\lambda|_F, \lambda(e)) \in \sigma\left(\operatorname{Con}_{\theta(e)}(\mathcal{C}^\bullet(\alpha|_F))\right)$. Bearing in mind that $\mathcal{C}^\bullet(\alpha|_F)$ is π-stable (see Definition 4.1), we conclude that $\lambda|_F \in \sigma(\mathcal{C}^\bullet(\alpha|_F))$ by Theorem 4.1, therefore $\lambda|_F \in \sigma(\alpha|_F)$.

Conversely, take $\mu \in \sigma(\alpha|_F)$ and assume that $\alpha([E, E])$ consists of quasinilpotent operators. Since $[E, E] \subseteq F$ and F is a quasinilpotent B-L algebra, it follows using Lemma 7.3 that $\mu(a) = 0$ for all $a \in [E, E]$. Thereby, arbitrary linear extension of the functional μ to E is a Lie character on E. By Theorem 4.1, $(\mu, c) \in \sigma\left(\operatorname{Con}_{\theta(e)} \mathcal{C}^\bullet(\alpha|_F)\right)$ for some $c \in \mathbb{C}$. Let

$$\lambda(ze + u) = zc + \mu(u), \quad z \in \mathbb{C}, \quad u \in F.$$

By Lemma 7.4, $\lambda \in \sigma(\alpha)$ and $\lambda|_F = \mu$.

Finally, let us assume that $\dim(E) < \infty$. Then E is a nilpotent Lie algebra and consequently each Lie subalgebra $F \subseteq E$ is subnormal [6, Ch. 1, Section 4], that is, there exists a sequence of Lie subalgebras $F = F_0 \subset F_1 \subset \cdots \subset F_{k-1} \subset F_k = E$ such that F_{s-1} is an ideal in F_s for each s, $s \geq 1$. Moreover, $\alpha([E, E])$ comprises quasinilpotent operators by virtue of Turovskii lemma 2.13. Then we could apply the projection property to each gap F_s/F_{s-1}. It follows that

$$\sigma(\alpha)|_F = \sigma(\alpha)|_{F_{k-1}}|_{F_0} = \sigma\left(\alpha|_{F_{k-1}}\right)|_{F_0} = \sigma\left(\alpha|_{F_{k-1}}\right)|_{F_{k-2}}|_{F_0} = \sigma\left(\alpha|_{F_{k-2}}\right)|_{F_0} = \cdots = \sigma(\alpha|_F),$$

that is, $\sigma(\alpha)|_F = \sigma(\alpha|_F)$. Thus the assertion has been proven for all $\sigma \in \mathfrak{S}_\delta \cup \mathfrak{S}^\pi$. Since E is a nilpotent Lie algebra, it follows using Corollary 7.2 that $\sigma(\alpha) = \overline{\sigma}(\alpha)$ for all $\sigma \in \mathfrak{S}^\delta \cup \mathfrak{S}_\pi$ ◄.

7.3. Projection property onto finite-dimensional Lie subalgebras

Now we prove the inclusion $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$ for finite-dimensional Lie subalgebras F of a B-L algebra E. First, we suggest necessary lemmas.

Lemma 7.5. *Let E be a B-L algebra, (X,α) a Banach module, $\sigma \in \mathfrak{S}^{\cdot}$, and let $Y \in \mathrm{Proj}$. Then $\sigma(L_\alpha) = \sigma(\alpha)$, where $L_\alpha : E \to \mathcal{B}(\mathcal{L}(Y,X))$, $L_\alpha(a) = L_{\alpha(a)}$, is the left regular representation.*

Proof. Note that there exists a canonical isomorphism between Banach space complexes $C^\bullet(L_\alpha)$ and $\mathcal{L}(Y, C^\bullet(\alpha))$. By appealing Theorem 3.2, we conclude that

$$\sigma(L_\alpha) = \sigma(\mathcal{C}^\bullet(L_\alpha)) = \sigma(\mathcal{L}(Y,\mathcal{C}^\bullet(\alpha))) = \sigma(\mathcal{C}^\bullet(\alpha)) = \sigma(\alpha),$$

that is, $\sigma(L_\alpha) = \sigma(\alpha)$. ◄

Exercise 14. *Let E be a B-L algebra, (X,α) a Banach module, $\sigma \in \mathfrak{S}.$, and let $Y \in \mathrm{Flat}$. Then $\sigma(\alpha \otimes 1) = \sigma(\alpha)$, where $\alpha \otimes 1 : E \to \mathcal{B}\left(X\widehat{\otimes}Y\right)$, $\alpha \otimes 1(a) = \alpha(a) \otimes 1_Y$.*

Lemma 7.6. *Let E be a quasinilpotent B-L algebra, F a finite-dimensional Lie subalgebra in E, $\sigma \in \mathfrak{S}^{\cdot}$, and let (X,α) be a Banach E-module. If $E \in \mathrm{Proj}$ then $\sigma(\theta_n|_F) = \sigma(\alpha|_F)$ for all $n \in \mathbb{Z}_+$.*

Proof. By Engel theorem, F is a nilpotent Lie subalgebra. Let us prove that $\sigma(\theta_n|_F) = \sigma(L_\alpha|_F)$. Let $\delta_1 = L_\alpha|_F$ and $\delta_2 = -R_{T_n}|_F$ be the left and right regular representations of the Lie algebra F on the space $\mathcal{L}(\wedge^n E, X)$. Since $[\delta_1(a), \delta_2(b)] = 0$ for all $a,b \in F$, it follows that the linear operator $\delta : F \times F \to \mathcal{B}(\mathcal{L}(\wedge^n E, X))$, $\delta(a,b) = \delta_1(a) + \delta_2(b)$, is a Lie representation of the Lie algebra $F \times F$. Let $M = \{(a,a) : a \in F\}$ be a Lie subalgebra in $F \times F$ and let $\iota : F \to M$, $\iota(a) = (a,a)$, be a canonical isomorphism of Lie algebras. Consider also a Lie subalgebra $F \times \{0\} \subseteq F \times F$ and a canonical isomorphism $\epsilon : F \to F \times \{0\}$, $\epsilon(a) = (a,0)$. Obviously, $\theta_n|_F = \delta|_M \cdot \iota$ and $\delta_1 = \delta|_{F\times\{0\}} \cdot \epsilon$.

Further, if $\lambda \in \sigma(\delta)$ then by lemmas 7.1, 7.3, $\lambda(0,a) \in \mathrm{sp}(\delta(0,a)) = -\mathrm{sp}(T_n(a)) = \{0\}$, $a \in F$. Therefore $\lambda(a,b) = \lambda(a,0)$ for each pair $(a,b) \in F \times F$. Thus $\lambda|_{F\times\{0\}} \cdot \epsilon = \lambda|_M \cdot \iota$. Using Theorem 7.1 (see also [46]), infer that

$$\sigma(\theta_n|_F) = \sigma(\delta|_M) \cdot \iota = \sigma(\delta)|_M \cdot \iota = \{\lambda|_M \cdot \iota : \lambda \in \sigma(\delta)\} = \left\{\lambda|_{F\times\{0\}} \cdot \epsilon : \lambda \in \sigma(\delta)\right\} =$$

$$= \sigma\left(\delta|_{F\times\{0\}}\right) \cdot \epsilon = \sigma\left(\delta|_{F\times\{0\}} \cdot \epsilon\right) = \sigma(\delta_1),$$

that is, $\sigma(\theta_n|_F) = \sigma(L_\alpha|_F)$. But, all $\wedge^n E \in \mathrm{Proj}$ by virtue of Lemma 2.8. Then $\sigma(L_\alpha|_F) = \sigma(\alpha|_F)$ by Lemma 7.5.◄

Exercise 15. *Let E be a quasinilpotent B-L algebra, F a finite-dimensional Lie subalgebra in E, $\sigma \in \mathfrak{S}.$, and let (X,α) be a Banach E-module. If $E \in \mathrm{Flat}$ then $\sigma(\vartheta_n|_F) = \sigma(\alpha|_F)$ for all n.*

Theorem 7.2. *Let E be a quasinilpotent B-L algebra, F a finite-dimensional Lie subalgebra in E, $\sigma \in \mathfrak{S}_\delta \cup \mathfrak{S}^\pi$, and let (X,α) be a Banach E-module. If $E \in \mathrm{Proj}$ then $\sigma(\alpha)|_F \subseteq \sigma(\alpha|_F)$.*

Proof. First, let us reduce the assertion to the case $\sigma \in \mathfrak{S}^\pi$. Indeed, if the assertion has been proven for all $\sigma \in \mathfrak{S}^\pi$, then we infer that $\sigma(\alpha)|_F = \sigma^*(\alpha^*)|_{F^{\mathrm{op}}} \subseteq \sigma^*(\alpha^*|_{F^{\mathrm{op}}}) = \sigma^*\left((\alpha|_{F^{\mathrm{op}}})^*\right) = \sigma(\alpha|_F)$, $\sigma \in \mathfrak{S}_\delta$.

Now let $\mathfrak{U}$ be an ultrafilter in a certain set S, $\mathcal{C}^\bullet(\alpha)_\mathfrak{U}$ the ultrapower of the cochain $\Delta(E)$-Banach complex $\mathcal{C}^\bullet(\alpha)$ and let $\mathcal{C}^\bullet(\alpha_\mathfrak{U}|_F)$ be $\Delta(F)$-Banach complex generated by the Lie representation $\alpha_\mathfrak{U} : F \to \mathcal{B}(X_\mathfrak{U})$, where $\alpha_\mathfrak{U}$ is the ultrapower of α. These complexes are connected by means of the following diagram $\mathcal{L}(\lambda,\mu)$:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & {\scriptstyle \beta(\mu)}\uparrow & & {\scriptstyle \beta(\mu)}\uparrow & & \\
\cdots & \overset{\varepsilon(\lambda)}{\to} & C^{q}\left(F,C^{n-1}(E,X)_{\mathfrak{U}}\right) & \overset{\varepsilon(\lambda)}{\to} & C^{q}\left(F,C^{n}(E,X)_{\mathfrak{U}}\right) & \overset{\varepsilon(\lambda)}{\to} & \cdots \\
 & & {\scriptstyle \beta(\mu)}\uparrow & & {\scriptstyle \beta(\mu)}\uparrow & & \\
\cdots & \overset{\varepsilon(\lambda)}{\to} & C^{q-1}\left(F,C^{n-1}(E,X)_{\mathfrak{U}}\right) & \overset{\varepsilon(\lambda)}{\to} & C^{q-1}\left(F,C^{n}(E,X)_{\mathfrak{U}}\right) & \overset{\varepsilon(\lambda)}{\to} & \cdots, \\
 & & {\scriptstyle \beta(\mu)}\uparrow & & {\scriptstyle \beta(\mu)}\uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

where $\varepsilon(\lambda)(\Phi)=d(\lambda)_{\mathfrak{U}}\Phi$, $\Phi\in C^{q}\left(F,C^{n}(E,X)_{\mathfrak{U}}\right)$ (here $d(\lambda)_{\mathfrak{U}}$ is the differential of $C^{\bullet}(\alpha-\lambda)_{\mathfrak{U}}$) and $\beta(\mu)$ is the differential of the complex $C^{\bullet}\left((\theta_n|_F)_{\mathfrak{U}}-\mu\right)$. As a simple consequence of the first cohomological formula (7.1), we obtain that the latter diagram is a commutative Banach space bicomplex with rows $\mathcal{L}\left(\wedge^{q}F,C^{\bullet}(\alpha-\lambda)_{\mathfrak{U}}\right)$, $q\in\mathbb{Z}_{+}$, and columns $C^{\bullet}\left((\theta_n|_F)_{\mathfrak{U}}-\mu\right)$, $n\in\mathbb{Z}_{+}$. Thus $\mathcal{B}=\{\mathcal{B}(\lambda,\mu):\lambda\in\Delta(E),\ \mu\in\Delta(F)\}$ is a $\Delta(E)\times\Delta(F)$-Banach bicomplex. Since $\dim(F)<\infty$, it follows that $\sigma\left(\mathcal{L}\left(\wedge^{q}F,\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}\right)\right)=\sigma\left(\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}\right)$ by Theorem 3.2. Moreover

$$\sigma\left(\mathcal{C}^{\bullet}\left((\theta_n|_F)_{\mathfrak{U}}\right)\right)=\sigma\left((\theta_n|_F)_{\mathfrak{U}}\right)=\sigma\left(\theta_n|_F\right)=\sigma\left(\alpha|_F\right)=\sigma\left(\alpha_{\mathfrak{U}}|_F\right)=\sigma\left(\mathcal{C}^{\bullet}\left(\alpha_{\mathfrak{U}}|_F\right)\right),$$

by lemmas 6.2, 7.6. Thus, by definition (see Section 5), $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}$ and $\mathcal{C}^{\bullet}\left(\alpha_{\mathfrak{U}}|_F\right)$ are π-spectrally connected (by dint of $\mathcal{B}$) complexes.

Now let $f:\Delta(E)\to\Delta(F)$, $f(\lambda)=\lambda|_F$, be the projection map. Let us prove that f is a π-prespectral mapping (see Definition 5.1) with respect to $\mathcal{B}$. Let $\mu=f(\lambda)$, $\beta(\mu):C^{n}(E,X)_{\mathfrak{U}}\to\mathcal{L}\left(F,C^{n}(E,X)_{\mathfrak{U}}\right)$, $(\beta(\mu)[\omega_s])a=[(\theta_n(a)-\mu(a))\omega_s]$, be the differential of the n-th column of $\mathcal{B}$, and let $[\omega_s]\in C^{n}(E,X)_{\mathfrak{U}}$ such that $d(\lambda)_{\mathfrak{U}}[\omega_s]=0$. If $I_{n-1}\in\mathcal{L}\left(F,C^{n-1}(E,X)_{\mathfrak{U}}\right)$, $I_{n-1}(a)=i_n(a)_{\mathfrak{U}}[\omega_s]$, then using (7.2), we infer

$$\begin{gathered}
[(\theta_n(a)-\mu(a))\omega_s]=[d(\lambda)i_n(a)\omega_s]+[i_{n+1}(a)d(\lambda)\omega_s]=d(\lambda)_{\mathfrak{U}}i_n(a)_{\mathfrak{U}}[\omega_s]+\\
+i_{n+1}(a)_{\mathfrak{U}}d(\lambda)_{\mathfrak{U}}[\omega_s]=d(\lambda)_{\mathfrak{U}}i_n(a)_{\mathfrak{U}}[\omega_s]=\varepsilon(\lambda)(I_{n-1})(a),
\end{gathered}$$

that is, the induced cohomology operator $\beta(\mu)^{\sim}:H^{n}\left(C^{\bullet}(\alpha-\lambda)_{\mathfrak{U}}\right)\to\mathcal{L}\left(F,H^{n}\left(C^{\bullet}(\alpha-\lambda)_{\mathfrak{U}}\right)\right)$ is trivial for all $n\in\mathbb{Z}_{+}$. Thus f is a π-prespectral mapping with respect to $\mathcal{B}$. By using Theorem 5.1, we obtain that $f\left(\sigma\left(\mathcal{C}^{\bullet}(\alpha)\right)\right)\subseteq\sigma\left(\mathcal{C}^{\bullet}\left(\alpha_{\mathfrak{U}}|_F\right)\right)$. But, $\sigma\left(\mathcal{C}^{\bullet}\left(\alpha_{\mathfrak{U}}|_F\right)\right)=\sigma\left(\alpha_{\mathfrak{U}}|_F\right)=\sigma\left(\alpha|_F\right)$ by Lemma 6.2, therefore $f(\sigma(\alpha))\subseteq\sigma\left(\alpha|_F\right)$. ◄

Let E be a closed subspace in $\mathcal{B}(X)$ generated by a family of mutually commuting operators $T=\{T_\iota:\iota\in\Lambda\}$, that is, E is a commutative B-L algebra. If T is a bounded family then there exists a bounded linear representation $\varepsilon:\ell_1(\Lambda)\to\mathcal{B}(X)$, $\varepsilon\left(\sum a_\iota e_\iota\right)=\sum a_\iota T_\iota$, where $\{e_\iota:\iota\in\Lambda\}$ is the canonical basis of the Banach space $\ell_1(\Lambda)$. The images of both spectra of the identity representation of E and the representation ε under the injective maps $E^{*}\to\mathbb{C}^{\Lambda}$, $\lambda\mapsto\left(\lambda(T_\iota)\right)_{\iota\in\Lambda}$, and $\ell_1(\Lambda)^{*}\to\mathbb{C}^{\Lambda}$, $\lambda\mapsto\left(\lambda(e_\iota)\right)_{\iota\in\Lambda}$, we denote by $\sigma(T)$ and $\ell_1\sigma(T)$, respectively. By Lemma 7.3, spectra $\sigma(T)$ and $\ell_1\sigma(T)$ are precompact in $\mathbb{C}^{\Lambda}$. Let $\sigma^{\mathrm{u}}(T)$ (resp., $\ell_1\sigma^{\mathrm{u}}(T)$) be the union of $\sigma(T_{\mathfrak{U}})$ (resp., $\ell_1\sigma(T_{\mathfrak{U}})$) over all ultrafilters $\mathfrak{U}$, where $T_{\mathfrak{U}}=\{T_{\iota\mathfrak{U}}:\iota\in\Lambda\}$. By Corollary 6.1, $\sigma^{\mathrm{u}}(T)\neq\emptyset$ and $\ell_1\sigma^{\mathrm{u}}(T_{\mathfrak{U}})\neq\emptyset$.

Corollary 7.3. *Let $T=\{T_\iota:\iota\in\Lambda\}$ be a bounded family in $\mathcal{B}(X)$ and let $T'=\{T_\iota:\iota\in\Xi\}$ with $\Xi\subseteq\Lambda$ having the finite complement $\Lambda\backslash\Xi=\{\iota_1,\dots,\iota_n\}$. Then*

$$\sigma(T')=\sigma(T)|_{\Xi},\quad \sigma^{\mathrm{u}}(T')=\sigma^{\mathrm{u}}(T)|_{\Xi},\quad \ell_1\sigma(T')=\ell_1\sigma(T)|_{\Xi},\quad \ell_1\sigma^{\mathrm{u}}(T')=\ell_1\sigma^{\mathrm{u}}(T)|_{\Xi}.$$

Moreover, $\ell_1\sigma^{\mathrm{u}}(T)|_{\Lambda\backslash\Xi}\subseteq\sigma(T_{\iota_1},\dots,T_{\iota_n})$.

Proof. It suffices to note that the closed subspace in E generated by the family T' has finite codimension and use theorems 7.1, 7.2.◀

If T is a finite family, then $\sigma(T) = \ell_1\sigma(T) = \sigma^{\mathrm{u}}(T) = \ell_1\sigma^{\mathrm{u}}(T)$ and we obtain the known [65], [62] projection properties for Taylor and Slodkowski spectra of finite commutative operator families.

8. The dominating algebras

In this section we apply suggested in Section 5 spectral mapping framework to spectral theory of B-L algebra representations.

In this section $\mathfrak{g}$ denotes a finite-dimensional nilpotent Lie algebra, $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, and (X,α) is a Banach $\mathfrak{g}$-module. Let $\mathcal{A}$ be a topological algebra. By a normed Lie subalgebra in $\mathcal{A}$ we mean a Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}$ equipped with a certain norm $\|\cdot\|$ such that $(\mathfrak{F}, \|\cdot\|)$ is a normed Lie algebra and the identity embedding $\mathfrak{F} \hookrightarrow \mathcal{A}$ is continuous. For instance, if $\mathcal{A}$ is a Banach algebra then a normed Lie algebra $(\mathfrak{F}, \|\cdot\|)$ is a normed Lie subalgebra in $\mathcal{A}$ whenever $\mathfrak{F} \subseteq \mathcal{A}$ and $\|u\| \geq \|u\|_{\mathcal{A}}$, $u \in \mathfrak{F}$.

8.1. Properties of the dominating algebras

The following definition generalizes the dominated Banach algebras suggested in [27, Section 7].

Definition 8.1. *Let $\mathcal{A}_{\mathfrak{g}} \in$ **LCA** with a fixed Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathcal{A}_{\mathfrak{g}}$. We say that $\mathcal{A}_{\mathfrak{g}}$ dominates over the module (X,α) and write $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$, if there is a continuous algebra homomorphism $\widehat{\theta} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ such that $\widehat{\theta}\cdot\pi = \theta$ and $\widehat{\theta}(\mathrm{im}(\widehat{\pi}))$ is dense in $\widehat{\theta}(\mathcal{A}_{\mathfrak{g}})$, where $\widehat{\pi} : \mathcal{R}_{\mathfrak{g},\pi} \to \mathcal{A}_{\mathfrak{g}}$ is the extension of the map π (see Section 2.8). The elements from the subalgebra $\mathrm{im}(\widehat{\pi})(= \mathcal{R}(\mathrm{im}(\pi)))$ are called rational functions in $\mathcal{A}_{\mathfrak{g}}$ acting on X.*

Example 8.1. *If $\mathcal{A}_{\mathfrak{g}}$ is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ equipped with the finest locally convex topology and π is the canonical embedding $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$, then $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$ for each $\mathfrak{g}$-module X.*

Example 8.2. *With respect to each $\mathfrak{g}$-module (X,α) one can suggest a dominating over that module Banach algebra $\mathcal{A}_{\theta}$ as the closure of the inverse closed subalgebra $\mathcal{R}(\theta(\mathfrak{g})) \subseteq \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ generated by the Lie subalgebra $\theta(\mathfrak{g})$, and the representation θ stands itself instead of a Lie homomorphism $\pi : \mathfrak{g} \to \mathcal{A}_{\theta}$. Undoubtedly, $\mathcal{A}_{\theta} \succ (X,\alpha)$.*

Other examples will be considered later (see also [27]).

Lemma 8.1. *If $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$ then $\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}} \succ (X^*,\alpha^*)$. Moreover, $\mathcal{A}_{\mathfrak{g}} \succ (X_{\mathfrak{U}}, \alpha_{\mathfrak{U}})$ for an ultrafilter $\mathfrak{U}$.*

Proof. By Corollary 7.1, the dual (to θ) representation $\theta^* : \mathfrak{g}^{\mathrm{op}} \to \mathcal{B}\left(\mathcal{L}(\wedge\mathfrak{g}, X)^*\right)$ is reduced (to within an isomorphism) to the representation $\theta' : \mathfrak{g}^{\mathrm{op}} \to \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X^*))$, $\theta'(a) = L_{\alpha^*(a)} - R_{T^{\mathrm{op}}(a)}$, extended the dual representation α^*. Then $\widehat{\theta}^*\pi(a) = \theta'(a)$ to within an isomorphism for all $a \in \mathfrak{g}$, where $\widehat{\theta}^* : \mathcal{A}_{\mathfrak{g}}^{\mathrm{op}} \to \mathcal{B}\left(\mathcal{L}(\wedge\mathfrak{g}, X)^*\right)$, $\widehat{\theta}^*(a) = \widehat{\theta}(a)^*$, is the dual to $\widehat{\theta} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ representation, the latter exists on the matter $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$. By Definition 8.1, $\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}} \succ (X^*,\alpha^*)$.

Further, consider the ultrapower $\widehat{\theta}_{\mathfrak{U}} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X)_{\mathfrak{U}})$ of the representation $\widehat{\theta}$. Since $\dim(\wedge\mathfrak{g}) < \infty$, we conclude that the canonical map $\mathcal{L}(\wedge\mathfrak{g}, X)_{\mathfrak{U}} \to \mathcal{L}(\wedge\mathfrak{g}, X_{\mathfrak{U}})$ is an isomorphism in **BS** [48]. Moreover, $\widehat{\theta}_{\mathfrak{U}}(\pi(u)) = \widehat{\theta}(\pi(u))_{\mathfrak{U}} = \theta(u)_{\mathfrak{U}}$ for each $u \in \mathfrak{g}$, and $\theta(u)_{\mathfrak{U}} =$

$\left(L_{\alpha(u)}-R_{T(u)}\right)_{\mathfrak{U}}=L_{\alpha_{\mathfrak{U}}(u)}-R_{T(u)}=\theta_{\mathfrak{U}}(u)$. Thus $\widehat{\theta}_{\mathfrak{U}}$ extends $\theta_{\mathfrak{U}}$, that is, $\widehat{\theta}_{\mathfrak{U}}=\widehat{\theta_{\mathfrak{U}}}$. Consequently, $\mathcal{A}_{\mathfrak{g}}\succ\left(X_{\mathfrak{U}}\ ,\alpha_{\mathfrak{U}}\ \right)$. ◄

Now let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$. By Definition 8.1, $\widehat{\theta}\left(\mathcal{R}\left(\operatorname{im}(\pi)\right)\right)$ is dense in $\widehat{\theta}\left(\mathcal{A}_{\mathfrak{g}}\right)$, therefore each subspace $C^{k}\left(\mathfrak{g},X\right)$ in $\mathcal{L}\left(\wedge\mathfrak{g},X\right)$ is a complemented $\mathcal{A}_{\mathfrak{g}}$-invariant subspace. We set $\widehat{\theta}_{k}\left(a\right)=\widehat{\theta}\left(a\right)|_{C^{k}(\mathfrak{g},X)}$, $a\in\mathcal{A}_{\mathfrak{g}}$. In particular, $X\in\mathcal{A}_{\mathfrak{g}}$-mod, for $X=C^{0}\left(\mathfrak{g},X\right)$. We denote the relevant bounded representation $\mathcal{A}_{\mathfrak{g}}\rightarrow\mathcal{B}\left(X\right)$ by $\alpha|^{\mathcal{A}_{\mathfrak{g}}}$, thus $\alpha|^{\mathcal{A}_{\mathfrak{g}}}\cdot\pi=\alpha$ and $\alpha|^{\mathcal{A}_{\mathfrak{g}}}\left(\mathcal{R}\left(\operatorname{im}\left(\pi\right)\right)\right)$ is dense in $\alpha|^{\mathcal{A}_{\mathfrak{g}}}\left(\mathcal{A}_{\mathfrak{g}}\right)$.

Let I be a Lie ideal in $\mathfrak{g}$. Then $C^{k}\left(I,X\right)$ is a $\mathfrak{g}$-module by the representation

$$\theta_{k,I}:\mathfrak{g}\rightarrow\mathcal{B}\left(C^{k}\left(I,X\right)\right),\quad\theta_{k,I}\left(u\right)=L_{\alpha(u)}-R_{T_{k,I}(u)},$$

(see Subsection 7.1) and the restriction map $C^{k}\left(\mathfrak{g},X\right)\rightarrow C^{k}\left(I,X\right)$, $\omega\mapsto\omega|_{I}$ (here $\omega|_{I}=\omega|_{\wedge^{k}I}$), is a $\mathfrak{g}$-module homomorphism.

Proposition 8.1. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$. Then $C^{k}\left(I,X\right)$ makes into a Banach $\mathcal{A}_{\mathfrak{g}}$-module extending its $\mathfrak{g}$-module structure such that the restriction map $C^{k}\left(\mathfrak{g},X\right)\rightarrow C^{k}\left(I,X\right)$ is a morphism in $\mathcal{A}_{\mathfrak{g}}$*-mod.

Proof. Since $\mathfrak{g}$ is a nilpotent Lie algebra, the ideal I can be included into a Jordan-Holder series of ideals having one dimensional gaps by virtue of Engel theorem [6, Ch. 1, Section 4]. Therefore, one suffices to prove the assertion for an ideal I of codimension 1. Take such an ideal I and let $e\notin I$. Note that the map

$$C^{k}\left(\mathfrak{g},X\right)\rightarrow C^{k}\left(I,X\right)\oplus C^{k-1}\left(I,X\right),\quad\omega\mapsto\left(\omega|_{I},\left(i_{k}\left(e\right)\omega\right)|_{I}\right),$$

implements a topological isomorphism in **BS** by Lemma 7.4. If we identify $C^{k}\left(\mathfrak{g},X\right)$ with the direct sum $C^{k}\left(I,X\right)\oplus C^{k-1}\left(I,X\right)$ by means of the isomorphism then the restriction map $C^{k}\left(\mathfrak{g},X\right)\rightarrow C^{k}\left(I,X\right)$ would be the projection onto the first summand. Fix $a\in\mathfrak{g}$. The operator $\theta_{k}\left(a\right)$ has the following matrix form

$$\left(\begin{array}{cc}\theta_{k,I}\left(a\right)&0\\G_{k}\left(\left[e,a\right]\right)&\theta_{k-1,I}\left(a\right)\end{array}\right),\tag{8.1}$$

with respect to the decomposition (see Exercise 12), where $G_{k}\left(b\right):C^{k}\left(I,X\right)\rightarrow C^{k-1}\left(I,X\right)$, $G_{k}\left(b\right)\left(\omega|_{I}\right)=\left(i_{k}\left(b\right)\omega\right)|_{I}$, $\omega\in C^{k}\left(\mathfrak{g},X\right)$, $b\in I$. Indeed, if A is the matrix (8.1) then using the third cohomological formula (7.3), we deduce

$$A\omega=A\left(\omega|_{I},\left(i_{k}\left(e\right)\omega\right)|_{I}\right)=\left(\theta_{k,I}\left(a\right)\left(\omega|_{I}\right),G_{k}\left(\left[e,a\right]\right)\left(\omega|_{I}\right)+\theta_{k-1,I}\left(a\right)\left(i_{k}\left(e\right)\omega\right)|_{I}\right)=$$

$$=\left(\left(\theta_{k}\left(a\right)\omega\right)|_{I},\left(i_{k}\left(\left[e,a\right]\right)\omega+\theta_{k-1}\left(a\right)i_{k}\left(e\right)\omega\right)|_{I}\right)=\left(\left(\theta_{k}\left(a\right)\omega\right)|_{I},\left(i_{k}\left(e\right)\theta_{k}\left(a\right)\omega\right)|_{I}\right)=\theta_{k}\left(a\right)\omega.$$

Now let us introduce the following operators

$$D_{k}\left(a\right)=\left(\begin{array}{cc}\theta_{k,I}\left(a\right)&0\\0&\theta_{k-1,I}\left(a\right)\end{array}\right),\quad N_{k}\left(b\right)=\left(\begin{array}{cc}0&0\\G_{k}\left(b\right)&0\end{array}\right),$$

where $a\in\mathfrak{g}$, $b\in I$. Using (7.3) again, we infer that

$$\left[D_{k}\left(a\right),N_{k}\left(b\right)\right]=N_{k}\left(\left[a,b\right]\right),\quad\left[D_{k}\left(a_{1}\right),D_{k}\left(a_{2}\right)\right]=D_{k}\left(\left[a_{1},a_{2}\right]\right),\quad N_{k}\left(b_{1}\right)N_{k}\left(b_{2}\right)=0,$$

for all $a_{i}\in\mathfrak{g}$ and $b_{i}\in I$, $i=1,2$. It follows that the Lie subalgebra $E\subseteq\mathcal{B}\left(C^{k}\left(I,X\right)\right)$ generated by these operators is a finite-dimensional nilpotent Lie algebra. By Turovskii lemma 2.13, the closure

B of the inverse closed subalgebra $\mathcal{R}(E) \subseteq \mathcal{B}\left(C^k(I,X)\right)$ generated by E is commutative modulo its radical $\operatorname{Rad} B$. Then $N_k(b) \in \operatorname{Rad} B$, $b \in I$, and $\mathcal{R}_{\mathfrak{g},D_k} \subseteq \mathcal{R}_{\mathfrak{g},\theta}$. Moreover, $\widehat{\theta}_k(r(\overline{\mathfrak{g}})) - r(D_k(\mathfrak{g})) \in \operatorname{Rad} B$, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},D_k}$, where $\overline{\mathfrak{g}} = \operatorname{im}(\pi)$. Taking into account that B is an inverse closed subalgebra, we deduce that

$$\operatorname{sp}(r(D_k(\mathfrak{g}))) = \operatorname{sp}_B(r(D_k(\mathfrak{g}))) = \operatorname{sp}_B\left(\widehat{\theta}_k(r(\overline{\mathfrak{g}}))\right) = \operatorname{sp}\left(\widehat{\theta}_k(r(\overline{\mathfrak{g}}))\right),$$

for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},D_k}$. By Lemma 2.11, $\mathcal{R}_{\mathfrak{g},D_k} = \mathcal{R}_{\mathfrak{g},\theta}$ and

$$\widehat{\theta}_k(r(\overline{\mathfrak{g}})) = \begin{pmatrix} r(\theta_{k,I}(\mathfrak{g})) & 0 \\ * & r(\theta_{k-1,I}(\mathfrak{g})) \end{pmatrix},$$

for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$. It follows that $C^k(I,X)$ is invariant under $\widehat{\theta}_k(r(\overline{\mathfrak{g}}))$ and $\left(\widehat{\theta}_k(r(\overline{\mathfrak{g}}))\omega\right)|_I = r(\theta_{k,I}(\mathfrak{g}))(\omega|_I)$, $\omega \in C^k(\mathfrak{g},X)$. But, $\widehat{\theta}_k(\operatorname{im}(\widehat{\pi}))$ is dense in $\widehat{\theta}_k(\mathcal{A}_{\mathfrak{g}})$ (see Definition 8.1), therefore $C^k(I,X)$ is a $\mathcal{A}_{\mathfrak{g}}$-submodule and the restriction map $C^k(\mathfrak{g},X) \to C^k(I,X)$ is a $\mathcal{A}_{\mathfrak{g}}$-module homomorphism.◄

Let $\widehat{\theta}_{k,I} : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}\left(C^k(I,X)\right)$ be the continuous representation defining $\mathcal{A}_{\mathfrak{g}}$-module structure on $C^k(I,X)$ suggested in Proposition 8.1. Then $\widehat{\theta}_{k,I} \cdot \pi = \theta_{k,I}$ and $\mathcal{B}(\wedge I, X)$ turns into a Banach $\mathcal{A}_{\mathfrak{g}}$-module by the representation $\widehat{\theta}_I = \oplus_{k\in\mathbb{Z}_+}\widehat{\theta}_{k,I}$, that is, $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha|_I)$. Note also that $\widehat{\theta}_{0,I} = \alpha|^{\mathcal{A}_{\mathfrak{g}}}$.

Corollary 8.1. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$ and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. Then $\sigma\left(\widehat{\theta}_k|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$, $\sigma \in \mathfrak{S}$, $k \in \mathbb{N}$. In particular, $\sigma\left(\widehat{\theta}|_{\mathfrak{F}}\right) = \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ for all $\sigma \in \mathfrak{S}$.*

Proof. One suffices to prove that $\sigma\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ for all ideals $I \subseteq \mathfrak{g}$. We proceed by induction on the pair $(k, \dim(I))$. Take an ideal $J \subset I$ such that $\dim(I/J) = 1$ and $[\mathfrak{g}, I] \subseteq J$. We have an admissible ($\mathbb{C}$-split) sequence

$$0 \to C^{k-1}(J,X) \longrightarrow C^k(I,X) \longrightarrow C^k(J,X) \to 0,$$

of Banach $\mathcal{A}_{\mathfrak{g}}$-modules by virtue of Proposition 8.1, and this in turn associates exact sequences of $\Delta(\mathfrak{F})$-Banach complexes

$$0 \to \mathcal{C}.\left(\widehat{\theta}_{k-1,J}|_{\mathfrak{F}}\right) \longrightarrow \mathcal{C}.\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \longrightarrow \mathcal{C}.\left(\widehat{\theta}_{k,J}|_{\mathfrak{F}}\right) \to 0,$$

and $0 \to \mathcal{C}^{\cdot}\left(\widehat{\theta}_{k-1,J}|_{\mathfrak{F}}\right) \longrightarrow \mathcal{C}^{\cdot}\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \longrightarrow \mathcal{C}^{\cdot}\left(\widehat{\theta}_{k,J}|_{\mathfrak{F}}\right) \to 0$ by Lemma 6.1. Then

$$\sigma\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\widehat{\theta}_{k-1,J}|_{\mathfrak{F}}\right) \cup \sigma\left(\widehat{\theta}_{k,J}|_{\mathfrak{F}}\right), \quad \sigma \in \mathfrak{S},$$

by Corollary 3.1. By induction hypothesis, $\sigma\left(\widehat{\theta}_{k-1,J}|_{\mathfrak{F}}\right) \cup \sigma\left(\widehat{\theta}_{k,J}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$, therefore $\sigma\left(\widehat{\theta}_{k,I}|_{\mathfrak{F}}\right) \subseteq \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$.

Finally, $\sigma\left(\widehat{\theta}|_{\mathfrak{F}}\right) = \sigma\left(\oplus_{k\in\mathbb{Z}_+}\widehat{\theta}_k|_{\mathfrak{F}}\right) = \bigcup_{k\in\mathbb{Z}_+}\sigma\left(\widehat{\theta}_k|_{\mathfrak{F}}\right) = \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$. ◄

Corollary 8.2. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$. Then $d(\lambda)\widehat{\theta}(a) = \widehat{\theta}(a)d(\lambda)$, $\operatorname{sp}\left(\widehat{\theta}(a)\right) = \operatorname{sp}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)$ for all $a \in \mathcal{A}_{\mathfrak{g}}$, where $d(\lambda)$ is the differential of the complex $C^{\bullet}(\alpha - \lambda)$, $\lambda \in \Delta(\mathfrak{g})$. In particular, $C^{\bullet}(\alpha - \lambda) \in \overline{\mathcal{A}_{\mathfrak{g}}\text{-}\mathrm{mod}}$.*

Proof. By Definition 8.1, $\operatorname{im}\left(\widehat{\theta}\right) \subseteq \mathcal{A}_\theta$, where $\mathcal{A}_\theta = \overline{\mathcal{R}\left(\theta\left(\mathfrak{g}\right)\right)}$ (see Example 8.2). Moreover, using the first cohomological formula (7.1), we conclude that $d\left(\lambda\right)\theta_\lambda\left(a\right) = \theta_\lambda\left(a\right)d\left(\lambda\right)$ for all $a \in \mathfrak{g}$, where $\theta_\lambda\left(a\right) = L_{(\alpha-\lambda)(a)} - R_{T(a)}$. Note that $\theta_\lambda\left(a\right) = \theta\left(a\right) - \lambda\left(a\right)$, whence $d\left(\lambda\right)\theta\left(a\right) = \theta\left(a\right)d\left(\lambda\right)$. The latter obviously implies that $d\left(\lambda\right)T = Td\left(\lambda\right)$ for all $T \in \mathcal{A}_\theta$.

To prove the equality $\operatorname{sp}\left(\widehat{\theta}\left(a\right)\right) = \operatorname{sp}\left(\alpha|^{\mathcal{A}_\mathfrak{g}}\left(a\right)\right)$, one suffices to set $\mathfrak{F} = \mathbb{C}a$ and $\sigma = \sigma_{\mathrm{t}}$ in Corollary 8.1.◄

Corollary 8.3. *Let (X, α) be a Banach $\mathfrak{g}$-module. Then $\mathcal{R}_{\mathfrak{g},\theta} = \mathcal{R}_{\mathfrak{g},\alpha}$.*

Proof. By definition, $\theta = \oplus_{k\in\mathbb{Z}_+}\theta_k$ and $\theta_0 = \alpha$. It follows that $\mathcal{R}_{\mathfrak{g},\theta} \subseteq \mathcal{R}_{\mathfrak{g},\alpha}$. Further, if $\mathcal{A}_\theta$ is the closed inverse closed subalgebra in $\mathcal{B}\left(\mathcal{L}\left(\wedge\mathfrak{g}, X\right)\right)$ generated by $\theta\left(\mathfrak{g}\right)$, then $\mathcal{A}_\theta \succ (X, \alpha)$ (see Example 8.2). By Corollary 8.2, $\operatorname{sp}\left(r\left(\theta\left(\mathfrak{g}\right)\right)\right) = \operatorname{sp}\left(r\left(\alpha\left(\mathfrak{g}\right)\right)\right)$ for all $r\left(\mathfrak{g}\right) \in \mathcal{R}_{\mathfrak{g},\theta}$. It remains to use Lemma 2.11.◄

Now let $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_\mathfrak{g}$ and let $\widehat{\mathfrak{F}}$ be the norm-completion of $\mathfrak{F}$. Let us introduce a bicomplex connecting parametrized Banach space complexes $\mathcal{C}^\bullet\left(\alpha\right)$ and $\mathcal{C}^\bullet\left(\alpha|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F}\right)$. By Corollary 8.2, the following diagram

$$\begin{array}{ccccc} & & \vdots & & \\ & & {\scriptstyle\beta_\mu}\uparrow & & \\ \cdots & \overset{\delta_\lambda}{\to} & C^s\left(\widehat{\mathfrak{F}}, C^k\left(\mathfrak{g}, X\right)\right) & \overset{\delta_\lambda}{\to} & \cdots, \\ & & {\scriptstyle\beta_\mu}\uparrow & & \\ & & \vdots & & \end{array}$$

is commutative, where $\delta_\lambda\left(\Phi\right) = d^k\left(\lambda\right)\cdot\Phi$, $\Phi \in C^s\left(\widehat{\mathfrak{F}}, C^k\left(\mathfrak{g}, X\right)\right)$ (here $d^k\left(\lambda\right)$ is the differential of the complex $C^\bullet\left(\alpha - \lambda\right)$) and β_μ is the differential of $C^\bullet\left(\widehat{\theta}_k|_\mathfrak{F} - \mu\right)$. Thus we deal with a parametrized Banach space bicomplex $\mathcal{B}_{\lambda,\mu}\left(\mathfrak{g}, \mathfrak{F}, X\right)$, $\lambda \in \Delta\left(\mathfrak{g}\right)$, $\mu \in \Delta\left(\mathfrak{F}\right)$, with rows $\mathcal{L}\left(\wedge^s\widehat{\mathfrak{F}}, C^\bullet\left(\alpha - \lambda\right)\right)$, $s \in \mathbb{Z}_+$, and columns $C^\bullet\left(\widehat{\theta}_k|_\mathfrak{F} - \mu\right)$, $k \in \mathbb{Z}_+$, for which we use the denotation $\mathcal{B}\left(\mathfrak{g}, \mathfrak{F}, X\right)$. The total complex of $\mathcal{B}_{\lambda,\mu}\left(\mathfrak{g}, \mathfrak{F}, X\right)$ is denoted by $\operatorname{Tot}_{\lambda,\mu}\left(\mathfrak{g}, \mathfrak{F}, X\right)$. Then

$$\operatorname{Tot}\left(\mathfrak{g}, \mathfrak{F}, X\right) = \left\{\operatorname{Tot}_{\lambda,\mu}\left(\mathfrak{g}, \mathfrak{F}, X\right) : (\lambda, \mu) \in \Delta\left(\mathfrak{g}\right)\times\Delta\left(\mathfrak{F}\right)\right\},$$

is a parametrized Banach space complex and their Slodkowski spectra are denoted by $\sigma\left(\mathfrak{g}, \mathfrak{F}, X\right)$, $\sigma \in \mathfrak{S}$.

Proposition 8.2. *Let $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_\mathfrak{g}$, $\mathfrak{U}$ an ultrafilter and let $\widetilde{\alpha} = \alpha|^{\mathcal{A}_\mathfrak{g}}$. If $\widehat{\mathfrak{F}} \in \operatorname{Proj}$ then the parametrized Banach space complexes $\mathcal{C}^\bullet\left(\alpha_\mathfrak{U}\right)$ and $\mathcal{C}^\bullet\left(\widetilde{\alpha}_\mathfrak{U}|_\mathfrak{F}\right)$ are π-spectrally connected by means of the $\Delta\left(\mathfrak{g}\right)\times\Delta\left(\mathfrak{F}\right)$-Banach bicomplex $\mathcal{B}\left(\mathfrak{g}, \mathfrak{F}, X_\mathfrak{U}\right)$.*

Proof. By Lemma 8.1, $\mathcal{A}_\mathfrak{g} \succ \underline{\left(X_\mathfrak{U}, \alpha_\mathfrak{U}\right)}$. Let $\widehat{\theta}_\mathfrak{U} : \mathcal{A}_\mathfrak{g} \to \mathcal{B}\left(\mathcal{L}\left(\wedge\mathfrak{g}, X_\mathfrak{U}\right)\right)$ be the representation extending $\alpha_\mathfrak{U}$. Since $C^\bullet\left(\alpha_\mathfrak{U}\right) \in \overline{\mathcal{A}_\mathfrak{g}\text{-}\operatorname{mod}}$, it follows that $\mathcal{B}\left(\mathfrak{g}, \mathfrak{F}, X_\mathfrak{U}\right)$ is a $\Delta\left(\mathfrak{g}\right)\times\Delta\left(\mathfrak{F}\right)$-Banach bicomplex. Further, $\sigma\left(\mathcal{C}^\bullet\left(\widehat{\theta}_\mathfrak{U}|_\mathfrak{F}\right)\right) = \bigcup_{k\in\mathbb{Z}_+}\sigma\left(\widehat{\theta}_{k\mathfrak{U}}|_\mathfrak{F}\right) \subseteq \sigma\left(\widetilde{\alpha}_\mathfrak{U}|_\mathfrak{F}\right)$, $\sigma \in \mathfrak{S}^\pi$, by virtue of Corollary 8.1. Moreover, $\bigcup_{s\in\mathbb{Z}_+}\sigma\left(\mathcal{L}\left(\wedge^s\widehat{\mathfrak{F}}, \mathcal{C}^\bullet\left(\alpha_\mathfrak{U}\right)\right)\right) \subseteq \sigma\left(\mathcal{C}^\bullet\left(\alpha_\mathfrak{U}\right)\right)$, $\sigma \in \mathfrak{S}^\pi$, by Lemma 2.8 and Theorem 3.2. It follows that the Banach space complexes $\mathcal{C}^\bullet\left(\alpha_\mathfrak{U}\right)$ and $\mathcal{C}^\bullet\left(\widetilde{\alpha}_\mathfrak{U}|_\mathfrak{F}\right)$ are π-spectrally connected (see Section 5) by means of $\mathcal{B}\left(\mathfrak{g}, \mathfrak{F}, X_\mathfrak{U}\right)$. ◄

Remark 8.1. *One can prove the chain version (using the chain complex $\mathcal{C}_\bullet(\alpha_{\mathfrak{U}})$) of this result replacing the requirement $\widehat{\widehat{\mathfrak{F}}} \in \mathrm{Proj}$ with $\widehat{\widehat{\mathfrak{F}}} \in \mathrm{Flat}$ and using Exercise 7.*

Let again (X, α) be a Banach $\mathfrak{g}$-module. As follows from the second cohomological formula (7.2), $\theta(u) - \lambda(u) = d(\lambda) i(u) + i(u) d(\lambda)$ for all $u \in \mathfrak{g}$, where $\lambda \in \Delta(\mathfrak{g})$, $d(\lambda) \in \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ is the differential of the complex $C^\bullet(\alpha - \lambda)$ and $i(u) \in \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ is the homotopy operator induced by u. The latter relation can be enlarged to all rational functions acting on X by the following way.

Proposition 8.3. *Let $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$, $\mu : \mathcal{A}_\mathfrak{g} \to \mathbb{C}$ a character and let a be a rational function in $\mathcal{A}_\mathfrak{g}$ acting on X. There exists an operator $i_\mu(a) \in \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ such that*

$$\widehat{\theta}(a) - \mu(a) = d(\mu \cdot \pi) i_\mu(a) + i_\mu(a) d(\mu \cdot \pi).$$

Moreover, if $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ then the assignment $\widetilde{\lambda} : \mathcal{R}(\theta(\mathfrak{g})) \to \mathbb{C}$, $r(\theta(\mathfrak{g})) \mapsto r(\lambda(\mathfrak{g}))$, defines a character $\widetilde{\lambda} \in \operatorname{Spec}\mathcal{A}_\theta$. In particular, $\widehat{\theta}(a) - \lambda|^{\mathcal{A}_\mathfrak{g}}(a) = d(\lambda) i_\lambda(a) + i_\lambda(a) d(\lambda)$, where $\lambda|^{\mathcal{A}_\mathfrak{g}} = \widetilde{\lambda} \cdot \widehat{\theta} \in \operatorname{Spec}\mathcal{A}_\mathfrak{g}$ and $i_\lambda(a) \in \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$.

Proof. To prove the first equality, one suffices to proceed by induction on the order of rational function a and use the cohomological formulae (7.2) and (7.1). Indeed, assume that the equality has been proven for all rational functions $a = \overline{r}(\mathfrak{g})$ of order $\leq n-1$, where $\overline{r}(\mathfrak{g}) = \widehat{\pi}(r(\mathfrak{g}))$, $r(\mathfrak{g}) \in \bigcup_{k<n} \mathcal{R}^k_{\mathfrak{g},\pi}$. Take $a = \overline{r}(\mathfrak{g}) \in \mathcal{R}(\operatorname{im}(\widehat{\pi}))$ such that $r(\mathfrak{g}) \in \mathcal{R}^n_{\mathfrak{g},\pi}$. By definition (see Section 2.8), $r(\mathfrak{g}) = p\left(r_\iota(\mathfrak{g}), r_\kappa^{-1}(\mathfrak{g})\right)$ is a polynomial of a finite set variables $\Phi = \{r_\iota(\mathfrak{g})\}$ and $\Psi = \left\{r_\kappa^{-1}(\mathfrak{g})\right\}$ such that all $r_\iota(\mathfrak{g}), r_\kappa(\mathfrak{g}) \in \mathcal{R}^{n-1}_{\mathfrak{g},\pi}$. Then $a = p\left(\overline{r_\iota}(\mathfrak{g}), \overline{r_\kappa}(\mathfrak{g})^{-1}\right)$ in $\mathcal{A}_\mathfrak{g}$. Now one suffices to prove that if the equality are true for some $a, b \in \mathcal{R}(\operatorname{im}(\widehat{\pi}))$ then the same is true for their multiplication ab, and if a is invertible then the equality remains true for its inverse a^{-1}. But these statements immediately follow from Corollary 8.2. Indeed,

$$\widehat{\theta}(ab) - \mu(ab) = \widehat{\theta}(a)\left(\widehat{\theta}(b) - \mu(b)\right) + \mu(b)\left(\widehat{\theta}(a) - \mu(a)\right) =$$

$$= d(\mu \cdot \pi)\left(\widehat{\theta}(a) i_\mu(b) + \mu(b) i_\mu(a)\right) + \left(\widehat{\theta}(a) i_\mu(b) + \mu(b) i_\mu(a)\right) d(\mu \cdot \pi), \tag{8.2}$$

further, if a is invertible then

$$\widehat{\theta}\left(a^{-1}\right) - \mu\left(a^{-1}\right) = \widehat{\theta}\left(a^{-1}\right)\left(\widehat{\theta}(a) - \mu(a)\right)\mu\left(-a^{-1}\right) =$$

$$= d(\mu \cdot \pi)\widehat{\theta}\left(a^{-1}\right) i_\mu(a)\mu\left(-a^{-1}\right) + \widehat{\theta}\left(a^{-1}\right) i_\mu(a)\mu\left(-a^{-1}\right) d(\mu \cdot \pi). \tag{8.3}$$

Now take $\lambda \in \sigma_{\mathrm{t}}(\alpha)$. Let us prove that $\mathcal{R}_{\mathfrak{g},\theta} \subseteq \mathcal{R}_{\mathfrak{g},\lambda}$ and

$$r(\theta(\mathfrak{g})) - r(\lambda(\mathfrak{g})) = d(\lambda) i_\lambda(r(\mathfrak{g})) + i_\lambda(r(\mathfrak{g})) d(\lambda), \tag{8.4}$$

for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$, where $i_\lambda(r(\mathfrak{g})) \in \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$. We proceed again by induction on the order of a rational function. As above one suffices to prove the statement for the multiplication $r_1 r_2(\theta(\mathfrak{g}))$ and the inverse $r^{-1}(\theta(\mathfrak{g}))$. With $r_1 r_2(\lambda(\mathfrak{g})) = r_1(\lambda(\mathfrak{g})) r_2(\lambda(\mathfrak{g}))$ in mind, the statement for $r_1 r_2(\theta(\mathfrak{g}))$ follows from (8.2). Now assume that $r(\theta(\mathfrak{g}))$ is invertible in $\mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$. By induction hypothesis, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\lambda}$ and (8.4) is valid. If $r(\lambda(\mathfrak{g})) = 0$ then $1 = d(\lambda) r(\theta(\mathfrak{g}))^{-1} i_\lambda(r(\mathfrak{g})) + r(\theta(\mathfrak{g}))^{-1} i_\lambda(r(\mathfrak{g})) d(\lambda)$ by Corollary 8.2, which in turn implies that the complex $C^\cdot(\alpha - \lambda)$ is admissible. Then $\lambda \notin \sigma_{\mathrm{t}}(\alpha)$, a contradiction. Hence $r(\lambda(\mathfrak{g})) \neq 0$ and $r^{-1}(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\lambda}$. By (8.3), we end the proof of (8.4).

Thus (8.4) is valid and consequently $r(\lambda(\mathfrak{g})) \in \operatorname{sp}(r(\theta(\mathfrak{g})))$ for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$. If $r_1(\theta(\mathfrak{g})) = r_2(\theta(\mathfrak{g}))$ for some $r_1(\mathfrak{g}), r_2(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\theta}$, then $(r_1 - r_2)(\lambda(\mathfrak{g})) \in \operatorname{sp}((r_1 - r_2)(\theta(\mathfrak{g}))) = \{0\}$, whence the assignment $\widetilde{\lambda} : \mathcal{R}(\theta(\mathfrak{g})) \to \mathbb{C}$, $r(\theta(\mathfrak{g})) \mapsto r(\lambda(\mathfrak{g}))$, is defined soundly. Moreover, $\widetilde{\lambda}$ is continuous owing to $\widetilde{\lambda}(r(\theta(\mathfrak{g}))) \in \operatorname{sp}(r(\theta(\mathfrak{g})))$. Hence $\widetilde{\lambda} \in \operatorname{Spec} \mathcal{A}_\theta$ and $\lambda|^{\mathcal{A}_\mathfrak{g}} = \widetilde{\lambda} \cdot \widehat{\theta} \in \operatorname{Spec} \mathcal{A}_\mathfrak{g}$.

The rest is clear.◀

Remark 8.2. *Note that* $i_\mu(a) = \sum_{s=1}^{k} \widehat{\theta}(a_1 \cdots a_{s-1})\, i(a_s)\, \mu(a_{s+1} \cdots a_k)$ *whenever* $a = a_1 \cdots a_k$, $a_i = \pi(u_i)$, $u_i \in \mathfrak{g}$, $i(a_s) \in \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ *is the homotopy operator. The latter immediately follows from the first two cohomological formulae (7.1) and (7.2).*

Corollary 8.4. *The assignment* $\sigma_{\mathrm{t}}(\alpha) \to \operatorname{Spec} \mathcal{A}_\mathfrak{g}$, $\lambda \mapsto \lambda|^{\mathcal{A}_\mathfrak{g}}$, *is a continuous mapping.*

Proof. Fix $a \in \mathcal{A}_\mathfrak{g}$. We have to prove that the function $f_a : \sigma_{\mathrm{t}}(\alpha) \to \mathbb{C}$, $f_a(\lambda) = \lambda|^{\mathcal{A}_\mathfrak{g}}(a)$, is continuous. If $a = \widehat{\pi}(r(\mathfrak{g}))$ for some $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\pi}$, then f_a is reduced to a usual rational function $\lambda \mapsto r(\lambda(\mathfrak{g}))$ by virtue of Proposition 8.3. Therefore f_a is continuous. In general case, $\widehat{\theta}(a) = \lim_k \widehat{\theta}(a_k)$ of some sequence $a_k = \widehat{\pi}(r_k(\mathfrak{g}))$, $\{r_k(\mathfrak{g})\} \subseteq \mathcal{R}_{\mathfrak{g},\pi}$, by Definition 8.1. Then $f_a(\lambda) = \widetilde{\lambda}\left(\widehat{\theta}(a)\right) = \lim_k \widetilde{\lambda}\left(\widehat{\theta}(a_k)\right) = \lim_k \lambda|^{\mathcal{A}_\mathfrak{g}}(a_k) = \lim_k f_k(\lambda)$ for each point $\lambda \in \sigma_{\mathrm{t}}(\alpha)$, where $f_k \in C(\sigma_{\mathrm{t}}(\alpha))$, $f_k = f_{a_k}$, $k \in \mathbb{N}$. Moreover, bearing in mind that the norm of all characters (in particular, $\widetilde{\lambda}$, $\lambda \in \sigma_{\mathrm{t}}(\alpha)$) of a Banach algebra are at most one, we infer that

$$\sup_{\lambda \in \sigma_{\mathrm{t}}(\alpha)} |f_a(\lambda) - f_k(\lambda)| = \sup_{\lambda \in \sigma_{\mathrm{t}}(\alpha)} \left|\widetilde{\lambda}\left(\widehat{\theta}(a) - \widehat{\theta}(a_k)\right)\right| \leq \sup_{\lambda \in \sigma_{\mathrm{t}}(\alpha)} \left\|\widetilde{\lambda}\right\| \left\|\widehat{\theta}(a) - \widehat{\theta}(a_k)\right\| \leq$$

$$\leq \left\|\widehat{\theta}(a) - \widehat{\theta}(a_k)\right\| \to 0, \quad n \to \infty.$$

Thus f_a as a uniform limit of the sequence $\{f_k\}$ of continuous functions on the compact space $\sigma_{\mathrm{t}}(\alpha)$ (see Lemma 7.3) is turning into a continuous mapping, that is, $f_a \in C(\sigma_{\mathrm{t}}(\alpha))$. ◀

The image of a Slodkowski spectrum $\sigma(\alpha)$, $\sigma \in \mathfrak{S}$, under the mapping from Corollary 8.4 is denoted by $\sigma(\alpha)|^{\mathcal{A}_\mathfrak{g}}$, thus $\sigma(\alpha)|^{\mathcal{A}_\mathfrak{g}} \subseteq \operatorname{Spec} \mathcal{A}_\mathfrak{g}$.

Proposition 8.4. *Let* $\mathcal{A}_\mathfrak{g} \succ (X, \alpha)$ *and let* $\mathfrak{F}$ *be a normed Lie subalgebra in* $\mathcal{A}_\mathfrak{g}$. *Then* $\sigma^{\mathrm{ap}}(\alpha)|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F} \subseteq \sigma^{\mathrm{ap}}\left(\alpha|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F}\right)$.

Proof. Let $\mathfrak{U}$ be an ultrafilter in a set S such that $\sigma^{\mathrm{p}}(\alpha_\mathfrak{U}) = \sigma^{\pi,0}(\alpha_\mathfrak{U}) = \sigma^{\mathrm{ap}}(\alpha)$. Such possibility is allowed under will of Theorem 6.2. Take $\lambda \in \sigma^{\mathrm{ap}}(\alpha)$ and let $\lambda|^{\mathcal{A}_\mathfrak{g}}|_\mathfrak{F} = \mu$. Taking into account that $\lambda \in \sigma^{\mathrm{p}}(\alpha_\mathfrak{U})$, we conclude that it follows that $(\alpha_\mathfrak{U}(u) - \lambda(u))\, y = 0$, $u \in \mathfrak{g}$, for a nonzero $y \in X_\mathfrak{U}$. The latter merely means that $d^0_\mathfrak{U}(\lambda)\, y = 0$, where $d^0_\mathfrak{U}(\lambda)$ is the differential of $C^\bullet(\alpha_\mathfrak{U} - \lambda)$. By Lemma 8.1, $\mathcal{A}_\mathfrak{g} \succ (X_\mathfrak{U}, \alpha_\mathfrak{U})$, therefore $\widehat{\theta_\mathfrak{U}} = \widehat{\theta}_\mathfrak{U}$, in particular, $\widehat{\theta}_{0\mathfrak{U}} = \alpha_\mathfrak{U}|^{\mathcal{A}_\mathfrak{g}} = \widetilde{\alpha}_\mathfrak{U}$, where $\widetilde{\alpha} = \alpha|^{\mathcal{A}_\mathfrak{g}}$. Using Proposition 8.3, we obtain that $\widehat{\theta}_{0\mathfrak{U}}(\overline{r}(\mathfrak{g})) - \lambda|^{\mathcal{A}_\mathfrak{g}}(\overline{r}(\mathfrak{g})) = \iota_\lambda(\overline{r}(\mathfrak{g}))\, d^0_\mathfrak{U}(\lambda)$ for all $\overline{r}(\mathfrak{g}) = \widehat{\pi}(r(\mathfrak{g}))$, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\pi}$. The latter in turn implies that $\left(\widetilde{\alpha}_\mathfrak{U}(\overline{r}(\mathfrak{g})) - \lambda|^{\mathcal{A}_\mathfrak{g}}(\overline{r}(\mathfrak{g}))\right) y = 0$ for all $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\pi}$. Take $a \in \mathcal{A}_\mathfrak{g}$. Bearing in mind that $\widehat{\theta}_\mathfrak{U}(\mathcal{R}(\operatorname{im}(\pi)))$ is dense in $\widehat{\theta}_\mathfrak{U}(\mathcal{A}_\mathfrak{g})$, one can choose a sequence $\{r_n(\mathfrak{g})\}_{n \in \mathbb{N}} \subset \mathcal{R}_{\mathfrak{g},\pi}$ such that $\widehat{\theta}_\mathfrak{U}(a) = \lim_n \widehat{\theta}_\mathfrak{U}(\overline{r_n}(\mathfrak{g}))$. Then $\widetilde{\alpha}_\mathfrak{U}(a) = \lim_n \widetilde{\alpha}_\mathfrak{U}(\overline{r_n}(\mathfrak{g}))$ and $\lambda|^{\mathcal{A}_\mathfrak{g}}(a) = \widetilde{\lambda} \cdot \widehat{\theta}_\mathfrak{U}(a) = \lim_n \lambda|^{\mathcal{A}_\mathfrak{g}}(\overline{r_n}(\mathfrak{g}))$. Moreover,

$$\widetilde{\alpha}_\mathfrak{U}(a)\, y = \lim_n \widetilde{\alpha}_\mathfrak{U}(\overline{r_n}(\mathfrak{g}))\, y = \lim_n \lambda|^{\mathcal{A}_\mathfrak{g}}(\overline{r_n}(\mathfrak{g}))\, y = \lambda|^{\mathcal{A}_\mathfrak{g}}(a)\, y,$$

that is, $\mu \in \sigma^{\mathrm{p}}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$. Thus $\mu \in \sigma^{\mathrm{up}}(\widetilde{\alpha}|_{\mathfrak{F}})$. Moreover, $\sigma_{\mathrm{u}}^{\pi,0}(\widetilde{\alpha}|_{\mathfrak{F}}) = \sigma^{\mathrm{up}}(\widetilde{\alpha}|_{\mathfrak{F}}) = \sigma^{\mathrm{ap}}(\widetilde{\alpha}|_{\mathfrak{F}})$ by Theorem 6.1, whence $\mu \in \sigma^{\mathrm{ap}}(\widetilde{\alpha}|_{\mathfrak{F}})$. ◀

Note that the forward spectral mapping property for the approximate point spectrum suggested in Proposition 8.4 can be rewritten as $\sigma^{\pi,0}(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_{\mathrm{u}}^{\pi,0}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$. Indeed, $\sigma^{\mathrm{ap}}(\alpha) = \sigma_{\mathrm{u}}^{\pi,0}(\alpha) = \sigma^{\pi,0}(\alpha)$ by Theorem 6.1 and Lemma 6.2. Moreover, $\sigma^{\mathrm{ap}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma_{\mathrm{u}}^{\pi,0}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ ($\mathfrak{F}$ may be infinite-dimensional Lie subalgebra). Below we generalize the latter inclusion for all Slodkowski spectra $\sigma \in \mathfrak{S}^{\pi}$ under the condition that the norm-completion of $\mathfrak{F}$ is a projective Banach space.

The splitting elements over a $\mathfrak{g}$-module play a fundamental role in the backward spectral mapping property.

Definition 8.2. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$. An element $a \in \mathcal{A}_{\mathfrak{g}}$ is said to be splitting over $\mathfrak{g}$-module X if for each $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ there exists $n \in \mathbb{N}$ (called splitting power with respect to λ) and an operator $i_{n,\lambda}(a) \in \mathcal{B}(\mathcal{L}(\wedge\mathfrak{g}, X))$ such that*

$$\left(\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n} = d(\lambda)\, i_{n,\lambda}(a) + i_{n,\lambda}(a)\, d(\lambda).$$

An element $a \in \mathcal{A}_{\mathfrak{g}}$ is said to be weakly splitting over $\mathfrak{g}$-module X if for each $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ the actions of $\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)$ on cohomologies $H^{k}C^{\bullet}(\alpha - \lambda)$, $k \in \mathbb{Z}_{+}$ (see Corollary 8.2), are nilpotent. The set of all (resp., weakly) splitting over $\mathfrak{g}$-module X elements is denoted by $\mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$).

Obviously, $\mathcal{A}_{\mathfrak{g}}(\alpha) \subseteq \mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$, and $\mathrm{im}(\widehat{\pi}) \subseteq \mathcal{A}_{\mathfrak{g}}(\alpha)$ by virtue of Proposition 8.3, where $\pi : \mathfrak{g} \to \mathcal{A}_{\mathfrak{g}}$ is the given Lie homomorphism. Moreover, all rational functions have splitting powers equal to 1 with respect to all characters λ taken from $\sigma_{\mathrm{t}}(\alpha)$. Let us note that a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ of those elements having splitting powers 1 with respect to all $\lambda \in \sigma_{\mathrm{t}}(\alpha)$ generates a subalgebra in $\mathcal{A}_{\mathfrak{g}}$ containing $\mathrm{im}(\widehat{\pi})$. Indeed, for a such couple a, $b \in \mathcal{A}_{\mathfrak{g}}(\alpha)$, using the same argument as in (8.2) and Corollary 8.2, we have

$$\widehat{\theta}(ab) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(ab) = d(\lambda)\left(\widehat{\theta}(a)\, i_{1,\lambda}(b) + \lambda|^{\mathcal{A}_{\mathfrak{g}}}(b)\, i_{1,\lambda}(a)\right) + \left(\widehat{\theta}(a)\, i_{1,\lambda}(b) + \lambda|^{\mathcal{A}_{\mathfrak{g}}}(b)\, i_{1,\lambda}(a)\right) d(\lambda),$$

that is, the splitting power of ab equals 1.

8.2. The forward spectral mapping property

Now let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$ and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. Evidently, the projection map $\operatorname{Spec}\mathcal{A}_{\mathfrak{g}} \to \Delta(\mathfrak{F})$, $\lambda \mapsto \lambda|_{\mathfrak{F}}$, is continuous, and using Corollary 8.4, we obtain a continuous mapping $\sigma_{\mathrm{t}}(\alpha) \to \Delta(\mathfrak{F})$, $\lambda \mapsto \lambda|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$. Consider an arbitrary continuous extension $f : \Delta(\mathfrak{g}) \to \Delta(\mathfrak{F})$ of the latter mapping.

Lemma 8.2. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$, $\mathfrak{U}$ an ultrafilter and let $\widetilde{\alpha} = \alpha|^{\mathcal{A}_{\mathfrak{g}}}$. If $\widehat{\mathfrak{F}} \in \mathrm{Proj}$ then $f : \Delta(\mathfrak{g}) \to \Delta(\mathfrak{F})$ is a π-prespectral mapping with respect to the bicomplex $\mathcal{B}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$ connecting $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$.*

Proof. First, note that the complexes $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$ are π-spectrally connected by means of the bicomplex $\mathcal{B}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$ due to Proposition 8.2. Now take $\lambda \in \Delta(\mathfrak{g})$. If $\lambda \notin \sigma_{\mathrm{t}}(\alpha)$ then noting is left to prove (see Definition 5.1). So, assume that $H^{m}C^{\bullet}(\alpha_{\mathfrak{U}} - \lambda)$ is a nontrivial Banach space, $\mu = f(\lambda)$, and let $\beta_{\mathfrak{U}\mu}^{\sim} : H^{m}C^{\bullet}(\alpha_{\mathfrak{U}} - \lambda) \to H^{m}\mathcal{L}\left(\widehat{\mathfrak{F}}, C^{\bullet}(\alpha_{\mathfrak{U}} - \lambda)\right)$ be the differential of the m-th vertical cohomology complex of the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$. One should prove that $\beta_{\mathfrak{U}\mu}^{\sim} = 0$. Take $\omega \in \ker\left(d_{\mathfrak{U}}^{m}(\lambda)\right) \setminus \mathrm{im}\left(d_{\mathfrak{U}}^{m-1}(\lambda)\right)$, where $d_{\mathfrak{U}}^{m}(\lambda)$ (resp., $d_{\mathfrak{U}}^{m-1}(\lambda)$) is the differential of the complex $C^{\bullet}(\alpha_{\mathfrak{U}}-\lambda)$. Then $\beta_{\mathfrak{U}\mu}(\omega)\in\mathcal{L}\left(\widehat{\mathfrak{F}},C^{m}(\mathfrak{g},X_{\mathfrak{U}})\right)$, $\beta_{\mathfrak{U}\mu}(\omega)a=\left(\widehat{\theta}_{\mathfrak{U}}(a)-\mu(a)\right)\omega$, $a\in\mathfrak{F}$. By Lemma 8.1, $\mathcal{A}_{\mathfrak{g}}\succ(X_{\mathfrak{U}},\alpha_{\mathfrak{U}})$, therefore $\widehat{\theta}_{\mathfrak{U}}(a)=\lim_k\widehat{\theta}_{\mathfrak{U}}(a_k)$ for a certain sequence $\{a_k\}_{k\in\mathbb{N}}$ of rational functions in $\mathcal{A}_{\mathfrak{g}}$ acting on $X_{\mathfrak{U}}$. Moreover, $\mu(a)=\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)$. Appealing Proposition 8.3, we deduce that

$$\left(\widehat{\theta}_{\mathfrak{U}}(a)-\mu(a)\right)\omega=\lim_k\left(\widehat{\theta}_{\mathfrak{U}}(a_k)-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a_k)\right)\omega=\lim_k\left(d_{\mathfrak{U}}^{m-1}(\lambda)\,i_{\lambda}(a_k)+i_{\lambda}(a_k)\,d_{\mathfrak{U}}^{m}(\lambda)\right)\omega=$$

$$=\lim_k d_{\mathfrak{U}}^{m-1}(\lambda)\,i_{\lambda}(a_k)\,\omega\in\overline{\operatorname{im}\left(d_{\mathfrak{U}}^{m-1}(\lambda)\right)}=\operatorname{im}\left(d_{\mathfrak{U}}^{m-1}(\lambda)\right),$$

that is, $\operatorname{im}(\beta_{\mathfrak{U}\mu}(\omega))\subseteq\operatorname{im}\left(d_{\mathfrak{U}}^{m-1}(\lambda)\right)$. With $\widehat{\mathfrak{F}}\in\mathrm{Proj}$ in mind, infer $\beta_{\mathfrak{U}\mu}(\omega)=d_{\mathfrak{U}}^{m-1}(\lambda)\cdot T=\delta_{\mathfrak{U}\lambda}(T)$ for a certain $T\in\mathcal{L}\left(\widehat{\mathfrak{F}},C^{m-1}(\mathfrak{g},X_{\mathfrak{U}})\right)$ (see Subsection 2.5), where $\delta_{\mathfrak{U}\lambda}$ is the row differential of $\mathcal{B}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$. But the latter merely means that $\beta_{\mathfrak{U}\mu}^{\sim}(\omega)=0$. Thus f is a π-prespectral mapping.◄

Theorem 8.1. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$ and let $\sigma\in\mathfrak{S}^{\pi}$. If $\widehat{\mathfrak{F}}\in\mathrm{Proj}$ then $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\subseteq\sigma_{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$.*

Proof. Let $\mathfrak{U}$ be an ultrafilter and let $\widetilde{\alpha}=\alpha|^{\mathcal{A}_{\mathfrak{g}}}$. By Lemma 6.2, $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}=\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$. Moreover, the parametrized Banach space complexes $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$ are π-spectrally connected by means of $\Delta(\mathfrak{g})\times\Delta(\mathfrak{F})$-Banach bicomplex $\mathcal{B}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$ due to Proposition 8.2. Now let $f:\Delta(\mathfrak{g})\to\Delta(\mathfrak{F})$ be arbitrary continuous extension of the map $\sigma_{\mathrm{t}}(\alpha)\to\Delta(\mathfrak{F})$, $\lambda\mapsto\lambda|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$. Then f is a π-prespectral mapping with respect to the bicomplex $\mathcal{B}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$ connecting $\mathcal{C}^{\bullet}(\alpha)_{\mathfrak{U}}$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$ by Lemma 8.2. Using Theorem 5.1, we deduce that

$$f(\sigma(\alpha))=f(\sigma(\mathcal{C}^{\bullet}(\alpha)))\subseteq\sigma(\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}}))=\sigma(\mathcal{C}^{\bullet}((\widetilde{\alpha}|_{\mathfrak{F}})_{\mathfrak{U}}))\subseteq\sigma_{\mathrm{u}}(\widetilde{\alpha}|_{\mathfrak{F}}),$$

(see Definition 6.2), that is, $f(\sigma(\alpha))\subseteq\sigma_{\mathrm{u}}(\widetilde{\alpha}|_{\mathfrak{F}})$. ◄

Corollary 8.5. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$ and let $\mathfrak{F}$ be a finite-dimensional Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. Then $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\subseteq\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ for all $\sigma\in\mathfrak{S}_{\delta}\cup\mathfrak{S}^{\pi}$.*

Proof. Taking into account that $\mathfrak{F}\in\mathrm{Proj}$, the inclusion for spectra $\sigma\in\mathfrak{S}^{\pi}$ immediately follows Theorem 8.1 and Lemma 6.2.

Now fix $\sigma\in\mathfrak{S}_{\delta}$. By Lemma 8.1, $\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}\succ(X^{*},\alpha^{*})$. Using Theorem 8.1, we obtain that

$$\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}=\sigma^{*}(\alpha^{*})|^{\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}}|_{\mathfrak{F}^{\mathrm{op}}}\subseteq\sigma_{\mathrm{u}}^{*}\left(\alpha^{*}|^{\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}}|_{\mathfrak{F}^{\mathrm{op}}}\right)=\sigma^{*}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right)=\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right),$$

that is, $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\subseteq\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$. ◄

Corollary 8.6. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$ and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}}\in\mathrm{Proj}$ then $\sigma_{\delta,k}(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\subseteq\sigma_{\mathrm{u}}^{*}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right)$, $\sigma\in\mathfrak{S}_{\delta}$. In particular, $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\subseteq\sigma^{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$, $\sigma\in\mathfrak{S}_{\delta}$, whenever X is superreflexive.*

Proof. Using Lemma 8.1 and Theorem 8.1, we infer that $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}=\sigma^{*}(\alpha^{*})|^{\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}}|_{\mathfrak{F}^{\mathrm{op}}}\subseteq\sigma_{\mathrm{u}}^{*}(\gamma^{*})$, where $\gamma=\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$. If X is super-reflexive then $(X^{*})_{\mathfrak{U}}=(X_{\mathfrak{U}})^{*}$ for a countably incomplete ultrafilter $\mathfrak{U}$ by Proposition 2.1. In particular, $(\gamma^{*})_{\mathfrak{U}}=(\gamma_{\mathfrak{U}})^{*}$ and $\sigma_{\mathrm{u}}^{*}(\gamma^{*})=\bigcup_{\mathfrak{U}}\sigma^{*}((\gamma^{*})_{\mathfrak{U}})=\bigcup_{\mathfrak{U}}\sigma^{*}\left((\gamma_{\mathfrak{U}})^{*}\right)=\bigcup_{\mathfrak{U}}\sigma(\gamma_{\mathfrak{U}})=\sigma^{\mathrm{u}}(\gamma)$ by Definition 6.2.◄

8.3. The backward spectral mapping property

Now we investigate the problem whether or not a continuous extension $f : \Delta(\mathfrak{g}) \to \Delta(\mathfrak{F})$ of the map $\sigma_{\mathrm{t}}(\alpha) \to \Delta(\mathfrak{F})$, $\lambda \mapsto \lambda|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$, is a π-spectral mapping with respect to the $\Delta(\mathfrak{g}) \times \Delta(\mathfrak{F})$-Banach bicomplex $\mathcal{B}(\mathfrak{g}, \mathfrak{F}, X_{\mathfrak{U}})$ connecting the complexes $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$, where $\widetilde{\alpha} = \alpha|^{\mathcal{A}_{\mathfrak{g}}}$.

Fix $m \in \mathbb{Z}_+$ and let $\alpha_\lambda = \alpha - \lambda$, $\lambda \in \Delta(\mathfrak{g})$. Then

$$0 \to H^m C^{\bullet}(\alpha_\lambda) \overset{\beta_\mu^{\sim}}{\to} H^m C^1\left(\widehat{\mathfrak{F}}, C^{\bullet}(\alpha_\lambda)\right) \to \cdots \to H^m C^s\left(\widehat{\mathfrak{F}}, C^{\bullet}(\alpha_\lambda)\right) \overset{\beta_\mu^{\sim}}{\to} \cdots, \tag{8.5}$$

is the m-th vertical cohomology complex of the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$. By Corollary 8.2, $\ker d^m(\lambda)$ is a closed $\mathcal{A}_{\mathfrak{g}}$ (in particular, $\mathfrak{F}$)-submodule in $C^m(\mathfrak{g}, X)$, where $d^m(\lambda)$ is the differential of (0-th row) the complex $C^{\bullet}(\alpha_\lambda)$. The cochain complex $C^{\bullet}\left(\widehat{\theta}|_{\mathfrak{F}} - \mu | \ker d^m(\lambda)\right)$ generated by the $\mathfrak{F}$-module $\left(\ker d^m(\lambda), \widehat{\theta}|_{\mathfrak{F}} - \mu\right)$ is a Banach space complex of $\mathfrak{F}$-modules and it is a subcomplex of the m-th column of $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$. The $\mathfrak{F}$-module structure on this complex is defined by the θ-representation (see Subsection 7.1)

$$\Theta_{s,\mu} : \mathfrak{F} \to \mathcal{B}\left(\mathcal{L}\left(\wedge^s \mathfrak{F}, \ker d^m(\lambda)\right)\right), \quad \Theta_{s,\mu}(a) = L_{\left(\widehat{\theta} - \mu\right)(a)} - R_{T_s(a)},$$

extending $\widehat{\theta}|_{\mathfrak{F}} - \mu$, and let $I_s(a)$ (here $a \in \mathfrak{F}$) be the homotopy operator on $C^{\bullet}\left(\widehat{\theta}|_{\mathfrak{F}} - \mu | \ker d^m(\lambda)\right)$. Using Corollary 8.2, one easily verify that $L_{\widehat{\theta}(a)} \delta_\lambda^{m-1} = \delta_\lambda^{m-1} L_{\widehat{\theta}(a)}$, $R_{T_s(a)} \delta_\lambda^{m-1} = \delta_\lambda^{m-1} R_{T_s(a)}$ and $I_s(a) \delta_\lambda^{m-1} = \delta_\lambda^{m-1} I_s(a)$, where δ_λ^{m-1} is the row differential of $\mathcal{B}_{\lambda,\mu}(\mathfrak{g}, \mathfrak{F}, X)$. In particular, $\Theta_{s,\mu}(a) \delta_\lambda^{m-1} = \delta_\lambda^{m-1} \Theta_{s,\mu}(a)$ and the image $\operatorname{im}\left(\delta_\lambda^{m-1}\right)$ is invariant under all operators $L_{\widehat{\theta}(a)}$, $R_{T_s(a)}$, $\Theta_{s,\mu}(a)$ and $I_s(a)$, $a \in \mathfrak{F}$, whence they induce operators on cohomologies

$$\begin{aligned} L^{\sim}_{\widehat{\theta}(a)}, \Theta^{\sim}_{s,\mu}(a), R^{\sim}_{T_s(a)} &\in \mathcal{B}\left(H^m C^s\left(\mathfrak{F}, C^{\bullet}(\alpha_\lambda)\right)\right), \\ I_s^{\sim}(a) &\in \mathcal{B}\left(H^m C^s\left(\mathfrak{F}, C^{\bullet}(\alpha_\lambda)\right), H^m C^{s-1}\left(\mathfrak{F}, C^{\bullet}(\alpha_\lambda)\right)\right), \end{aligned}$$

by the canonical way. Using the first and second cohomological formulae (7.1), (7.2) for the complex $C^{\bullet}\left(\widehat{\theta}|_{\mathfrak{F}} - \mu | \ker d^m(\lambda)\right)$ and by passing to cohomologies, we obtain that

$$\beta_\mu^{\sim} \Theta^{\sim}_{s,\mu}(a) = \Theta^{\sim}_{s,\mu}(a) \beta_\mu^{\sim}, \tag{8.6}$$

$$\beta_\mu^{\sim} I_s^{\sim}(a) + I_{s+1}^{\sim}(a) \beta_\mu^{\sim} = \Theta^{\sim}_{s,\mu}(a). \tag{8.7}$$

The following lemma describes the operator $\Theta^{\sim}_{s,\mu}(a)$ when $a \in \mathcal{A}_{\mathfrak{g}}(\alpha)$.

Lemma 8.3. *Assume that* $a \in \mathfrak{F} \cap \mathcal{A}_{\mathfrak{g}}(\alpha)$, *or* $a \in \mathfrak{F} \cap \mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ *and* $\dim(\mathfrak{F}) < \infty$. *Then* $\Theta^{\sim}_{s,\mu}(a) = \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a) - \mu(a) - R^{\sim}_{T_s(a)} + N$, *where* N *is a nilpotent operator. In particular,* $0 \notin \operatorname{sp}\left(\Theta^{\sim}_{s,\mu}(a)\right)$ *whenever* $\operatorname{sp}\left(\operatorname{ad}(a)|_{\mathfrak{F}}\right) = \{0\}$ *and* $\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a) \neq \mu(a)$.

Proof. By definition of $\Theta^{\sim}_{s,\mu}(a)$, one suffices to prove that $N = L^{\sim}_{\widehat{\theta}(a)} - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)$ is a nilpotent operator on the cohomology $H^m C^s\left(\mathfrak{F}, C^{\bullet}(\alpha_\lambda)\right)$. Take $\Phi \in C^s\left(\mathfrak{F}, \ker d^m(\lambda)\right)$. By Definition 8.2, $\left(\widehat{\theta}(a) - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^n = d^{m-1}(\lambda) i_{n,\lambda}(a) + i_{n,\lambda}(a) d^m(\lambda)$ for a certain $n \in \mathbb{N}$ and some operator $i_{n,\lambda}(a) \in \mathcal{B}\left(\mathcal{L}\left(\wedge \mathfrak{g}, X\right)\right)$ whenever $a \in \mathfrak{F} \cap \mathcal{A}_{\mathfrak{g}}(\alpha)$. Then

$$\left(L_{\widehat{\theta}(a)} - \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^n \Phi = d^{m-1}(\lambda) i_{n,\lambda}(a) \Phi + i_{n,\lambda}(a) d^m(\lambda) \Phi = \delta_\lambda^{m-1}\left(L_{i_{n,\lambda}(a)} \Phi\right),$$

whence $\left(L^{\sim}_{\widehat{\theta}(a)}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi^{\sim}=\left(\delta^{m-1}_{\lambda}\left(L_{i_{n,\lambda(a)}}\Phi\right)\right)^{\sim}=0$. If $a\in\mathfrak{F}\cap\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ and $\dim(\mathfrak{F})<\infty$, then $\left(\left(L_{\widehat{\theta}(a)}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi\right)(\underline{u})=\left(\widehat{\theta}(a)-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}(\Phi(\underline{u}))\in\operatorname{im}\left(d^{m-1}(\lambda)\right)$ for all $\underline{u}\in\wedge^{s}\mathfrak{F}$, and therefore $\left(L_{\widehat{\theta}(a)}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi=d^{m-1}(\lambda)\cdot\Psi$ for some $\Psi\in C^{s}\left(\mathfrak{F},C^{m-1}(\mathfrak{g},X)\right)$. It follows again that $\left(L^{\sim}_{\widehat{\theta}(a)}-\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)^{n}\Phi^{\sim}=0$.

Now let us assume that $\operatorname{sp}(\operatorname{ad}(a)|_{\mathfrak{F}})=\{0\}$ and $z_a=\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)-\mu(a)\neq 0$. Obviously, the operator of the adjoint representation $\operatorname{ad}(a)\in\mathcal{B}\left(\widehat{\mathfrak{F}}\right)$ is quasinilpotent, thereby so are all operators $T_s(a)\in\mathcal{B}\left(\wedge^{s}\widehat{\mathfrak{F}}\right)$, $s\in\mathbb{Z}_+$, by virtue of Lemma 7.1. The latter involves that $\operatorname{sp}\left(R_{T_s(a)}\right)=\{0\}$ for all s. But, $R_{T_s(a)}\in\mathcal{B}(C^{s}(\mathfrak{F},\ker d^{m}(\lambda)))$ is a Banach space operator, so, the series $G_s(a)=\sum_{k=0}^{\infty}\left(z_a^{-1}R_{T_s(a)}\right)^{k}$ converges absolutely in $\mathcal{B}(C^{s}(\mathfrak{F},\ker d^{m}(\lambda)))$ and $\delta^{m-1}_{\lambda}G_s(a)=G_s(a)\,\delta^{m-1}_{\lambda}$. Moreover, the operator $G^{\sim}_s(a)\in\mathcal{L}(H^{m}C^{s}(\mathfrak{F},C^{\bullet}(\alpha_\lambda)))$ commutes with N. Indeed, $[G^{\sim}_s(a),N]=\left[G_s(a),L_{\widehat{\theta}(a)}\right]^{\sim}=\left(\sum_{k=0}^{\infty}z_a^{-1}\left[R^{k}_{T_s(a)},L_{\widehat{\theta}(a)}\right]\right)^{\sim}=0^{\sim}$. Note also that $z_a^{-1}G_s(a)=\left(z_a-R_{T_s(a)}\right)^{-1}$ and $z_a^{-1}G^{\sim}_s(a)=\left(z_a-R^{\sim}_{T_s(a)}\right)^{-1}$. Finally

$$z_a^{-1}G^{\sim}_s(a)\,\Theta^{\sim}_{s,\mu}(a)=z_a^{-1}\Theta^{\sim}_{s,\mu}(a)\,G^{\sim}_s(a)=z_a^{-1}G^{\sim}_s(a)\left(z_a-R^{\sim}_{T_s(a)}+N\right)=$$

$$=1+z_a^{-1}G^{\sim}_s(a)\,N.$$

It is clear that $1+z_a^{-1}G^{\sim}_s(a)\,N$ is invertible and

$$\left(1+z_a^{-1}G^{\sim}_s(a)\,N\right)^{-1}=\sum_{k=0}^{n-1}(-1)^{k}\left(z_a^{-1}G^{\sim}_s(a)\,N\right)^{k}.$$

Thus $0\notin\operatorname{sp}\left(\Theta^{\sim}_{s,\mu}(a)\right)$.◄

Proposition 8.5. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$ and let S be a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$) generating a quasinilpotent normed (resp., finite-dimensional) Lie subalgebra $\mathfrak{F}\subseteq\mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}}\in\operatorname{Proj}$ then $f:\Delta(\mathfrak{g})\to\Delta(\mathfrak{F})$ is a π-spectral mapping with respect to the bicomplex $\mathcal{B}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$ connecting $\mathcal{C}^{\bullet}(\alpha_{\mathfrak{U}})$ and $\mathcal{C}^{\bullet}(\widetilde{\alpha}_{\mathfrak{U}}|_{\mathfrak{F}})$, where $\widetilde{\alpha}=\alpha|^{\mathcal{A}_{\mathfrak{g}}}$.*

Proof. First, note that if $S\subseteq\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ generates a finite-dimensional Lie subalgebra $\mathfrak{F}\subseteq\mathcal{A}_{\mathfrak{g}}$ then automatically $\mathfrak{F}=\widehat{\mathfrak{F}}\in\operatorname{Proj}$. We have already proven (see Lemma 8.2) that f is a π-prespectral mapping whenever $\widehat{\mathfrak{F}}\in\operatorname{Proj}$. So, it remains (see Definition 5.1) to prove that all vertical cohomology complexes (8.5) for the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$ are exact, whenever $f(\lambda)\neq\mu$. Note that $f(\lambda)=\lambda|^{\mathcal{A}_{\mathfrak{g}}}$ and if $f(\lambda)\neq\mu$ then $\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\neq\mu(a)$ for a certain $a\in S$, for S is a topological Lie generator set of $\mathfrak{F}$ (or $\widehat{\mathfrak{F}}$). By assumption, $a\in\mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $a\in\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$) and $\operatorname{sp}(\operatorname{ad}(a)|_{\mathfrak{F}})=\{0\}$. Using Lemma 8.3, we conclude that all operators $\Theta^{\sim}_{s,\mu}(a)$, $s\in\mathbb{Z}_+$, acting on the vertical cohomology complexes of the bicomplex $\mathcal{B}_{\lambda,\mu}(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})$ are invertible. But, the latter implies that all vertical cohomology complexes are exact by virtue of (8.6) and (8.7).◄

As follows from Proposition 8.5, to prove the backward spectral mapping property for normed Lie subalgebras of the dominating algebra one remains to establish the projection property suggested in Theorem 5.2. The following assertion implements our aim, it is a topological version of Fainshtein's lemma [46, Lemma 5.2].

Lemma 8.4. *Let* $\mathcal{A}_{\mathfrak{g}} \succ (X, \alpha)$*,* $\mathfrak{F}$ *a normed Lie subalgebra in* $\mathcal{A}_{\mathfrak{g}}$*,* I *and* J *ideals in* $\mathfrak{g}$*, such that* $J \subseteq I$*,* $\dim(I/J) = 1$*,* $\lambda \in \Delta(I)$*,* $\mu \in \Delta(\mathfrak{F})$*, and let* $u \in I \backslash J$*. There exists a bounded endomorphism* $\kappa(u)$ *of the Banach space complex* $\mathrm{Tot}_{\lambda|_J,\mu}(J, \mathfrak{F}, X)$ *such that*

$$\mathrm{Tot}_{\lambda,\mu}(I, \mathfrak{F}, X) = \mathrm{Con}\left(\mathrm{Tot}_{\lambda|_J,\mu}(J, \mathfrak{F}, X), \kappa(u) - \lambda(u)\right),$$

Moreover, $\mathrm{sp}(\kappa(u)) = \mathrm{sp}(\alpha(u))$*.*

Proof. We denote the restriction of $\omega \in C^k(I, X)$ onto $\wedge^k J$ by $\omega|_J$. The map

$$\varkappa_{k,s} : C^s\left(\mathfrak{F}, C^k(I, X)\right) \to C^s\left(\mathfrak{F}, C^k(J, X)\right) \oplus C^s\left(\mathfrak{F}, C^{k-1}(J, X)\right),$$

$$\varkappa_{k,s}(\Phi) = \left(\Phi|_J, (i(u)\Phi)|_J\right),$$

is a topological isomorphism by virtue of Lemma 7.4, where $\Phi|_J(h) = \Phi(h)|_J$, $h \in \wedge^k \mathfrak{F}$. By definition, the differential $\gamma_{\lambda,\mu}$ of $\mathrm{Tot}_{\lambda,\mu}(I, \mathfrak{F}, X)$ is given by the rule $\gamma_{\lambda,\mu}(\Phi) = \delta_\lambda(\Phi) + (-1)^k \beta_\mu(\Phi)$, $\Phi \in C^s\left(\mathfrak{F}, C^k(I, X)\right)$. Let us find the components of $\varkappa_{k+1,s}(\delta_\lambda(\Phi))$ and $\varkappa_{k,s+1}\left((-1)^k \beta_\mu(\Phi)\right)$ in the relevant decompositions. Note that $\delta_\lambda(\Phi) = d_\lambda \cdot \Phi$ and

$$(d_\lambda \cdot \Phi)|_J(h) = d_\lambda(\Phi(h))|_J = d_{\lambda|_J}(\Phi(h)|_J) = \left(\delta_{\lambda|_J}(\Phi|_J)\right)(h), \quad h \in \wedge^k \mathfrak{F},$$

where d_λ is differential of $C^\bullet(\alpha|_I - \lambda)$. To transform the second term of $\varkappa_{k+1,s}(\delta_\lambda(\Phi))$, we use the second cohomological formula (7.2):

$$(i(u)(d_\lambda \cdot \Phi))|_J(h) = (i(u) d_\lambda(\Phi(h)))|_J = -d_{\lambda|_J}(i(u)\Phi(h))|_J + (\theta - \lambda)(u)(\Phi(h)|_J).$$

Thus, $\varkappa_{k+1,s}(\delta_\lambda(\Phi)) = \left(\delta_{\lambda|_J}(\Phi|_J), -\delta_{\lambda|_J}((i(u)\Phi)|_J) + (\theta - \lambda)(u)(\Phi|_J)\right)$. Let us transform components of $\varkappa_{k,s+1}(\beta_\mu(\Phi)) = (\beta_\mu(\Phi)|_J, (i(u)\beta_\mu(\Phi))|_J)$:

$$\beta_\mu(\Phi)|_J(\underline{a}) = \sum_{i=1}^{s+1} (-1)^{i+1}\left(\widehat{\theta} - \mu\right)(a_i)\Phi(\underline{a}_i)|_J + \sum_{i<j} (-1)^{i+j}\Phi\left([a_i, a_j] \wedge \underline{a}_{i,j}\right)|_J =$$

$$= \beta_\mu(\Phi|_J)(\underline{a}),$$

where $\underline{a} = a_1 \wedge \ldots \wedge a_{s+1} \in \wedge^{s+1}\mathfrak{F}$. Thus $\beta_\mu(\Phi)|_J = \beta_\mu(\Phi|_J)$. To transform the second term, we introduce the operator

$$\Gamma : \mathcal{A}_{\mathfrak{g}} \to \mathcal{B}(\mathcal{L}(\wedge I, X)), \quad \Gamma(a) = \left[i(u), \widehat{\theta}(a)\right].$$

Demonstrate that if $\omega \in C^k(I, X)$, $\omega|_J = 0$, then $(\Gamma(a)\omega)|_J = 0$ for all $a \in \mathcal{A}_{\mathfrak{g}}$. Taking into account that $\widehat{\theta}(\mathrm{im}(\widehat{\pi}))$ is dense in $\widehat{\theta}(\mathcal{A}_{\mathfrak{g}})$ (see Definition 8.1), one suffices to prove the latter for the rational functions $a = r(\pi(\mathfrak{g}))$, $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\pi}$, in $\mathcal{A}_{\mathfrak{g}}$ acting on X. We proceed by induction on the order of rational functions $r(\mathfrak{g})$. We divide our inductive arguments into several steps.

Step 1. If the assertion is valid for some $r(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\pi}$ and $a = r(\pi(\mathfrak{g}))$ is invertible in $\mathcal{A}_{\mathfrak{g}}$ then the same is true for $r^{-1}(\mathfrak{g})$. Indeed, let $\omega|_J = 0$. Obviously

$$\left(\Gamma\left(a^{-1}\right)\omega\right)|_J = -\left(\widehat{\theta}(a)^{-1}\Gamma(a)\widehat{\theta}(a)^{-1}\omega\right)|_J.$$

By Proposition 8.1, the restriction map $C^k(I, X) \to C^k(J, X)$, $\omega \mapsto \omega|_J$, is a morphism in $\mathcal{A}_{\mathfrak{g}}$-mod, therefore $\left(\widehat{\theta}(a)^{-1}\omega\right)|_J = 0$, for $\omega|_J = 0$. By assumption, $\left(\Gamma(a)\widehat{\theta}(a)^{-1}\omega\right)|_J = 0$. Then again $\left(\widehat{\theta}(a)^{-1}\Gamma(a)\widehat{\theta}(a)^{-1}\omega\right)|_J = 0$, whence $\left(\Gamma\left(a^{-1}\right)\omega\right)|_J = 0$.

Step 2. If the assertion is valid for some subset $S \subseteq \mathcal{R}_{\mathfrak{g},\pi}$ then the same is true for a polynomial p in variables S. One suffices to prove the assertion when p is a monomial. Let $\omega|_J = 0$ and let $p = r_1(\mathfrak{g}) \cdots r_m(\mathfrak{g}) \in \mathcal{R}_{\mathfrak{g},\pi}$, $r_i(\mathfrak{g}) \in S$, be a monomial. We set $a_i = r_i(\pi(\mathfrak{g}))$ and $b = a_1 \cdots a_m$. As in Step 1, we have $\left(\widehat{\theta}(a_2 \cdots a_m)\omega\right)|_J = 0$. Further

$$(\Gamma(b)\omega)|_J = \widehat{\theta}(a_1)(\Gamma(a_2 \cdots a_m)\omega)|_J + \Gamma(a_1)\left(\widehat{\theta}(a_2 \cdots a_m)\omega\right)|_J =$$

$$= \widehat{\theta}(a_1)(\Gamma(a_2 \cdots a_m)\omega)|_J.$$

By induction on length m of the monomial, we obtain that $(\Gamma(b)\omega)|_J = 0$.

Step 3. The assertion is valid for all $a \in \mathfrak{g}$. Indeed, in this case $\Gamma(a) = i([u,a])$ by the third cohomological formula (7.3), and $[u,a] \in J$, for $\mathfrak{g}$ is nilpotent. Thus $(i([u,a])\omega)|_J = 0$ because of $\omega|_J = 0$.

Now using statements from Step 2 and 3, we obtain that the assertion is valid for all polynomials $r(\mathfrak{g}) \in \mathcal{R}^0_{\mathfrak{g},\pi}$. By induction hypothesis, the assertion is valid for all $r(\mathfrak{g})$, $r(\mathfrak{g}) \in \mathcal{R}^{n-1}_{\mathfrak{g},\pi}$. Using Step 1 and 2, we obtain that the assertion is valid for all $r(\mathfrak{g}) \in \mathcal{R}^n_{\mathfrak{g},\pi}$ too.

Thus $(\Gamma(a)\omega)|_J = 0$ for all $a \in \mathcal{A}_{\mathfrak{g}}$, whenever $\omega \in C^k(I,X)$ and $\omega|_J = 0$. We set

$$\Gamma \wedge \Phi \in C^{s+1}\left(\mathfrak{F}, C^{k-1}(I,X)\right), \quad (\Gamma \wedge \Phi)(\underline{a}) = \sum_{i=1}^{s+1}(-1)^{i+1}\Gamma(a_i)\Phi(\underline{a}_i),$$

where $\Phi \in C^s\left(\mathfrak{F}, C^k(I,X)\right)$ and $\underline{a} = a_1 \wedge \ldots \wedge a_{s+1} \in \wedge^{s+1}\mathfrak{F}$. Using the assertion just proved above, we conclude that the assignment

$$C^s\left(\mathfrak{F}, C^k(J,X)\right) \to C^{s+1}\left(\mathfrak{F}, C^{k-1}(J,X)\right), \quad \Phi|_J \mapsto (-1)^k(\Gamma \wedge \Phi)|_J,$$

is a bounded linear operator denoted by Γ_k. Then

$$(i(u)\beta_\mu(\Phi))|_J(\underline{a}) = \sum_{i=1}^{s+1}(-1)^{i+1}\left(i(u)\left(\widehat{\theta} - \mu\right)(a_i)\Phi(\underline{a}_i)\right)|_J +$$

$$+\sum_{i<j}(-1)^{i+j}\left(i(u)\Phi\left([a_i,a_j] \wedge \underline{a}_{i,j}\right)\right)|_J = \beta_\mu((i(u)\Phi)|_J)(\underline{a}) + (\Gamma \wedge \Phi)|_J(\underline{a}).$$

Thus $\varkappa_{k,s+1}(\beta_\mu(\Phi)) = \left(\beta_\mu(\Phi|_J), \beta_\mu((i(u)\Phi)|_J) + (-1)^k\Gamma_k(\Phi|_J)\right)$ and for the differential $\gamma_{\lambda,\mu} = \delta_\lambda + (-1)^k\beta_\mu$ of the complex $\mathrm{Tot}_{\lambda,\mu}(I,\mathfrak{F},X)$ we obtain (to within an isomorphism) the following expression

$$\gamma_{\lambda,\mu}(\Phi|_J, (i(u)\Phi)|_J) = \left(\gamma_{\lambda|_J,\mu}(\Phi|_J), -\gamma_{\lambda|_J,\mu}((i(u)\Phi)|_J) + (\kappa(u) - \lambda(u))(\Phi|_J)\right),$$

where $\kappa(u) = L_{\theta(u)} + \sum_k \Gamma_k$. The operator $\kappa(u)$ is presented by a triangular operator matrix with diagonal elements $L_{\theta(u)}$. Then $\mathrm{sp}(\kappa(u)) = \mathrm{sp}(\theta(u))$, and $\mathrm{sp}(\theta(u)) = \mathrm{sp}(\alpha(u))$ by virtue of Lemma 7.1. The condition $\gamma^2_{\lambda,\mu} = 0$ implies that $\kappa(u)$ is an endomorphism of the complex $\mathrm{Tot}_{\lambda|_J,\mu}(J,\mathfrak{F},X)$, and the expression for $\gamma_{\lambda,\mu}$ demonstrates that it is the differential of the cone (see Subsection 2.4).◄

Theorem 8.2. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$, I and J ideals in $\mathfrak{g}$, such that $J \subseteq I$, $\dim(I/J) = 1$, and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}} \in \mathrm{Proj}$ then*

$$\sigma(I,\mathfrak{F},X)|_{J\times\mathfrak{F}} = \sigma(J,\mathfrak{F},X),$$

for all $\sigma \in \mathfrak{S}^\pi$. In particular, $\sigma(\mathfrak{g},\mathfrak{F},X)|_{\{0\}\times\mathfrak{F}} = \sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$.

Proof. By Lemma 8.4, $\sigma(I,\mathfrak{F},X)=\sigma(\mathrm{Tot}(I,\mathfrak{F},X))\subseteq\sigma\left(\mathrm{Con}_{\kappa(u)}\mathrm{Tot}(J,\mathfrak{F},X)\right)$, where $u\in I\backslash J$. Conversely, take $(\tau,c)\in\sigma\left(\mathrm{Con}_{\kappa(u)}\mathrm{Tot}(J,\mathfrak{F},X)\right)$. Using projection property stated in Theorem 4.1, we infer that $\tau\in\sigma(\mathrm{Tot}(J,\mathfrak{F},X))$. Note that the parametrized Banach space complexes $\mathcal{C}^{\bullet}(\alpha|_J)$ and $\mathcal{C}^{\bullet}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ are π-spectrally connected by means of $\Delta(J)\times\Delta(\mathfrak{F})$-Banach bicomplex $\mathcal{B}(J,\mathfrak{F},X)$ due to Proposition 8.2 (since $\widehat{\mathfrak{F}}\in\mathrm{Proj}$). Then $\tau_J\in\sigma_{\mathrm{t}}(\alpha|_J)$ by virtue of Proposition 5.1, where $\tau_J=\tau|_{J\times\{0\}}$. By Turovskii lemma 2.13, $\alpha([I,I])$ consists of quasinilpotent operators, and $[I,I]\subseteq J$, for $\mathfrak{g}$ is a nilpotent Lie algebra. It follows that $\tau_J([I,I])=0$ by Theorem 7.1. Thus each linear extension of τ_J up to a functional on I is a Lie charter. Take $\lambda\in I^*$ such that $\lambda|_J=\tau_J$ and $\lambda(u)=c$. By Lemma 8.4

$$\mathrm{Tot}_{\lambda,\tau_{\mathfrak{F}}}(I,\mathfrak{F},X)=\mathrm{Con}\left(\mathrm{Tot}_{\tau_J,\tau_{\mathfrak{F}}}(J,\mathfrak{F},X),\kappa(u)-c\right),$$

where $\tau_{\mathfrak{F}}=\tau|_{\{0\}\times\mathfrak{F}}$. Thereby, $(\tau,c)\in\sigma(I,\mathfrak{F},X)$. Consequently

$$\sigma(I,\mathfrak{F},X)=\sigma\left(\underset{\kappa(u)}{\mathrm{Con}}\,\mathrm{Tot}(J,\mathfrak{F},X)\right),\sigma\in\mathfrak{S}^{\pi}.$$

Now we can deduce that $\sigma(I,\mathfrak{F},X)|_{J\times\mathfrak{F}}=\sigma(J,\mathfrak{F},X)$ by virtue of Theorem 4.1. Further, let

$$\mathfrak{g}=I_1\supset I_2\supset\cdots\supset I_n\supset I_{n+1}=\{0\},$$

be a chain of ideals in $\mathfrak{g}$ such that $\dim(I_i/I_{i+1})=1$, $1\leq i\leq n$. Then $\sigma(I_k,\mathfrak{F},X)|_{I_{k+1}\times\mathfrak{F}}=\sigma(I_{k+1},\mathfrak{F},X)$ for all k. Finally

$$\sigma(\mathfrak{g},\mathfrak{F},X)|_{\{0\}\times\mathfrak{F}}=\sigma(I_2,\mathfrak{F},X)|_{\{0\}\times\mathfrak{F}}=\cdots=\sigma(I_n,\mathfrak{F},X)|_{\{0\}\times\mathfrak{F}}=\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right),$$

that is, $\sigma(\mathfrak{g},\mathfrak{F},X)|_{\{0\}\times\mathfrak{F}}=\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$.◄

Corollary 8.7. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$, $\mathfrak{U}$ an ultrafilter and let $\mathfrak{F}$ be a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}}\in\mathrm{Proj}$ then*

$$\sigma(\mathfrak{g},\mathfrak{F},X_{\mathfrak{U}})|_{\{0\}\times\mathfrak{F}}=\sigma\left(\alpha_{\mathfrak{U}}|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right),$$

for all $\sigma\in\mathfrak{S}$.

Proof. One suffices to apply Theorem 8.2 to the ultrapower $X_{\mathfrak{U}}$.◄

Now we are in a position to prove the spectral mapping theorem.

Theorem 8.3. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$ and let S be a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ (resp., $\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$) generating a quasinilpotent normed (resp., finite-dimensional) Lie subalgebra $\mathfrak{F}\subseteq\mathcal{A}_{\mathfrak{g}}$. If $\widehat{\mathfrak{F}}\in\mathrm{Proj}$ then*

$$\sigma_{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)=\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}},\quad\sigma\in\mathfrak{S}^{\pi}.$$

Proof. The inclusion $\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\subseteq\sigma_{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ was proved in Theorem 8.1. The reverse inclusion follows from Proposition 8.5, Corollary 8.7 and Theorem 5.2.◄

Corollary 8.8. *Let $\mathcal{A}_{\mathfrak{g}}\succ(X,\alpha)$, and let $\mathfrak{F}\subseteq\mathcal{A}_{\mathfrak{g}}\langle\alpha\rangle$ be a finite-dimensional nilpotent Lie subalgebra. Then*

$$\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)=\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}},\quad\sigma\in\mathfrak{S}.$$

Proof. If $\sigma \in \mathfrak{S}^{\pi}$ then result follows from Theorem 8.3. To prove the equality for spectra $\sigma \in \mathfrak{S}_{\delta}$ we use the same argument carried out in the proof of Corollary 8.5. Namely using Lemma 8.1 and Theorem 8.3, we obtain that

$$\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma^{*}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right) = \sigma_{\mathrm{u}}^{*}\left(\alpha^{*}|^{\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}}|_{\mathfrak{F}^{\mathrm{op}}}\right) \subseteq \sigma^{*}\left(\alpha^{*}\right)|^{\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}}|_{\mathfrak{F}^{\mathrm{op}}} = \sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}},$$

and by Corollary 8.5, $\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$.

Thus the equality has been proven for all $\sigma \in \mathfrak{S}_{\delta} \cup \mathfrak{S}^{\pi}$. For other spectra one suffices to use Corollary 7.2. Namely, if $\sigma \in \mathfrak{S}^{\delta} \cup \mathfrak{S}_{\pi}$ then $\overline{\sigma} \in \mathfrak{S}_{\delta} \cup \mathfrak{S}^{\pi}$ and $\sigma\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \overline{\sigma}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \overline{\sigma}\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} = \sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$.◀

Corollary 8.9. *Let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$ and let S be a subset in $\mathcal{A}_{\mathfrak{g}}(\alpha)$ generating a quasinilpotent normed Lie subalgebra $\mathfrak{F} \subseteq \mathcal{A}_{\mathfrak{g}}$ such that $\widehat{\mathfrak{F}} \in \mathrm{Proj}$. Then $\sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} = \sigma_{\mathrm{u}}^{*}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right)$, $\sigma \in \mathfrak{S}_{\delta}$. In particular, $\sigma^{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}$, $\sigma \in \mathfrak{S}_{\delta}$, whenever X is super-reflexive.*

Proof. The inclusion $\sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} \subseteq \sigma_{\mathrm{u}}^{*}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right)$ was proved in Corollary 8.6. The reverse inclusion follows from Theorem 8.3. Namely,

$$\sigma_{\mathrm{u}}^{*}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right) = \sigma_{\mathrm{u}}^{*}\left(\alpha^{*}|^{\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}}|_{\mathfrak{F}^{\mathrm{op}}}\right) \subseteq \sigma^{*}\left(\alpha^{*}\right)|^{\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}}|_{\mathfrak{F}^{\mathrm{op}}} = \sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}.$$

If X is super-reflexive then $\sigma_{\mathrm{u}}^{*}\left(\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)^{*}\right) = \sigma^{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)$ (see Corollary 8.6). Therefore

$$\sigma\left(\alpha\right)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}} = \sigma^{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right), \sigma \in \mathfrak{S}_{\delta}. \blacktriangleleft$$

To obtain the classical version $(\sigma(f(a)) = f(\sigma(a)))$ of our spectral mapping theorem, we use the following standard argument as in Subsection 7.3. Let $\mathfrak{F}$ be a B-L algebra, S a set of Lie generators, and let (X,α) be a Banach $\mathfrak{F}$-module. The assignment $\widehat{S} : \Delta(\mathfrak{F}) \to \mathbb{C}^{S}$, $\widehat{S}(\lambda) = (\lambda(s))_{s\in S}$, is an injective continuous linear map. We set $\sigma(\alpha(S)) = \widehat{S}(\sigma^{\mathrm{u}}(\alpha))$, $\sigma \in \mathfrak{S}$, which we call Slodkowski $\mathfrak{F}$-spectra of the operator family $\alpha(S)$. Now let $\mathcal{A}_{\mathfrak{g}} \succ (X,\alpha)$, w a set of Lie generators of $\mathfrak{g}$, $\mathfrak{F}$ a normed Lie subalgebra in $\mathcal{A}_{\mathfrak{g}}$ with a set of Lie generators f and let $T = \alpha(u)$, $f(T) = \alpha|^{\mathcal{A}_{\mathfrak{g}}}(f)$. Consider a continuous algebra homomorphism $\mathcal{A}_{\mathfrak{g}} \to C(\sigma_{\mathrm{t}}(T))$, $a \mapsto \varphi_a$, where $\varphi_a(\widehat{w}(\lambda)) = \lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)$, $\lambda \in \sigma_{\mathrm{t}}(\alpha) (= \sigma_{\mathrm{t}}^{\mathrm{u}}(\alpha))$. We identify φ_a with a. Under the assumptions of spectral mapping theorem (Theorem 8.3), we obtain that

$$\sigma(f(T)) = \widehat{f}\left(\sigma^{\mathrm{u}}\left(\alpha|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right)\right) = \widehat{f}\left(\sigma(\alpha)|^{\mathcal{A}_{\mathfrak{g}}}|_{\mathfrak{F}}\right) = \left\{\left(\lambda|^{\mathcal{A}_{\mathfrak{g}}}(a)\right)_{a\in f} : \lambda \in \sigma(\alpha)\right\} =$$

$$= \left\{\left(a(\mu)\right)_{a\in f} : \mu \in \sigma(T)\right\} = f(\sigma(T)),$$

thus $\sigma(f(T)) = f(\sigma(T))$ for all $\sigma \in \mathfrak{S}^{\pi}$.

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